A SPHERICAL REPRESENTATION OF THE REAL DIRECTIONAL DERIVATIVE

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Let $f : A \to R$ where A is some open subset of the finite dimensional vector space E_n $(n \geqslant 2)$. If f is differentiable at $\mathbf{x} = (x_1, \ldots, x_n) \in A$, and $\mathbf{u}$ is a unit vector in E_n, the directional derivative of f at $\mathbf{x}$, in the direction specified by $\mathbf{u}$, is the scalar given by

(1) $$\rho = D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$

where $\nabla f = (f_{x_1}, \ldots, f_{x_n})$ is the gradient of f. We propose to seek a geometric representation of ρ *as a function* of $\mathbf{u}$ at a fixed point $\mathbf{x}$. For convenience, we introduce also a vector $\mathbf{v}$ defined by the equation

(2) $$\mathbf{v} = \rho\, \mathbf{u}$$

If $\rho \neq 0$ this is a vector in the direction of $\mathbf{u}$, or in the opposite direction, depending on whether $\rho > 0$ or $\rho < 0$. In any case, $|\mathbf{v}| = |\rho|$.

If $f_{x_i} = 0$ $(i = 1, \ldots, n)$ at $\mathbf{x}$, then $\nabla f = 0$ and $\rho = 0$ for all $\mathbf{u}$.

Assuming

$$|\nabla f| = \left(\sum_1^n f_{x_i}{}^2\right)^{\frac{1}{2}} \neq 0,$$

and letting $\omega = \sphericalangle (\mathbf{u}, \nabla f)$, equation (1) can be written in the form

(3) $$\rho = |\nabla f| \cos \omega = \text{proj}_{\mathbf{u}} \nabla$$

If the point P is the tip of the vector $\mathbf{v} = \rho\, \mathbf{u}$, Q is the tip of the vector ∇f, and 0 denotes the coordinate origin, it follows that $\sphericalangle 0PQ = \pi/2$. Hence the locus of P as $\mathbf{u}$ varies in direction is the $(n - 1)$-dimensional spherical surface G (a circle if $n = 2$) with center at $\frac{1}{2} \nabla f$ and radius $\frac{1}{2} |\nabla f|$. The graph of G is given in Cartesian coordinates X_i by

$$(4) \qquad \sum_{i=1}^{n} \left(X_i - \frac{1}{2} \tfrac{1}{2} f_{x_i} \right)^2 = \frac{1}{4} \sum_{i=1}^{n} f^2{}_{x_i}$$

However, it must be noted that the spherical surface defined by (3) is described twice by the tip of **v** as **u** takes once every possible direction in the space E_n. To see this, consider the unit vector $\mathbf{u}_1 = -\mathbf{u}$. Then $\omega_1 = \sphericalangle (\mathbf{u}_1, \nabla f) = \omega + \pi$, and

$$\rho_1 = |\nabla f| \cos \omega_1 = - |\nabla f| \cos \omega = - \rho$$

so that

$$\mathbf{v}_1 = \rho_1 \mathbf{u}_1 = \rho \mathbf{u} = \mathbf{v}$$

Also, $\mathbf{v}_2 = \mathbf{v}$ or $\rho_2 \mathbf{u}_2 = \rho \mathbf{u}$ implies either $\rho_2 = \rho$ and $\mathbf{u}_2 = \mathbf{u}$ or $\rho_2 = -\rho$ and $\mathbf{u}_2 = -\mathbf{u}$. Thus, the same vector **v** is obtained only for opposite orientations of **u**.

For any vector **u** orthogonal to ∇f (i.e. for any unit vector lying in the tangent hyperplane to G at 0) we have

