HOMOLOGICAL TRANSFORMATIONS IN FOUR-DIMENSIONAL SPACE

by

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The purpose of this work is to present the identifcation of this type of geometric transformation among the elements of the orthographic projections of a figure belonging to a given space without entering into the details of the complete study of the projective geometry of four dimensions. It would be almost impossible to try to justify everything related to this study without actually dedicating extensive chapters on the subject. For this reason, if our conclusions concerning the matter as applied to descriptive geometry seem to be of an intuitive nature, the best we can recommend is a separate investigation of the problems of projective geometry of four dimensions by the reader. Unfortunately, we cannot make any reference to such a study, since it seems that there is none available for consultation. However, we think that the foregoing study is self-explanatory, because we base and support our conclusions on descriptive conditions, and also because they involve nothing more than the familiar three-dimensional Euclidean space.

To visualize what type of problems we have in mind to examine, we will take the time to present how the study of projective geometry of three dimensions is applied in descriptive geometry. Not to be mentioned are the problems related to the study of the rotation of planes.

We think that the more familiar problem is that where the existence of a homological relationship between the two projections of a plane figure is demonstrated. The projections are the homological figures; the axis of homology is the intersection of the plane of the figure with the bisector plane of the second and fourth dihedral angles of the Mongean system of reference. For orthographic projections, this line is the geometric locus of the points of the plane with coinciding projections, and so this line satisfies the conditions of the definition of an axis of homology. The center of homology and two homologue points are colinear, and this is satisfied in orthographic projections where the two projections are on the line perpendicular to the reference line. Because these lines of projections are parallel, the center of homology is a point of the line of the infinity of the plane of the orthographic projections. This homological transformation receives the particular name of affinity due to the location of the center of homology.

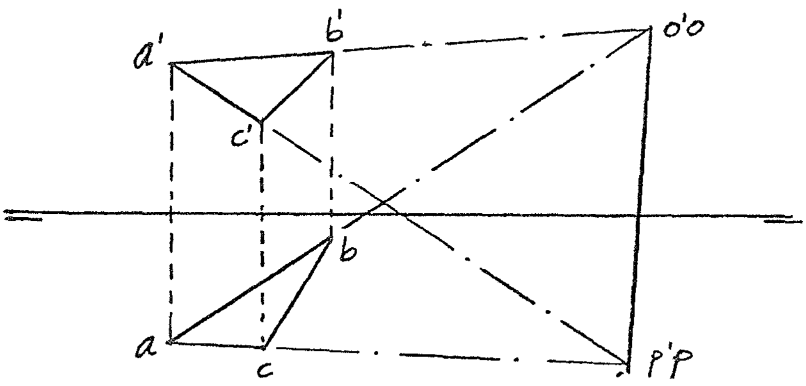

Figure 1

Note: We have used here, for the orthographic projections in descriptive geometry of three dimensions, the standard notation originally adapted by Monge's disciples. Since there is no uniformity or convention regulating this matter, we chose to use this one with we are more familiar.

The line OP is the intersection of the plane of the triangle ABC with the bisector of the second and fourth dihedral angles. OP is the axis of affinity. The lines of projection aa', bb', cc' are the rays of affinity.

Assuming that the horizontal projection S of a point $S(s', s)$ is given, the point S belonging to the plane of the triangle ABC, its vertical projection s' is obtained by using the conditions set in the study of homology. A line AS will have on the axis OP a point M whose projections coincide. The point S belonging to the line AM will have its projections on the projection of the line with the same name. Therefore, the vertical projection s' is found on the intersection of the line of projection containing s (or on the ray of affinity of the point s) and the vertical projection $a'm'$ which is the homologue of am.

We shall now examine the existence of similar properties in a plane referred to the system of reference in four-dimensional space. We shall do this in steps, first discussing the relations among the elements of that system of reference and then taking the opportunity to analyse the more common methods of representation in four-dimensional descriptive geometry. This analysis should prove useful to those who wish to make similar investigations concerning homological transformations in four-dimensional space, taking into consideration the particularities of each method. Our discussion will be made by studying the homological transformations in the plane as represented through application of the Mongean-type method.

