A THEOREM CONCERNING THE MEANS OF AN ENTIRE FUNCTION AND ITS DERIVATIVES

by

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1. Let f(z) be an entire function of order ρ and lower order λ.

For $0 < \delta < \infty$, and $z = re^{i\theta}$, let

$$\{ M_\delta (r, f) \}^\delta = \mu_\delta (r, f) = \frac{1}{2\pi} \int_0^{2\pi} | f(re^{i\theta}) |^\delta \, d\theta$$

and

$$\{ M_\delta (r, f^{(m)}) \}^\delta = \mu_\delta (r, f^{(m)}) = \frac{1}{2\pi} \int_0^{2\pi} | f^{(m)} (re^{i\theta}) |^\delta \, d\theta,$$

where $f^{(m)}(z)$ denotes the mth derivative of $f(z)$. It is known [3] that

$$\lim_{r \to \infty} {}^{\sup}_{\inf} \frac{\log\log M_\delta (r, f)}{\log r} = {}^{\rho}_{\lambda} \quad . \tag{1.1}$$

It was proved by Juneja [1] that, *'for every entire function* f(z), *other than a polynomial,*

$$\limsup_{r \to \infty} \frac{\log r \, [\mu_\delta (r, f^{(m)})/\mu_\delta (r, f)]^{1/m\delta}}{\log r} = \rho \ (0 < \delta < 1),$$

where r *tends to infinity through values excluding an exceptional set of at most finite measure.'*

In this note, we generalise the above result of Juneja [1]. Our theorem is not only more general than Juneja's theorem, but has the different proof and more widely applicable.

2. First of all we point out an important point (*) over looked by Juneja

In the proof of the theorem he has used two lemmas. His Lemma 3 is based on the following result of Valiron [4; p. 106]

$$\frac{f^{(m)}(z)}{f(z)} = \left(\frac{\nu(r)}{z}\right)^m [1 + h_m(z)\ \nu(R)^{-1/16}],\ |\ h_m(z)\ | < k$$

i.e.

$$|\ f^{(m)}(z)/f(z)\ | \geqslant \left(\frac{\nu(r)}{r}\right)^m [1 - k\ \nu(R)^{1/16}]. \tag{2.1}$$

This result is valid at the points on the circle $|\ z\ | = r$ at which one of the functions

$$f(z),\ \ldots\ldots,\ \left(\frac{z}{\nu(r)}\right)^m f^{(m)}(z)$$

is greater in modulus than $M(r)\ \{\ \nu(r)\ \}^{-1/8}$. At remaining points, the validity of this result is not known. Now, $\mu_\delta(r, f)$ is the mean of $|\ f(re^{i\theta})\ |^\delta$ taken over the circle $|\ z\ | = r$. We can carry out integration in (2.1) to form the means of $f(z)$ and $f^{(m)}(z)$ over the circle $|\ z\ | = r$, provided of course the set of omission points has a measure zero.

Now, consider the entire function $\exp(z)$. For this function $M(r) = \exp(r)$, $|\ \exp(z)\ | = \exp(r \cos\theta)$ and $\nu(r) = n$, for $E_n = n \leqslant r < n+1$. Let

$$J(r) = \{\ re^{i\theta}\ |\ r \in E_n\ ;\ \theta \underset{\leqslant}{\geqslant} \cos^{-1}\left(1 - \frac{\log r}{8\, r}\right)$$

according as $0 \leqslant \theta \leqslant \pi$, or $\pi \leqslant \theta \leqslant 2\pi$ }.

Clearly the total variation of logr in E_n tends to infinity with n. Also, at all points of the above set J, we have both the numbers

$$|\ \exp(z)\ |\ \text{and}\ |\ \left(\frac{z}{\nu(r)}\right)^m \exp(z)\ |$$

are less than or equal to $M(r)\ \{\ \nu(r)\ \}^{-1/8}$ and $m(J) > 0$. This establishes the fact that entire functions with $|\ f(z)\ |$, $|\ (z/\nu(r))^m f^m(z)\ |$ less than or equal to $M(r)\ \{\ \nu(r)\ \}^{-1/8}$ over a set of measure greater than zero exists. Thus, for all entire functions the integration carried out by Juneja is not justified. Hence, his proof is incorrect.

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* This point is not observed in the review of Juneja's paper [1] MR [33 # 2814].

Lemma 4 of Juneja [ibid] is based on the following unproved assertion

$$\left| \frac{f^{(1)}(z)}{f(z)} \right| \leqslant (r^{(\rho+\varepsilon-1)}) .$$

3. Now, we state and prove our Theoren.

THEOREM *For every entire function,*

$$\lim_{r \to \infty} {}^{\sup}_{\inf} \frac{\log [r \{M_\delta (r, f^{(m)}/M_\delta (r, f)\}^{1/m}}{\log r} = {}^{\rho}_{\lambda} ,$$

where $m = 1,2 \ldots\ldots m$ *and* $0 < \delta < 1$.

