SOME EXTENSIONS OF THE MULTIPLICATION THEOREMS

by

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PREFACE

Numerous transcendents —the so— called special functions and their properties due to usefulness as a tool in Applied Mathematics have been extensively developed during the course of investigation by several authors. In a sequence of papers, the author has afforded Generalizarion and Extension of a large number of results in the branches of Special Functions, Operational Calculus, Integral Transforms and Integral Equations etc. Here the author establishes a set of most general multiplication theorems related to Fox's H-functions of two variables.

A short dsscription of two theorems runs as under:

Multiplication Theorems:

(i)
$$H\left[\begin{matrix}\lambda u|\\ \lambda v|\end{matrix}\,\theta\,|\,\Psi\,|\,\Phi\,|\right]$$
$$= \lambda^{\frac{a_{p1}-1}{A_{p1}}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\lambda^{-\frac{1}{A_{p1}}} - 1\right)^k H\left[\begin{matrix}u|\\ v|\end{matrix}\,\theta_1\,|\,\Psi\,|\,\Phi\,|\right]; \text{ and}$$

(ii)
$$H\left[\begin{matrix}u\,|\\ \lambda v|\end{matrix}\,\theta\,|\,\Psi\,|\,\Phi\,|\right]$$
$$= \lambda^{\frac{f_1}{F_1}} \sum_{\lambda=0}^{\infty} \frac{1}{k!}\left(1 - \lambda^{\frac{1}{F_1}}\right)^k H\left[\begin{matrix}u|\\ v|\end{matrix}\,\theta\,|\,\Psi\,|\,\Phi_1\,|\right]$$

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in terms of SHAH's notation

$$H\begin{bmatrix} x \\ y \end{bmatrix}$$

for Fox's H- function in two arguments.

By means of suitable substitutions, the theorems are associated with many results proved carlier by MEIJER, MACROBERT, SHAH and several others.

Also, they yield corollaries of special interest cihich may prove to be useful in the derivation of some problems in Science, Thechology and Molecular Quantum Mechanics.

§.1 PREREQUISITES

MUNOT and KALLA [6, p. 68, (2)] aroused interest in what they define a generalized H-function of two variables. Further, SHAH [7, p. 1, (1-1)] has given it in a slightly variant computable form

$$(1.1) \quad H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H\left[\begin{matrix} x \\ y \end{matrix}\middle| \theta \mid \Psi \mid \Phi \mid\right]$$

$$= \frac{1}{(2 \pi i)^2} \int_{L_1} \int_{L_2} \Delta_3 [\xi + \eta] \, \Delta_1 (\xi) \, \Delta_2 (\eta) \, x^{\xi} y^{\eta} \, d\xi \, d\eta$$

where an empty product is to be interpreted as unity.

Here

$$(i) \begin{cases} \theta = \begin{bmatrix} m_1, 0 \\ p_1, q_1 \end{bmatrix} \begin{matrix} \{(a_{p_1}, A_{p_1})\} \\ \{(b_{q_1}, B_{q_1})\} \end{matrix} ; \Psi = \begin{pmatrix} m_2, n_2 \\ p_2, q_2 \end{pmatrix} \begin{matrix} \{(c_{p_2}, C_{p_2})\} \\ \{(d_{q_2}, D_{q_2})\} \end{matrix} ; \\ \Phi = \begin{pmatrix} m_3, n_3 \\ p_3, q_3 \end{pmatrix} \begin{matrix} \{(e_{p_3}, E_{y_3})\} \\ \{(f_{q_3}, F_{q_3})\} \end{matrix} ; \text{ and} \end{cases}$$

(ii)

$$\Delta_3[\xi+\eta] = \frac{\prod_{j=1}^{m_1} \Gamma[1-aj+Aj\,\xi+Aj\,\eta]}{\prod_{j=m+1}^{p_1} \Gamma[a_j - A_j\,\xi - A_j\,\eta_j \prod_{j=1}^{q_1} \Gamma\,[bj + B_j\,\xi + B_j\,\eta]}$$

$$\Delta_1(\xi) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - D_j\,\Gamma) \prod_{j=1}^{n_2} \Gamma(1-c_j+C_j\,\xi)}{\prod_{j=m_2+1}^{q_2} \Gamma(1-d_j+D_j\,\xi) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - C_j\xi)}$$

