# SOME EXTENSIONS OF THE MULTIPLICATION THEOREMS

by

MANILAL SHAH

*(Continuación)*

## §.3 MULTIPLICATION THEOREMS

This section derives a set of General Multiplication Theorems for Fox's H-functions of two variables. They are as follows:

Theorem 1. Assumptions: Let

(*i*) $\operatorname{Re}\lambda > \frac{1}{2}$, $q_1 < p_1$, $\left| \lambda^{-\frac{1}{A_{p1}}} - 1 \right| < 1$;

(*ii*) $\gamma_1 \leqslant 0$, $\gamma_2 > 0$, $|\arg u| < \frac{1}{2}\pi\gamma_2$;

(*iii*) $\mu_1 \leqslant 0$, $\mu_2 > 0$, $|\arg v| < \frac{1}{2}\pi\mu_2$; and

(*iv*) $p_1 \geqslant m_1 \geqslant 0$, $q_1 \geqslant 0$, $p_2 \geqslant n_2 \geqslant 0$, $q_2 \geqslant m_2 \geqslant 0$, $p_3 \geqslant n_3 \geqslant 0$, $q_3 \geqslant m_3 \geqslant 0$, $p_1 + p_2 \leqslant q_1 + q_2$, $p_1 + p_3 \leqslant q_1 + q_3$.

Then

$$\text{(3.1)} \qquad H\left[\begin{matrix}\lambda u\\ \lambda v\end{matrix}\,\middle|\, 0 \mid \Psi \mid \Phi \right]$$

$$= \lambda^{\frac{a_{p1}-1}{A_{p1}}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\lambda^{-\frac{1}{A_{p1}}} - 1\right)^k H\left[\begin{matrix}u\\ v\end{matrix}\,\middle|\, \theta_1 \mid \Psi \mid \Phi \right]$$

where

$$\theta_1 = \begin{bmatrix} m_1 , 0 \\ p_1 , q_1 \end{bmatrix} \begin{matrix} \{(a_{p_1-1} , A_{p_1-1})\} , (a_{p_1} - k , A_{p_1}) \\ \{(b_{q_1} , B_{q_1})\} \end{matrix} \text{ ; and}$$

$\Psi$, $\Phi$ have retained the same values as referred to above.

Theorem 2. If

$(i)$ $m_3 \geqslant 1$ , $|1 - \lambda^{\frac{1}{F_1}}| < 1$;

$(ii)$ $\gamma_1 \leqslant 0$ , $\gamma_2 > 0$ , $|\arg u| < \frac{1}{2} \pi \gamma_2$ ; and

$(iii)$ $\mu_1 \leqslant 0$ , $\mu_2 > 0$ , $|\arg v| < \frac{1}{2} \pi \mu_2$.

Then

$$\text{(3.2)} \qquad H\left[\begin{matrix} u \\ \lambda v \end{matrix}\middle| \theta \mid \Psi \mid \Phi \mid\right]$$

$$= \lambda^{\frac{f_1}{F_1}} \sum_{k=0}^{\infty} \frac{1}{k!} (1 - \lambda^{\frac{1}{F_1}})^k \, H\left[\begin{matrix} u \\ v \end{matrix} \; \theta \mid \Psi \mid \Phi_1 \mid\right]$$

provided

$(iv)$ Let

(a) $p_1 \geqslant m_1 \geqslant 0$ , $q_1 \geqslant 0$ , $p_2 \geqslant n_2 \geqslant 0$ , $q_2 \geqslant m_2 \geqslant 0$ , $p_3 \geqslant n_3 \geqslant$ $\geqslant 0$ , $q_3 \geqslant m_3 \geqslant 0$;

(b) $q_1 + q_2 > p_1 + p_2$ , $q_1 + q_3 \geqslant p_1 + p_3$ ; and

$$\text{(v)} \qquad \Phi_1 = \begin{pmatrix} m_3 , n_3 \\ p_3 , q_3 \end{pmatrix} \begin{matrix} \{(e_{p_3} , E_{p_3})\} \\ (f_1 + k , F_1) , \{({}_2f_{q_3} , {}_2F_{q_3})\}. \end{matrix}$$