At least three methods for four-dimensional descriptive geometry can be accounted for as the ones which will prove to be better tools in the representation, determination, and analysis of the geometric properties of forms embedded in a four-dimensional Euclidean space.

These methods, which are an outgrowth of those proposed by geometers such as Schoute, Eckhart, Jouffret, Maurin, and others can also serve as the basis for the extension to representation of forms in spaces of higher order. In addition, the new format of these methods indicates better possibilities for practical use and for the translation into computer language of the constructions which they outline.

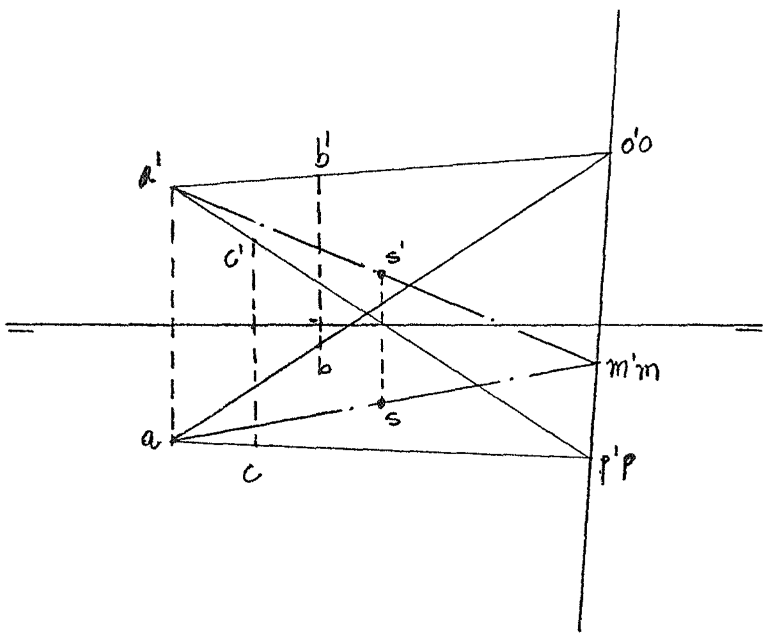

Figure 2

These three methods, the mongean-type, the direct-method, and the Bonfiglioli method, have been properly developed under the light of projective concepts. The resulting orthographic projections or views at once permit the indentification of the relations among the projections of a point in hyper-space upon the elements of the system of reference.

In this presentation, no attempt is made to demonstrate the validity of statements having to do with relationships among geometric elements in four-dimensional space. For example, there will be reference to perpendicularity between line and 3-D space, between two 3-D spaces, between plane and space, to the intersection of two planes in a single point, to the perpendicularity of four lines having a line in common, and to the perpendicularity of four lines having a point in common. All these relationships and others belong to the realm of synthetic geometry, and the reader should accept them as valid.

MONGEAN-TYPE METHOD

Let us take the basic system of reference as constituted of four lines mutually perpendicular and belonging to a point. Figure 3.

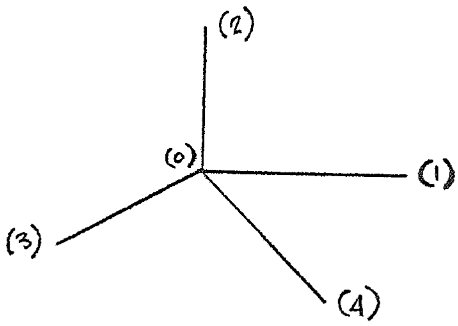

Figure 3

These four lines determine:

a. Six planes which can be combined in two manners: theree pairs of planes intersecting at point (0) —these planes are said to be absolutely perpendicular; twelve pairs of planes in groups of three having a line in common. Figure 4.

$$\eta \times \pi_1 \rightrightarrows \text{Point}\,(0)$$

$$\delta \times \pi_2 \rightrightarrows \text{Point}\,(0)$$

$$\gamma \times \pi_3 \rightrightarrows \text{Point}\,(0)$$

$$\pi_1 \times \pi_2 \rightrightarrows \text{Line}\,(1)$$

$$\pi_2 \times \pi_3 \rightrightarrows \text{Line}\,(1)$$

$$\pi_1 \times \pi_3 \rightrightarrows \text{Line}\,(1)$$

Etc.