$$\rho = \nabla f \cdot \mathbf{u} = 0$$

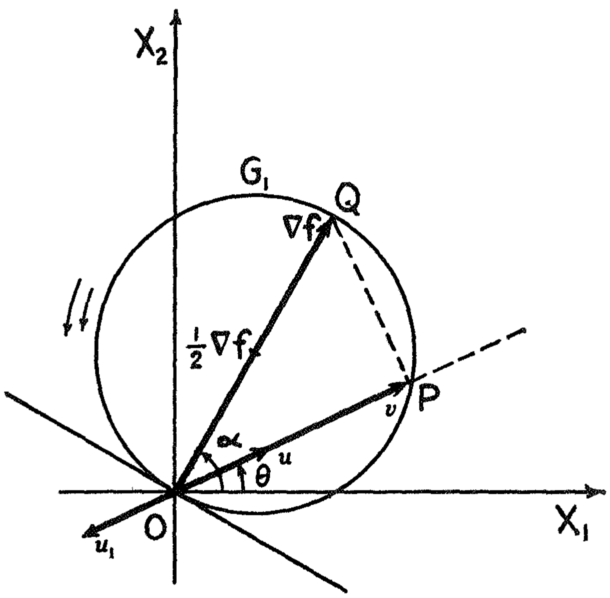

Fig. 1.

Clearly, from (3) it follows, for $\omega = 0$,

$$\rho_{max} = |\nabla f|$$

and, for $\omega = \pi$,

$$\rho_{min} = - |\nabla f|$$

Hence, the maximum value of the directional derivative is attained when the orientation of **u** is the same as that of the gradient of f, and the minimum value is attained when the orientation of **u** is opposite to that of the gradient.

EXAMPLES. 1. *The plane case.* Letting $\theta = \text{Arg}\,\mathbf{u}$ $(0 \leqslant \theta \leqslant 2\pi)$, $\alpha = \text{Arg}\,\nabla f$, we have

$$\rho = |\nabla f| \cos(\theta - \alpha), \qquad 0 \leqslant \theta \leqslant 2\pi$$

This is the equation of the circle G, with center at $\frac{1}{2}\nabla f$ and radius $\frac{1}{2}|\nabla f|$ described twice in the positive (counterclockwise) direction (Fig. 1).

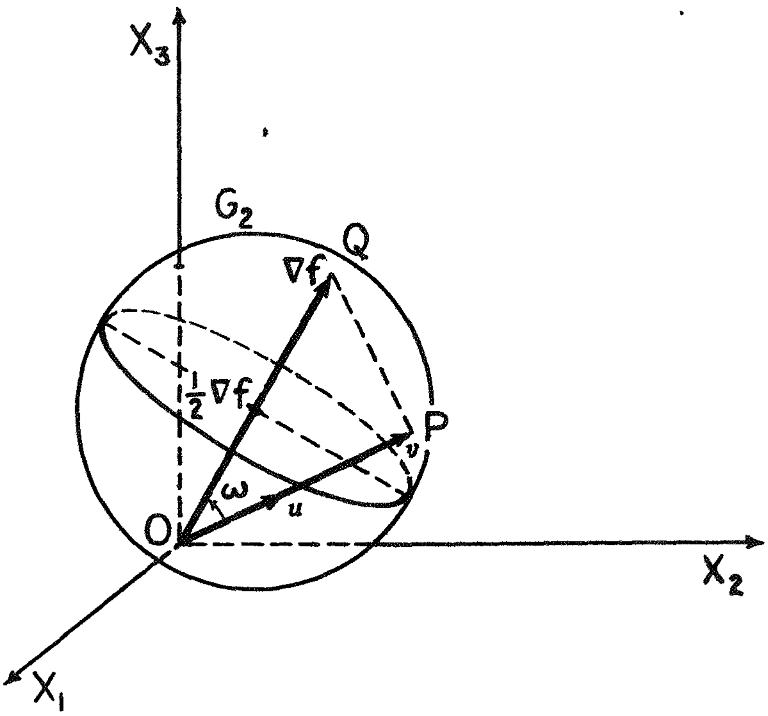

Fig. 2.

2. *The 3-dimensional case.* Similarly, for all positions of the vector **u** in the space E_3 the vector **v** describes twice the ordinary sphere G_2 (Fig. 2).

3. *The Kasner circle.* The Kasner circle [1, 2] of a complex function $f = U + iV$, with differentiable components U, V at a point z, represents geometrically the values of the complex directional derivative $f'_\theta(z)$ at z for values of the direction angle θ in the interval $[0, 2\pi]$. Its equation is given by

$$\xi = f_z + f_{\bar z}\, e^{-2i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi$$

and so it has center at f_z and radius $|f_{\bar z}|$, the circle being described twice in the negative (clockwise) direction as θ varies from 0 to 2π.

