# A NEW GENERALIZED THEOREM ON FOX'S H-FUNCTION

by

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## SUMMARY

In a series of papers, the author continues his studies in the direction of (i) unification, (ii) generalization, (iii) co-ordination, (iv) extension, and (v) co-relation of certain results appearing in the branch of Mathematics, The author obtains here a ne*E* generalized expansion-theorem on Fox's H-function in terms of similar functions. The technique selected for investigation is based on the application of «Differential Operators». On appropriately specializing the parameters, the theorem leads to generalizations of many mathematical functions scattered throughout literature.

§1. Prerequisites:

A generalization of the well-known Meijer's G-function [4, p. 207, (1)] has been given by Charles Fox [5, p. 408], viz.,

$$\mathrm{H}_{p,q}^{m,n}\left[x\right] = \mathrm{H}_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix}\{a_p, \alpha_p\} \\ \{b_q, \beta_q\}\end{matrix}\right] \qquad [1.1]$$

$$= \frac{1}{2\pi i}\int_{\mathrm{L}} \Phi(s)\, x^s\, ds,$$

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where

(i) $\{(a_p, \alpha_p)\}$ represents the set of $p$-ordered pairs $(a_1, \alpha_1)$ $(a_2, \alpha_2), \ldots,$ $(a_p, \alpha_p)$ and similarly for $\{(b_q, \beta_q)\}$,

(ii) $$\Phi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + \beta_j s) \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j s)},$$

(iii) $x \neq 0$ and an empty product is interpreted as unity,

(iv) $p, q, m$ and $n$ are integers satisfying $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$; $\alpha_j (1 \leqslant j \leqslant p)$, $\beta_j (1 \leqslant j \leqslant q)$ are positive numbers and $a_j (1 \leqslant j \leqslant p)$, $b_j (1 \leqslant j \leqslant q)$ are complex numbers;

(v) L is a suitable contour and all the poles of the integrand [1.1] are simple.

In view of Braaksma [2, pp. 239-240], the H-function makes sense and defines an analytic function of $x$ in the following two cases:

(vi) $\delta = \sum_1^q \beta_j = \sum_1^p \alpha_j > 0$, where $\delta > 0$. $x \neq 0$, and

(vii) $\sum_1^q \beta_j = \sum_1^p \alpha_j$ and $0 < |x| < D^{-1}$ where $D = \prod_1^p (\alpha_j)^{\alpha_j} \prod_1^q (\beta_j)^{-\beta_j}$.

Asymptotic Expansions for the H-function: According to Braaksma [2, 0. 279, (6.5) and p. 246, (2.16)]:

(viii) $H_{p,q}^{m,n} [x] = o\,(|x|^{\sigma})$ for small $x$, where $\sigma \geqslant 0$,

$$\sum_1^q \beta_j - \sum_1^p \alpha_j \geqslant 0,\ \sigma = \min \operatorname{Re}\left(\frac{b_h}{\beta_h}\right)\ (1 \leqslant h \leqslant m); \text{ and}$$

(ix) $H_{p,q}^{m,n} [x] = o\,(|x|^{\rho})$ for large $x$, where

$$\sum_1^q \beta_j - \sum_1^p \alpha_j > 0\ ;\ \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j + \sum_1^m \beta_j - \sum_{m+1}^q \beta_j \equiv \lambda > 0,$$

$$|\arg x| < \frac{1}{2} \lambda \Pi \text{ and } \rho = \max \operatorname{Re}\left(\frac{a_i - 1}{\alpha_i}\right) (1 \leqslant i \leqslant n).$$

In the same paper, Braaksma has studied the behaviour of the H-function for large $x$ at full length and also considered different sets of conditions for convergence of the integral [1.1]. However we restrict ourselves throughout this note to the above conditions of validity.

In this note, the Differential Operators procedure has been applied to establish a general theorem for Fox's H-function in a series of the H-functions. We expect that this note will prove to be useful in the search of a systematic development for the derivation of some more general results. Due to the most general character of the H-function, many interesting corollaries follow as particular cases of the theorem.