$$\Delta_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(f_j - F_j\,\eta) \prod_{j=1}^{n_3} \Gamma(1-e_j+E_j\,\eta)}{\prod_{j=m_3+1}^{q_3} \Gamma(1-f_j+F^{\prime}\,\eta) \prod_{j=n_3+1}^{p_3} \Gamma(e_j - E_j\,\eta)}$$

in which $p_1 \geqslant m_1 \geqslant 0$, $q_1 \geqslant 0$, $p_2 \geqslant n_2 \geqslant 0$, $q_2 \geqslant m_2 \geqslant 0$, $p_3 \geqslant n_3 \geqslant 0$, $q_3 \geqslant m_3 \geqslant 0$, $q_1 + q_2 \geqslant p_1 + P_3$, $q_1 + q_3 \geqslant p_1 + p_3$ and cach of p's q's, m's and n's is a non-negative integers.

The integral (1.1) converges if

$$\gamma_1 = \sum_1^{p_1} Aj + \sum_1^{p_2} Cj - \sum_1^{q_1} Bj - \sum_1^{q_2} Dj \leqslant 0\ ;$$

$$\mu_1 = \sum_1^{p_1} Aj + \sum_1^{p_3} Ej - \sum_1^{q_1} Bj - \sum_1^{q_3} Fj \leqslant 0;$$

$$\gamma_2 = \sum_1^{m_1} Aj - \sum_{m_1+1}^{p_1} Aj - \sum_1^{q_1} Bj + \sum_1^{n_2} Cj - \sum_{n+1}^{p_2} Cj + \sum_1^{m_2} Dj -$$

$$- \sum_{m_2+1}^{q_2} Dj > 0,\ |\arg x| < \frac{q}{2} \overline{\Lambda}\,\gamma_2\ ;$$

$$\mu_2 = \sum_1^{m_1} Aj - \sum_{m_1+1}^{p_1} Aj - \mathrm{E}_1^{q_1} Bj + \sum_1^{n_3} Ej - \sum_{n_3+1}^{p_3} Ej + \sum_1^{m_3} Fj -$$

$$- \sum_{m_3+1}^{q_3} Fj > 0,\ |\arg y| < \frac{1}{2} \overline{\Lambda}\,\mu_2.$$

For more references to

$$H\begin{bmatrix} x \\ y \end{bmatrix}$$

see MUNOT and KALLA [6].

In this paper, the author presents a systematic achievement to obtain an elegant unification of several extensions of interesting formulas dueto FOX's H-, MEIJER's G- and MACROBERT's E-functions etc.

Section 2 incorporates preliminaries employed in the investigation. Section 3 contains the derivation of Multiplication Theorems for Fox's H-functions of two variables. Several interesting corollaries are presented in Section 4.

Often, as a space saver and printing convenience, γ_1. γ_2, μ_1, μ_2, θ, Ψ, Φ etc have the same meaning throughont as cited above.

§.2 PRELIMINARIES

Notations, Formulas and Results:

(i) a_p or $|a|_p = a_1, a_2, \ldots, a_p$;

(ii) $\pi(a_p)_s$ stands for the product $(a_1)_s\,(a_2)_s \ldots (a_p)_s$;

(iii) $\{({}_pa_q, {}_pA_q)\} = (a_p, A_p), (a_{p+1}, A_{p+1}), \ldots, (a_q, A_q)$;

(iv) $[a]_n = \dfrac{\Gamma[a+n]}{\Gamma[a]} = a(a+1)\ldots(a+n-1)$; $n = 1, 2, 3, \ldots$;

(2.1) $${}_1F_0(-n\,;\,-\,;\,-\zeta) = (1+\zeta)^n = \sum_{k=0}^{\infty} \frac{(-1)^k[-n]_k}{k!}\zeta^k \text{ where}$$ $|\zeta| < 1$,

(2.2) $$(-1)^k[-n]_k = \frac{\Gamma[1+n]^{k=0}}{\Gamma[1+n-k]},$$

(2.3) $${}_1F_0(a\,;\,-\,;\,\zeta) = (1-\zeta)^{-a} = \sum_{n=0}^{\infty} \frac{[a]_n}{n!}\zeta^n, \text{ where } |\zeta| < 1.$$

When $A's = B's = C's = D's = E's = F's = 1$, (1.1) reduces to MEIJER's G function of two variables:

(2.4) $$G\begin{bmatrix} x \\ y \end{bmatrix} = G\begin{bmatrix} x \\ y \end{bmatrix}\left|\begin{bmatrix} m_1, & 0 \\ p_1, & q_1 \end{bmatrix}\begin{matrix} a_{p_1} \\ b_{q_1} \end{matrix}\right|\left.\begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix}\begin{matrix} c_{p_2} \\ d_{q_2} \end{matrix}\right|\left.\begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix}\begin{matrix} e_{p_3} \\ f_{q_3} \end{matrix}\right|$$

introduced carlier by AGARWAL [2, p. 537].