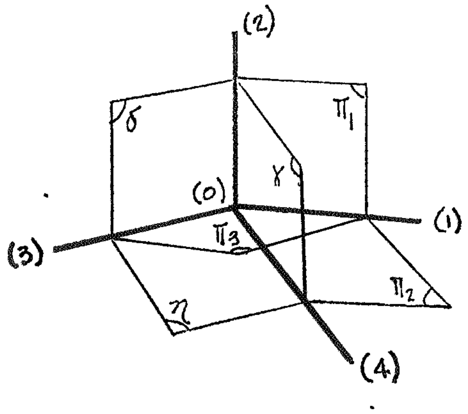

Figure 4

b. Four 3-D spaces determined by combining the four lines in groups of three (Figure 5).

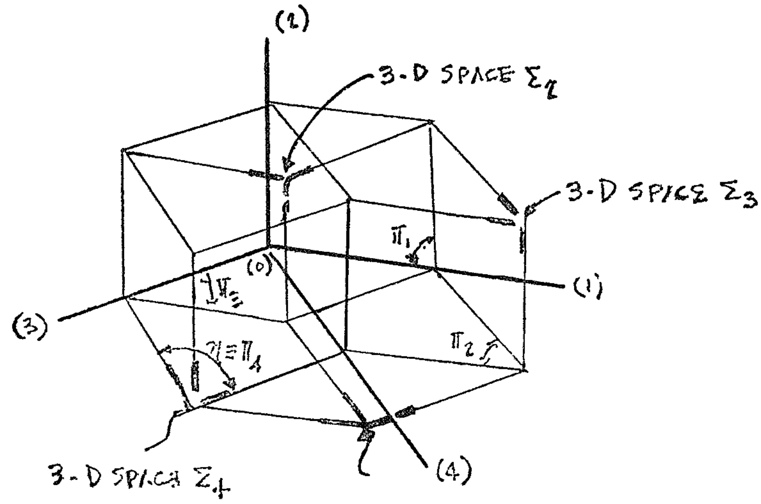

Figure 5

In the nomenclature of these elements and in the indication of their relationships, $\times$ is used to indicate intersections, one arrow $\rightarrow$ to indicate the resulting geometric element by combining two or three others, two arrows $\rightrightarrows$ indicate the result of the intersection of two or three geometric elements. (See figure 4).

The projections of a point (p) of the four-dimensional space are obtained as follows: (Figure 6).

a. Project (p) upon the four 3-D spaces of the system: we obtain points (p_1), (p_2), (p_3), and (p_4).

b. Project (p_1) upon planes π_2 and π_3 (which determine 3-D space Σ_1) to obtain p_1 and p_2; project (p_2) upon π_1 and π_3 (determine Σ_2) to obtain p_1 and p_3; project (p_3) upon π_1 and π_2 (determine Σ_3) to obtain p_1 and p_2; project (p_4) upon plane π_4 to obtain p_4.

