SOME EXTENSIONS OF THE MULTIPLICATION THEOREMS

by

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(Continuación)

§.4 COROLLARIES

This section discusses certain illustrations on important corollaries for our master or key formulas which will give rise to generalizations of some well-known results for FOX's H- and MEIJER's G-functions etc available in the theory of Special-Functions.

However, some illustrative cases are recorded below:

(i) If we substitute $A_j = B_i = \ldots$ etc $= 1$ $(1 \leqslant j \leqslant p_1, 1 \leqslant i \leqslant q_1, \ldots$ etc.) in (3.1) and by specific adjustment of parameters etc., we arrive at the multiplication theorem for MEIJER's G-function of two variables as

$$(4.1) \qquad G\begin{bmatrix} \lambda\, u \\ \lambda\, v \end{bmatrix} \left[\begin{matrix} m_1, & 0 \\ p_1, & q_1 \end{matrix}\right]\begin{matrix} a_{p_1} \\ b_{q_1} \end{matrix} \left| \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix}\begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \right| \left. \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix}\begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right\|$$

$$= \lambda^{a_{p_1}-1} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\lambda} - 1\right)^k$$

$$G\begin{bmatrix} u \\ v \end{bmatrix} \left[\begin{matrix} m_1, & 0 \\ p_1, & q_1 \end{matrix}\right]\begin{matrix} a_{p_1}-1, \; a_{p_1}-k \\ b_{q_1} \end{matrix} \left| \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix}\begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix} \right| \left. \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix}\begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix} \right\|$$

where

$$\mathrm{Re}\, \lambda > \frac{1}{2}, \; q_1 < p_1, \; \left|\frac{1}{\lambda} - 1\right|^{f_{q_3}} < 1$$

and other conditions of validity are obtainable from (3.1) with necessary changes.

Next specializing $m_1 = p_1 = \sigma$, $q_1 = \nu$, $m_2 = m_3 = p_2 = p_3 = \mu$, $n_2 = n_3 = 1$, $q_2 = q_3 = \rho + 1$, $d_1 = f_1 = 0$ and replacing $1 - a_j$, b_j, $1 - e_j$, $1 - d_j$ and $1 - f_j$ *by* α_j, γ_j, β_j, δ_j, β'_j and δ_j etc., and then operating (2.6), our formula (4.1) would reduce to the elegant Multiplication Theorem for KAMPÉ DE FÉRIET's function which, inturn yields for WHITTAKER functions in consequences of (2.8)-(2.10).

(*ii*) Yet another special case of (3.1) would seem to occur when (2.5) is afforded. Indeed we thus find

$$\text{(4.2)}\qquad H^{m,\,n}_{p,\,q}\left[\lambda x \,\middle|\, \begin{matrix}\{(a_p,\, A_p)\}\\ \{(b_q,\, B_q)\}\end{matrix}\right]$$

$$= \lambda^{\frac{a_p - 1}{A_p}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda^{-\frac{1}{A_p}} - 1\right)^k H^{m,\,n}_{p,\,q}\left[x \,\middle|\, \begin{matrix}\{(a_{p-1},\, A_{p-1})\},(a_p - k,\, A_p)\\ \{(b_q,\, B_q)\}\end{matrix}\right]$$

provided

$$\text{Re}\ \lambda > \frac{1}{2},\ q < p,\ \left|\ \lambda^{-\frac{1}{A_p}} - 1\ \right| < 1;$$

and valid by analytic continuation for the conditions referred to (3.1) with obvious alterations.

A corollary of special interest:

If we make use of (2.11), our formula (4.2) would involve a known result of MEIJER [4, p. 213, (4)]:

$$G^{m,\,n}_{p,\,q}\left(\lambda x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right)$$

$$= \lambda^{a_p - 1} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\lambda} - 1\right)^k G^{m,\,n}_{p,\,q}\left(x \,\middle|\, \begin{matrix}a_1, \ldots, a_{p-1}, a_p - k\\ b_1, \ldots, b_q\end{matrix}\right)$$

where

$$n < p,\ \text{Re}\ \lambda > \frac{1}{2}.$$

(*iii*) If we put $p_1 = m_1 = 0$, the H-function reduces to the product of two Fox's H-functions and delete

$$H^{m_2,\,n_2}_{p_2,\,q_2}\left[u\right]$$

etc., from (3.2), we find

(4.3) $$\mathrm{H}_{p,q}^{m,n}\left[\lambda x \,\middle|\, \begin{matrix}\{(a_p, \mathrm{A}_p)\}\\ \{(b_q, \mathrm{B}_q)\}\end{matrix}\right]$$