Proof. We know that for $\varepsilon > 0$ and large r

$$\frac{M_\delta (r, f^{(1)})}{M_\delta (r, f)} \leqslant r^{(\rho+\varepsilon-1)} , 0 < \delta < 1 , \rho < \infty. \qquad [2]$$

Since order function is invariant under differentiation, therefore we have

$$M_\delta (r, f^{(p)})/M_\delta (r, f^{(p-1)}) \leqslant r^{(\rho+\varepsilon-1)}.$$

Giving p the values 1, 2, 3, ... , m and multiplying the m inequalities thus obtained, we get,

$$M_\delta (r, f^{(m)})/M_\delta (r, f) \leqslant r^{(\rho+\varepsilon-1)m}.$$

Further, if $f(z)$ is an entire function of finite lower order λ, then we get similarly, for a sequence of values of r tending to infinity,

$$M_\delta (r, f^{(m)})/M_\delta (r, f) \leqslant r^{(\lambda+\varepsilon-1)m} . \qquad \lambda < \infty.$$

Consequently, we get

$$\lim_{r \to \infty} {}^{\sup}_{\inf} \frac{\log [r \{ M_\delta (r, f^{(m)})/ M_\delta (r, f) \}^{1/m}]}{\log r} \leqslant . {}^{\rho}_{\lambda} \qquad (3.1)$$

By the property of a derivative, we have

$$| f^{(p-1)} (re^{i\theta}) | \leqslant | f^{(p)} (re^{i\theta}) | \varepsilon r + \varepsilon^2 r + | f^{(p-1)} (\overline{r - r\varepsilon}\, e^{i\theta}) |$$

$$\sim | f^{(p-1)} (re^{i\theta}) | \varepsilon r + | f^{(p-1)} (\overline{r - r\varepsilon}\, e^{i\theta}) | .$$

Owing to $(a + b)^\delta \leqslant a^\delta + b^\delta$, $(0 < \delta < 1)$, we have

$$| f^{(p)} (re^{i\theta}) |^\delta (r\varepsilon)^\delta \geqslant | f^{(p-1)} (re^{i\theta}) |^\delta - | f^{(p-1)} (\overline{r - r\varepsilon}\, e^{i\theta}) |^\delta$$

or

$$\frac{1}{2\pi}\int_0^{2\pi} | f^{(p)}(re^{i\theta}) |^{\delta} d\theta \geqslant$$

$$\geqslant \left\{ \frac{1}{2\pi}\int_0^{2\pi} | f^{(p-1)}(re^{i\theta}) |^{\delta} d\theta - \frac{1}{2\pi}\int_0^{2\pi} | f^{(p-1)}(\overline{r - r\varepsilon}\, e^{i\theta}) |^{\delta} d\theta/(\varepsilon r)^{\delta} \right\}.$$

Hence

$$\mu_{\delta}(r, f^{(p)}) \geqslant \frac{\delta r^{(1-\delta)}}{\{ \mu_{\delta}(r, f^{(p-1)}) \}^{(1/\delta-1)}} \frac{d}{dr} M_{\delta}(r, f^{(p-1)}), \qquad (3.2)$$

by virtue of the fact $(a^{1/\delta} - b^{1/\delta} \leqslant (1/\delta)\, a^{(1/\delta)-1}(a - b),\ 1/\delta \geqslant 1)$. Let

$$S(r) = \frac{\log M_{\delta}(r, f^{(p-1)})}{\log r}.$$

As $S(r)$ is an increasing function of r, we have $S'(r) > 0$ and hence

$$\frac{d}{dr} \log M_{\delta}(r, f^{(p-1)}) = \left(\frac{S(r)}{r}\right) + S'(r) \log r$$

$$> \frac{S(r)}{r} = \frac{\log M_{\delta}(r, f^{(p-1)})}{r \log r}.$$

This together with (3.2) gives us

$$M_{\delta}(r, f^{(p)}) > \alpha \left\{ \frac{M_{\delta}(r, f^{(p-1)})}{r} \right\} \left(\frac{\log M_{\delta}(r, f^{(p-1)})}{\log r} \right)^{1/\delta},$$

$$0 < \alpha < 1 \qquad (3.3)$$

Owing to the fact that

$$\frac{\log M_{\delta}(r, f^{(p-1)})}{\log r}$$

tends to ∞ with r, we find that

$$\log M_{\delta}(r, f^{(p)}) > \log \alpha + \{1 + 0(1)\} \log M_{\delta}(r, f^{(p-1)}) - \{1 + 0(1)\} \log r$$

$$\sim \log M_{\delta}(r, f^{(p-1)}) \quad (r \to \infty).$$

Hence

$$\log M_{\delta}(r, f^{(p-1)}) > O(1) \{ \log M_{\delta}(r, f) \}.$$

This together with (3.3) gives us

$$M_\delta(r, f^{(p)}) > O(1) \left\{ \frac{M_\delta(r, f^{(p-1)})}{r} \right\} \left(\frac{\log M_\delta(r, f)}{\log r} \right).$$

Finally, we have

$$M_\delta(r, f^{(m)}) > O(1) \left\{ \frac{M_\delta(r, f)}{r^m} \right\} \left(\frac{\log M_\delta(r, f)}{\log r} \right)^m \text{ for all}$$

$$r > r_0 > 0. \tag{3.4}$$

Combining (3.4) and (3.1), we obtain

$$\lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log [r \{M_\delta(r, f^{(m)})/M_\delta(r, f)\}^{1/m}]}{\log r} \geqslant \begin{matrix}\rho\\ \lambda\end{matrix} . \tag{3.5}$$

The desired conclusion follows at one from (3.1) and (3.5) viz.

$$\lim_{r \to \infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log [r \{M_\delta(r, f^{(m)})/M_\delta(r, f)\}^{1/m}]}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix} , 0 < \delta < 1 .$$

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