Theorem 3:

$$\text{(3.3)} \qquad H\left[\begin{matrix} \lambda u \\ \lambda v \end{matrix} \; \theta \mid \Psi \mid \Phi \mid\right]$$

$$= \lambda^{\frac{a_1 - 1}{A_1}} \sum_{k=0}^{\infty} \frac{1}{k!} (1 - \lambda^{-\frac{1}{A_1}})^k \, H\left[\begin{matrix} u \\ v \end{matrix}\middle| \theta_1 \mid \Psi \mid \Phi \mid\right]$$

wehre

$$\theta_1 = \begin{bmatrix} m_1 , 0 \\ p_1 , q_1 \end{bmatrix} \begin{matrix} (a_1 - k , A_1) , \{({}_2a_{p_1} , {}_2A_{p_1})\} \\ \{(b_{q_1} , B_{q_1})\} \end{matrix}$$

and the conditions for validity are:

(*i*) $q_1 \geqslant 1$ , $\mathrm{Re}\,\lambda > \frac{1}{2}$ ; $|1 - \lambda^{-1/A_1}| < 1$;

(*ii*) $\gamma_1 \leqslant 0$ , $\gamma_2 > 0$ , $|\arg u| < \frac{1}{2} \pi \gamma_2$;

(*iii*) $\mu_1 \leqslant 0$ , $\mu_2 > 0$ , $|\arg v| < \frac{1}{2} \pi \mu_2$ ; and

(*iv*) $p_1 \geqslant m_1 \geqslant 0 , q_1 \geqslant 0 , p_2 \geqslant n_2 \geqslant 0 , q_2 \geqslant m_2 \geqslant 0 , p_3 \geqslant n_3 \geqslant 0 ,$ $q_3 \geqslant m_3 \geqslant 0 , p_1 + p_2 \leqslant q_1 + q_2 , p_1 + p_3 \leqslant q_1 + q_3.$

Theorem 4. Assumptions:

(*i*) Let $\gamma_1 \leqslant 0$ , $\gamma_2 > 0$ , $|\arg u| < \frac{1}{2} \pi \gamma_2$;

(*ii*) Let $\mu_1 \leqslant 0$ , $\mu_2 > 0$ , $|\arg v| < \frac{1}{2} \pi \mu_2$;

(*iii*) Let

(a) $p_1 \geqslant m_1 \geqslant 0 , q_1 \geqslant 0 , p_2 \geqslant n_2 \geqslant 0 , q_2 \geqslant m_2 \geqslant 1 , p_3 \geqslant n_3 \geqslant$ $\geqslant 0 , q_3 \geqslant m_3 \geqslant 0;$

(b) $p_1 + p_2 \leqslant q_1 + q_2 , p_1 + p_3 \leqslant q_1 + q_3.$

Then

$$\text{(3.4)} \qquad H \left[ \begin{matrix} \lambda u \\ v \end{matrix} \middle| \theta \mid \Phi \mid \Psi \right]$$

$$= \lambda^{\frac{c_1 - 1}{C_1}} \sum_{k=0}^{\infty} \frac{1}{k!} (1 - \lambda^{-\frac{1}{C_1}})^k \, H \left[ \begin{matrix} u \\ v \end{matrix} \middle| \theta \mid \Psi_1 \mid \Phi \right]$$

provided

$$\Psi_1 = \begin{pmatrix} m_2 , n_2 \\ p_2 , q_2 \end{pmatrix} \begin{matrix} (c_1 - k , C_1) , \{({}_2c_{p_2} , {}_2C_{p_2})\} \\ \{(d_{q_2} , D_{q_2})\} \end{matrix} \quad ; \text{ and}$$

(*iv*) $\mathrm{Re}\,\lambda > \frac{1}{2}$, $|1 - \lambda^{-1/C_1}| < 1$.

Theorem 5: If

(*i*) $m_3 < q_3$, $|\lambda^{\frac{1}{F_3}} - 1| < 1$;

(*ii*) $\gamma_1 \leqslant 0$, $\gamma_2 > 0$, $|\arg u| < \frac{1}{2}\pi\gamma_2$; and

(*iii*) $\mu_1 \leqslant 0$, $\mu_2 > 0$, $|\arg v| < \frac{1}{2}\pi\mu_2$.