Notations used in the present work:

i) $\Delta(n,m) = \frac{m}{n}, \frac{m+1}{n}, \ldots, \frac{m+n-1}{n},$

(ii) $[a]_m = \frac{\Gamma[a+m]}{\Gamma[a]} = a(a+1)\ldots(a+m-1); m = 1, 2, 3, \ldots$

The binomial coefficient:

(iii) $C_{n,k} = \overset{n}{C}_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$

(i) The differential operator $D \equiv \frac{d}{dx}.$

Results required in the proof:

(a) Functional equation [4, p. 3, (3)]:

$$(z-1)(z-2)\ldots(z-n) = \frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n(-z+1)_n \qquad [1.2]$$

(b) Gauss' multiplication formula [4, p.4, (11)]:

$$\Gamma[mz] = (2\pi)^{1/2(m-1)}\, m^{mz-1/2} \prod_{r=0}^{m-1} \Gamma\left[z+\frac{r}{m}\right], m = 2,3,4,\ldots \qquad [1.3]$$

(c) Leibnitz' rule for the $k$-th derivative of a product:

$$D^k(uv) = \sum_{r=0}^{k} C_{k,r}(D^r u)(D^{k-r} v) \qquad [1.4]$$

in which $u$ and $v$ are to be functions of $x$.

## §2. Basic Expansion Theorem.

The theorem to be proved is as follows:

Assumptions:

(i) $m, n, p$ and $q$ are positive integers such that $1 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$;

(ii) Let $\sum_1^q \beta_j - \sum_1^p \alpha_j > 0$, when $x^\rho \neq 0$; $\sum_1^q \beta_j - \sum_1^p \alpha_j = 0$;

and $0 < |x^\rho| < D^{-1}$ where $D = \prod_1^p (\alpha_j)^{\alpha_j} \prod_1^q (\beta_j)^{-\beta_j}$;

(iii) Let $\sum_1^m \beta_j - \sum_{m+1}^q \beta, + \sum_1^n \alpha_j - \sum_{n+1}^p \alpha_j \equiv \Psi > 0$, $|\arg x^\rho| <$

$< \frac{1}{2} \Psi \pi$;

(iv) $\rho$ is a positive integer $> 0$, and $r = 0,1,2,\ldots.$

(v) Let $\operatorname{Re}\left[k + \mu + \rho \frac{b_j}{\beta_j}\right] > 1$, $(1 \leqslant j \leqslant m)$,

then

$$[2.1] \quad \rho^k \, H^{m,\, n+\rho}_{p+\rho,\, q+\rho}\left[zx^\rho \left| \begin{matrix} \Delta(\rho, -k-\mu+1), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, \Delta(\rho, -\mu+1) \end{matrix} \right.\right]$$

$$= \sum_{r=0}^{k} C_{k,r} \frac{[\mu]_k}{[\mu]_r} \rho^r H^{m,n+\rho}_{p+\rho,\, q+\rho}\left[zx^\rho \left| \begin{matrix} \Delta(\rho, 0), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, \Delta(\rho, r) \end{matrix} \right.\right].$$

Proof:-

In order to derive [2.1], we begin by considering the function in the form

$$[2.2] \qquad f(x) = x^{k+\mu-1} H^{m,n}_{p,q}\left[zx^\rho \left| \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right.\right].$$

We now use the definition of the H-function in terms of Mellin-Barnes integral from [1.1[ on the right hand side and then differentiate each member of equation [2.2] with respect to $x$ to obtain

$$[2.3] \quad D^k [f(x)] = \frac{1}{2\pi i} \int_L \Phi(s)\, z^s \{(k+\mu+\rho s-1)(k+\mu+\rho s - 2) \ldots (k+\mu+\rho s-k)\}\, x^{\mu+\rho s-1}\, ds$$

provided that

$\mathrm{Re}\left[k + \mu + \rho \frac{b_h}{\beta_h}\right] > 1$ $(h = 1,2, \ldots, n)$ and $k^{th}$ derivative of

$x^{k+\mu-1} H^{m,n}_{p,q} [zx^{\rho}]$ exists.