We may associate to any Kasner circle K the circle H defined by the equation

$$\xi_1 = f_{\bar z} + f_z e^{2i\theta}, \qquad 0 \leqslant \theta \leqslant 2\pi$$

This circle has its center at $f_{\bar z}$ and radius $|f_z|$. It is described twice in the positive direction as θ varies in $[0, 2\pi]$.

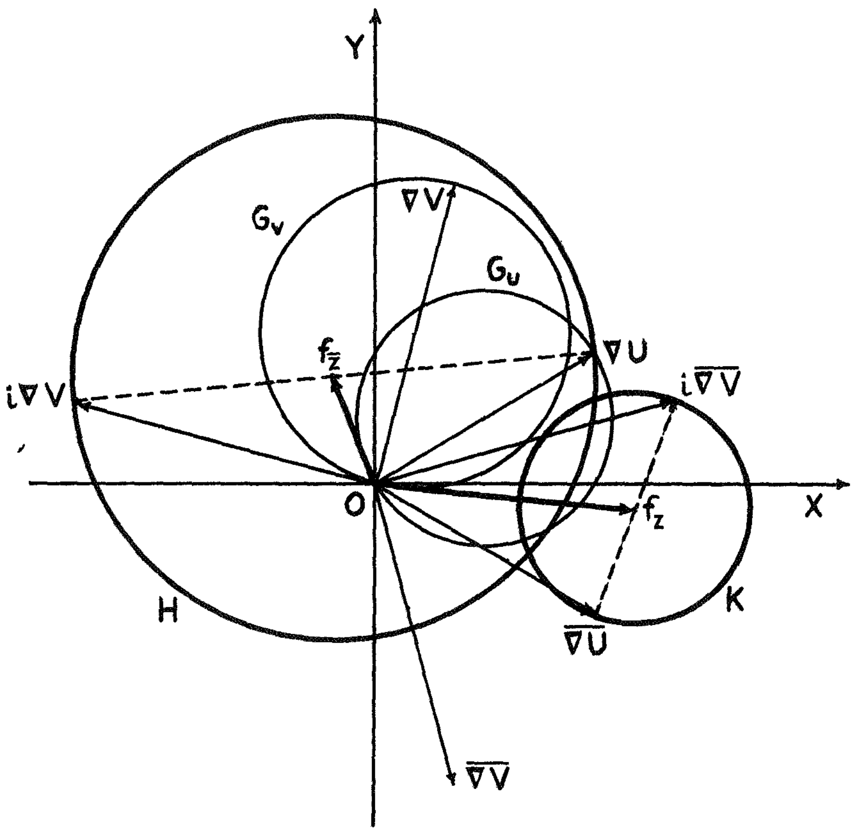

Fig. 3.

It could be asked whether the circles K and H can be constructed geometrically from the circles G_U and G_V representing the real directional derivatives of the functions U and V. To answer this question we note that the circles G_U and G_V are determined by the corresponding gradients ∇U and ∇V, which in the notation of Complex Analysis are written

$$\nabla U = U_x + iU_y \quad \text{and} \quad \nabla V = V_x + iV_y$$

Also, we note that the position and size of the circles H and K are determined by the partial derivatives f_z and $f_{\bar{z}}$. It is an easy exercise to show that

$$f_z = \tfrac{1}{2}(\overline{\nabla U} + i\,\overline{\nabla V})$$

and

$$f_{\bar{z}} = \tfrac{1}{2}(\nabla U + i\,\nabla V)$$

Hence, the point f_z is the midpoint of the line segment joining the tips of $\overline{\nabla U}$ and $i\,\overline{\nabla V}$, while $f_{\bar{z}}$ is the midpoint of the line segment joining the tips of ∇U and $i\,\nabla V$. Furthermore, the points $\overline{\nabla U}$ and $i\,\overline{\nabla V}$ are the endpoints of a diameter of the Kasner circle, since

$$|\overline{\nabla U} - i\,\overline{\nabla V}| = |\nabla U + i\,\nabla V| = 2\,|f_{\bar{z}}|$$

Similarly, ∇U and $i\,\nabla V$ are the endpoints of a diameter of the H circle. These observations lead to the construction shown in Figure 3 for the Kasner circle and its associate.

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