Then

$$\text{(3.5)} \qquad H\left[\begin{matrix} u \\ \lambda v \end{matrix}\middle| \theta \mid \Psi \mid \Phi \mid\right]$$

$$= \lambda^{\frac{f_{q_3}}{F_{q_3}}} \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda^{\frac{1}{F_{q_3}}} - 1)^k \, H\left[\begin{matrix} u \\ v \end{matrix}\middle| \theta \mid \Psi \mid \Phi_1 \mid\right]$$

where

$$\Phi_1 = \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix}^{\{(e_{p_3}, E_{p_3})\}}_{\{(f_{q_3-1}, F_{q_3-1})\}, (f_{q_3} + k, F_{q_3})} \text{; and}$$

(*iv*) $p_1 \geqslant m_1 \geqslant 0$, $q_1 \geqslant 0$, $p_2 \geqslant n_2 \geqslant 0$, $q_2 \geqslant m_2 \geqslant 0$, $p_3 \geqslant n_3 \geqslant 0$, $q_3 \geqslant m_3 \geqslant 0$, $p_1 + p_2 \leqslant q_1 + q_2$, $p_1 + p_3 \leqslant q_1 + q_3$.

Proof of (3.1):

Denoting the left member

$$H\left[\begin{matrix} \lambda u \\ \lambda v \end{matrix}\right]$$

of (3.1) by its Mellin-Baranes integral representation (1.1), we get

$$\frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Delta_3 [\xi + \eta] \, \Delta_1(\xi) \, \Delta_2(\eta) \, \lambda^{\xi+\eta} \, u^{\xi} v^{\eta} \, d\xi \, d\eta$$

Next $\lambda^{\xi+\eta}$ after replacing $\lambda$ by $(1+h)^{-A_{p_1}}$ can be written as

$$(1+h)^{1 - a_{p_1} + a_{p_1} - A_{p_1}(\xi + \eta) - 1}.$$

Application of (2.1) transforms the latter into

$$\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \Delta_3[\xi+\eta]\,\Delta_1(\xi)\,\Delta_2(\eta)\,u^{\xi}\,v^{\eta}$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,[-\{a_{p1}-A_{p1}(\xi+\eta)-1\}]_k}{k!}\,h^k\,d\xi\,d\eta.$$

Now employ (2.2) and then change the order of integration and summation, the expression becomes

$$(1+h)^{1-a_{p1}}\sum_{k=0}^{\infty}\frac{h^k}{k!}\,\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}\Delta_3[\xi+\eta]\,\Delta_1(\xi)\,\Delta_2(\eta)$$

$$\frac{\Gamma[a_{p1}-A_{p1}(\xi+\eta)]}{\Gamma[a_{p1}-k-A_{p1}(\xi+\eta)]}\,u^{\xi}\,v^{\eta}\,d\xi\,d\eta$$

where the paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(d_j - D_j\,\xi)$ $(1 \leqslant j \leqslant m_2)$, $\Gamma(f_j - F_j\,\eta)$ $(1 \leqslant j \leqslant m_3)$ and $\Gamma[a_{p1} - A_{p1}(\xi+\eta)]$ lie to the right and those of $\Gamma(1 - c_j + C_j\,\xi)$ $(1 \leqslant j \leqslant n_2)$, $\Gamma(1 - e_j + E_j\,\eta)$ $(1 \leqslant j \leqslant n_3)$ and $\Gamma[1 - a_j + A_j(\xi+\eta)]$ $(1 \leqslant j \leqslant m_1)$ lie to the left of the imaginary axis.

To justify the inversion of order of integration and summation involved in the process, we observe that.

(*i*) the term- by-term integration is valid as it is independent of the argument $h$ and conditions stated in (3.1) justify the existence of every H-function in the series;

(*ii*) moreover the series

$$\sum_{k=0}^{\infty}\frac{(-1)^k\,[-\{a_{p1}-A_{p1}(\xi+\eta)-1\}]_k}{k!}\,h^k$$

is uniformly convergent as long as $|h| < 1$ [4, p. 68, §.2.1.6];

(*iii*) Fox's H-function is an analytic function and continuous for all finite values of $u \geqslant u_0 > 0$; and $v \geqslant v_0 > 0$; and

(*iv*) the double-contour integral converges in view of the conditions referred to (1.2).

Therefore, the inversion is justified by an appeal of Bromwich [3, p. 500].

Finally we freely substitute the value of $h$ and performance of (1.1) in the latter which obviously leads to the R. H. S. of (3.1).

This concludes the proof of (3.1).

Evaluation of (3.2) is based upon an analysis similar to the one utilized in (3.1); and it is readily seen that (3.2) follows at once with the aid of (2.3). Hence, we omit such details.

Derivation of (3.3)-(3.5) are much akin to (3.1) and (3.2). We conveniently delete repetitions General Remark:

The change of order of integration and summation in all cases seems to be most thoroughly justified by virtue of De :a Vallée Poussin's theorem [3, p. 500) and by principle of analytic continuation subject to the conditions imposed with the results.

*(Continuará.)*