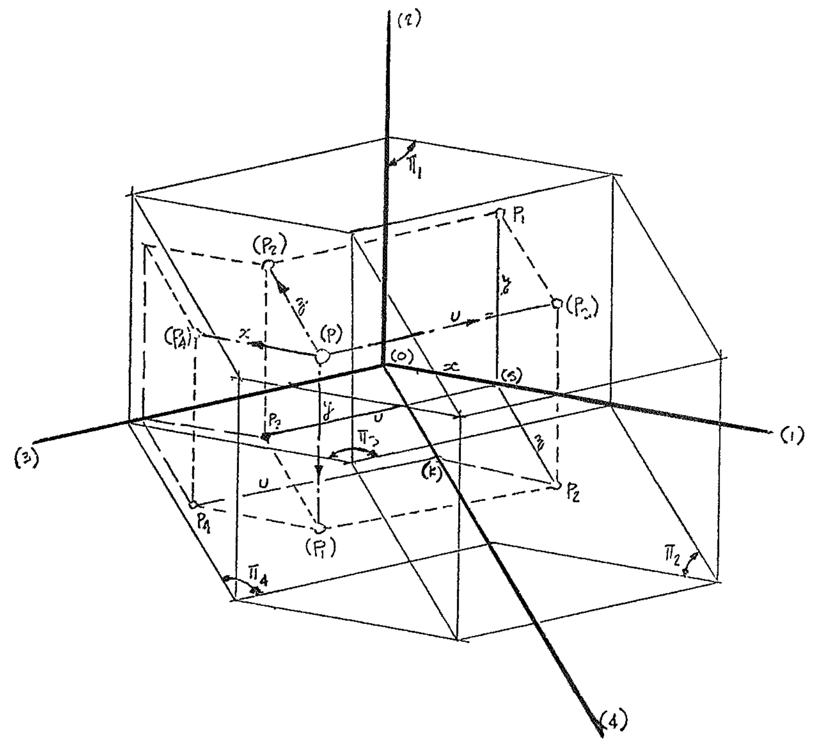

Figure 6

Projections upon a 3-D space indicated by ——— — ——— — ———
Projections upon a plane indicated by — — — — — — —
Relation between two projections ———— or —— —— ——

The relationships among projections upon 3-D spaces and upon planes, as well as proper identification of the four coordinates of the point x, y, z, and u as shown in Figure 6, can all be demonstrated.

To attain the plane representation of the projections, one rotation of 90° is made about a line, causing two of the 3-D spaces to be superimposed upon a third (Σ_2 and Σ_3 upon Σ_1). As a result, Σ_2 and Σ_4 are «opened up», similar to what is done in 3-D descriptive geometry with the horizontal and frontal planes (Figure 7).

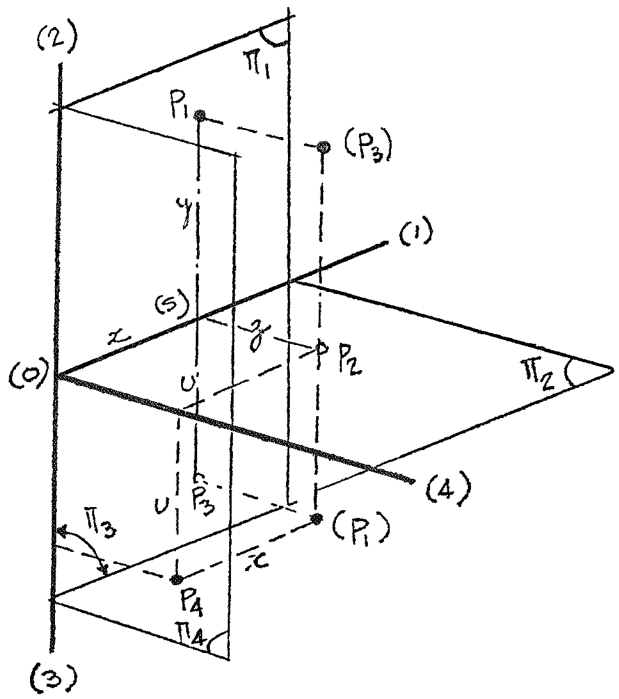

Figure 7

A second rotation of 90° is made about line (1) until the superimposition of π_2 with $\pi_1 = \pi_3$. The result is shown in figure 8.

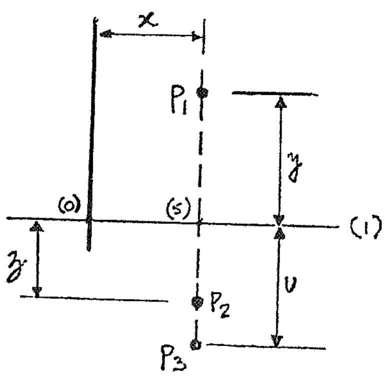

Figure 8

This is the orthographic presentation of the point (p) of the 4-D space. We disregard projection p_4, because, as it can be seen in figures 6 and 7, p_4 is related to projections p_1, p_2 and p_3 through coordinate x. Since this coordinate can be identified as (os), this fourth projection p_4 is not necessary in orthographic projections.