A worth mentioning special case of the confluence principle of (1.1) is

$$\lim_{y \to 0} H\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} m_1, 0 \\ p_1, q_1 \end{bmatrix} \begin{matrix} \{(a_{p_1}, A_{p_1})\} \\ \{(b_{q_1}, B_{q_1})\} \end{matrix} \Bigg| \Psi \Bigg| \begin{pmatrix} 1, 0 \\ 0, 1 \end{pmatrix} (0, 1) \Bigg| \tag{2.5}$$

$$= H^{m_1 + m_2, n_2}_{p_1 + p_2, q_1 + q_2} \Bigg[x$$

$$x \left| \begin{matrix} (1 - a_{m_1}, A_{m_1}), \{(c_{p_2}, C_{p_2})\}, (1 - a_{w_1+1}, A_{m_1+1}), \ldots, (1 - a_{p_1}, A_1) \\ \{(d_{q_2}, D_{q_2})\}, \{(b_{q_1}, B_{q_1})\} \end{matrix} \right|$$

where $p_1 + p_2 \leqslant q_1 + q_2$, and $H[x]$ represents Fox's H- function [5, p. 408].

A generalized hypergeometric function of two variables due to Kampé de Fériet J. [1, p. 150] has been defined in computable representation

$$F^{m, l}_{n, p} \left[\begin{matrix} |a|_m : |b, b'|_q \\ |c|_n : |d, d'|_p \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right] \tag{2.6}$$

Again, if parameters in (2.6) are selected in view of [1, p. 14 (125)], the Appell function F_2 and its confluent function are indicated by

$$F^{1, 1}_{0, 1} \left[\begin{matrix} a : b, b' \\ - : d, d' \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right] = F_2 (a : b, b'; d, d'; x, y); \tag{2.7}$$

$$\Psi_2 (a, d, d'; x, y) = \sum_{m, n=0}^{\infty} \frac{(a)_{m+n}\, x^m\, y^n}{(d)_m\, (d|)_n\, m!\, n!}. \tag{2.8}$$

Also, from [1, p. 132, (133)]:

$$M_{k, \mu, \nu} (x, y) = x^{\mu + \frac{1}{2}}\; y^{\nu + \frac{1}{2}}\; e^{-\frac{1}{2}(x+y)} \tag{2.9}$$

$$\Psi_2 (\mu + \nu - k + 1, 2\mu + 1, 2\nu + 1; x, y), \text{ and}$$

$$\lim_{\epsilon \to 0} M_{k, \mu, \epsilon^{-\frac{2}{1}}} (x, \epsilon^2 y) = M_{k, \mu}(x) \tag{2.10}$$

where $M_{1, \mu, \nu}$ (x, y) and $M_{k, \mu}(x)$ are the Whittaker functions in two and one variables respectively.

The following special cases of H-function are worthy of note:

$$H^{m,\,n}_{p,\,q}\left[x\middle|\begin{matrix}\{(a_p,\,1)\}\\ \{(b_q,\,1)\}\end{matrix}\right] = G^{m\;n}_{p\;q}\left(x\middle|\begin{matrix}a_1,\,\ldots,\,a_p\\ b_1,\,\ldots,\,b_q\end{matrix}\right) \tag{2.11}$$

where G is MEIJER's G function [4, p. 207 (1)].

$$H^{p,\,1}_{q+1,\,p}\left[x\middle|\begin{matrix}(1,\,1),\,\{(b_q,\,1)\}\\ \{(a_p,\,1)\}\end{matrix}\right] = E\,(p\;;\,a_r:q\;;\,b_s:x) \tag{2.12}$$

where E is MAC ROBERT,S E-function [4, p. 215 (2)].

(Continuará.)