$$= \lambda^{\frac{b_1}{\mathrm{B}_1}} \sum_{k=0}^{\infty} \frac{1}{k!} (1-\lambda^{\frac{1}{\mathrm{B}_1}})^k \, \mathrm{H}_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{(a_p, \mathrm{A}_p)\}\\ (b_1+k, \mathrm{B}_1), \{({}_2b_q, {}_2\mathrm{B}_q)\}\end{matrix}\right]$$

which provides a generalization of MEIJER's formula [4, p. 213, (1)]:

$$G_{p,q}^{m,n}\left[\lambda x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1, \ldots, b_q\end{matrix}\right]$$

$$= \lambda^{b_1} \sum_{k=0}^{\infty} \frac{1}{k!} (1-\lambda)^k \, G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix}a_1, \ldots, a_p\\ b_1+k, b_2, \ldots, b_q\end{matrix}\right)$$

where $m \geqslant 1$, $|\lambda - 1| < 1$.

(iv) Evidently, with the aid of (2.5) and let $v \to 0$ etc., then (3.3) shows that

(4.4) $$\mathrm{H}_{p,q}^{m,n}\left[\lambda x \,\middle|\, \begin{matrix}\{(a_p, \mathrm{A}_p)\}\\ \{(b_q, \mathrm{B}_q)\}\end{matrix}\right]$$

$$= \lambda^{\frac{a_1-1}{\mathrm{A}_1}} \sum_{k=0}^{\infty} \frac{1}{k!} (1-\lambda^{-\frac{1}{\mathrm{A}_1}})^k \, \mathrm{H}_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}(a_1-k, \mathrm{A}_1), \{({}_2a_p, {}_2\mathrm{A}_p)\}\\ \{(b_q, \mathrm{B}_q)\}\end{matrix}\right]$$

valid render the following assumptions:

(1) m, n, p, q are positive integers such that $1 \leqslant m \leqslant q$, $1 \leqslant n < p$;

(2) Let $\sum_1^q \mathrm{B}_j - \sum_1^p \mathrm{A}_j > 0$, when $x \neq 0$; $\sum_1^q \mathrm{B}_j - \sum_1^p \mathrm{A}_j = 0$;

and

$0 < |x| < \mathrm{D}^{-1}$ where $\mathrm{D} = \prod_1^p (\mathrm{A}_j)^{\mathrm{A}_j} \prod_1^q (\mathrm{B}_j)^{-\mathrm{B}_j}$;

(3) Let $\sum_1^m \mathrm{B}_j - \sum_{m+1}^q \mathrm{B}_j + \sum_1^n \mathrm{A}_j - \sum_{n+1}^p \mathrm{A}_j \equiv \Psi > 0$, $|\arg x| <$

$< \frac{1}{2} \pi \Psi$; and

(4) Let $\mathrm{Re}\, \lambda > \frac{1}{2}$, $q \geqslant 1$, $|1 - \lambda^{-\frac{1}{\mathrm{A}_1}}| < 1$.

An important deduction:

When the parameters are adjusted with the help of (2.11), our formula (4.4) affords a relation

$$G_{p,q}^{m,n}\left(\lambda x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$= \lambda^{a_1-1} \sum_{k=0}^{\infty} \frac{1}{k!}\left(1 - \frac{1}{\lambda}\right)^k G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1 - k, a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

where $n \geqslant 1$, $Re\ \lambda > {}^1/_2$.

This is a special case of a result due to MEIJER [4, p. 213, (3)].

Obviously, (3.4) also provides a generalization of MEIJER's formula [4, p. 213, (3)] to which it would reduce when (2.5) is operated.

v) Finally, we take $p_1 = m_1 = 0$, and cancell

$$H_{p_2, q_2}^{m_2, n_2}\left[u\right]$$

etc. from both the sides of (3.5), our theorem becomes

$$H_{p,q}^{m,n}\left[\lambda x \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_q, B_q)\} \end{matrix}\right] \tag{4.5}$$

$$= \lambda^{\frac{b_q}{B_q}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\lambda^{\frac{1}{B_q}} - 1\right)^k H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} \{(a_p, A_p)\} \\ \{(b_{q-1}, B_{q-1})\}, (b_q + k, B_q) \end{matrix}\right]$$

which would further reduce to a well-known result by MEIJER [4, p. 213, (2)]:

$$G_{p,q}^{m,n}\left(\lambda x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$$

$$= \lambda^{b_q} \sum_{k=0}^{\infty} \frac{1}{k!}(\lambda - 1)^k G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_{q-1}, b_q + k \end{matrix}\right)$$

where $m < q$, $|\lambda - 1| < 1$.

It may be of interest to conclude with the remark that formulas (4.2)-(4.5) yield the results for MACROBERT's E-functions by virtue of (2.12).

In conclusion, we find that very recently SHAH [8-15] has given a detailed account of various interesting and important Generalization, Extension, Unification, Co-ordination and Co-relation for many generalized functions and their basic properties.

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