In [2.3], if we employ [1.2] and [1.3], we find that

$$[2.4] \quad D^k [f(x)] = x^{\mu-1} \rho^k \frac{1}{2 \pi i} \int_L \Phi(s) \frac{\prod_{j=0}^{\rho-1} \Gamma\left[\frac{k+\mu+j}{\rho} + s\right]}{\prod_{j=0}^{\rho-1} \Gamma\left[\frac{\mu+j}{\rho} + s\right]} z^{s} x^{\rho s} ds$$

where the contour L in the complex $s$-plane runs from $\sigma - i \infty$ *to* $\sigma + i \infty$ such that the poles of $\Gamma(b_j - \mu_j s)$ $(1 \leqslant j \leqslant m)$. lie on the right and those of

$$\Gamma(1 - a_j + \alpha_j s), (1 \leqslant j \leqslant n), \Gamma\left[\frac{k+\mu+j}{\rho} + s\right],$$

$\{j = 0,1,2, \ldots, (\rho - 1)\}$, on the left of the contour.

Finally, making use of [1.1] on the R.H.S. of [2.4], it transforms into

$$[2.5] \quad D^k [f(x)] = x^{\mu-1} \rho^k H^{m,\, n + \rho}_{p+\rho,\, q+\rho} \left[ zx^{\rho} \,\middle|\, \begin{matrix} \Delta(\rho, -k+\mu+1), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, \Delta(\rho, -\mu+1) \end{matrix} \right].$$

Next, we treat $f(x)$ as the product of two functions:

$$x^{k+\mu-1} \text{ and } H^{m,n}_{p,q} [zx^{\rho}]$$

and the application of [1.4] in [2.2] readily yields

$$[2.6] \quad D^k [f(x)] =$$

$$= \sum_{r=0}^{k} C_{k,r} D^{k-r} [x^{k+\mu-1}] \quad D^r \left\{ H^{m,n}_{p,q} \left[ zx^{\rho} \,\middle|\, \begin{matrix} \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\} \end{matrix} \right] \right\}.$$

By performing on the same lines as above and the use of [1.3] in [2.6] concludes that

$$[2.7] \quad D^k [f(x)] = x^{\mu-1} \sum_{r=0}^{k} C_{k,r} \frac{[\mu]_k}{[\mu]_r} \rho^r \frac{1}{2\pi i} \int_L \Phi(s)\, z^s \frac{\prod_{j=0}^{\rho-1} \Gamma\left[\frac{1+j}{\rho} + s\right]}{\prod_{j=0}^{\rho-1} \Gamma\left[\frac{-r+1+j}{\rho} + s\right]} x^{\rho s}\, ds$$

from which it follows that

$$[2.8] \quad D^k [f(x)] = x^{\mu-1} \sum_{r=0}^{k} C_{k,r} \frac{[\mu]_k}{[\mu]_r} \rho^r \, H^{m,\, n+\rho}_{p+\rho,\, q+\rho}\left[ z x^{\rho} \,\middle|\, \begin{matrix} \Delta(\rho, 0), \{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}, \Delta(\rho, r) \end{matrix} \right].$$

Therefore, a comparison of $D^k [f(x)]$ from [2.5] and [2.8] arrives at the result [2.1]. This completes the proof of the theorem.

## §3. Corollaries.

In [2.1], substituting $\alpha_j = \beta_h = 1 (1 \leqslant j \leqslant p, 1 \leqslant h \leqslant q)$ and adjusting the parameters, we can obtain the illustrative results associated with Maijer's G-,MacRobert's E-and generalized hypergeometric functions ${}_pF_q$ etc. with the aid of the known relation [4, p. 215].

Several researchers [1,3.6-12] have contributed a large amount of work on Fox's H-functions incorporating most of the frequently used functions-the-so-called special functions.

Recently, in a sequence of subsequent papers [13-26], the author has investigated various interesting properties associated with classical orthogonal polynomials, integral transforms, Fourier integrals-series, expansions, summation-multiplication theorems, integral equations, recurrence relations, derivatives and applications in the applied field of mathematics etc.

It may of interest to note that achievements on Fox's H-functions are more general than even Meijer's G-functions. The result recorded here is very important since each expression formulated becomes master or key formula from which a vast number of relations can be deduced for the functions commonly utilized in the analysis of many problems of mathematics, both pure and applied, and in mathematical physics.

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