The direct method

Noticing in figure 7 that projection p_4 is related to p_2 through coordinates x and u, we may rotate plane π_4 about line (4) until superimposition with π_2 and then rotate $\pi_2 = \pi_4$ about line (1) until superimposition with π_1. In this case, we shall work with projections p_1, p_2, and p_4, since these three projections are sufficient for the proper identification of all four coordinates. The same result can be attained starting with the rotation involving the 3-D spaces or in the selection of the planes upon which the projections are made once we have determined the projections (p_1), (p_2), (p_3), and (p_4) upon Σ_1, Σ_2, Σ_3, and Σ_4.

A thorough analysis of these possibilities was done by Professor Steve M. Slaby, Princeton University, arriving at the arrangement shown in figure 9.

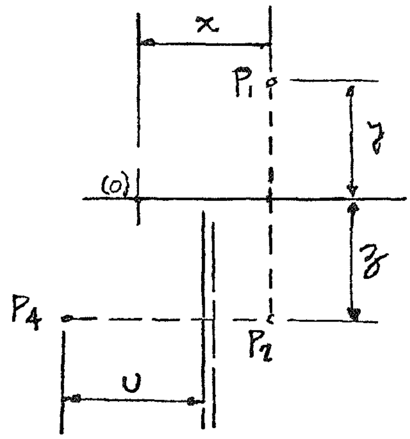

Figure 9

The Bonfiglioli method

Dr. Luisa Bonfiglioli, Israel Institute of Technology, Haifa, Isarel, devised this method of representation using parallel and axonometric projections.

Starting with the system of reference O(X, Y, Z, T), make an isometric projection of O(X, Y, Z) and an axonometric projection of

O(X, Y, T) so that the projections of (OX) and (OY) in one are parallel those in the other and so that the projection of (OY) falls perpendicular to (OZ). Due to the isometric projection (*ox*) and (*oy*) make, in projection, an angle of 120° with (*oz*). Combining the two projections, the plane representation of the system of reference is shown in figure 10.

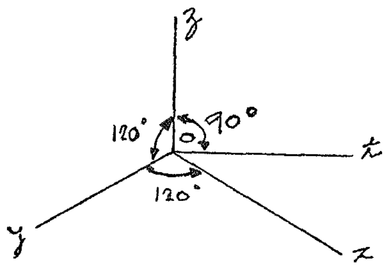

Figure 10

Upon each axis, we mark equal segments selected as unit. The four edges OX, OY, OZ, and OT are the projections, parallel and axonometric, of a pentahedroid (a four-dimensional solid). In four-dimensional space, the pentahedroid consists of five individual tetrahedrons, OXYZ, OXYT, OYZT, OXZT, XYZT, called cells, and they are the five cells in the system of reference for the method.

The cell XYZT is at infinity (X, Y, Z, and T are points at infinity of the direction of each axis). Since four coordinates are involved, two cells, OXYZ and OXYT, are sufficient for their identification. Therefore we determine the coordinates x, y, z in cell OXYZ and coordinates x, y, t in cell OXYT. These two cells have a face in common, OXY, which is called zeroplane (O-p). (Figure 11).

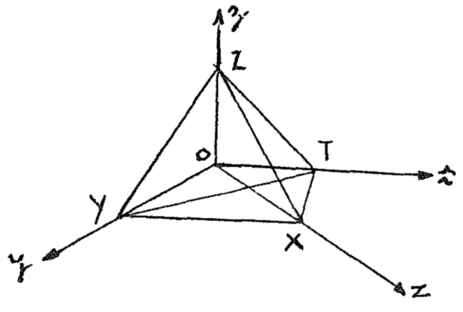

Figure 11

A point $p(x, y, z, t)$ is represented by means of three projections, p_0, p_1, and p_2. Projection p_0 lies on the zero-plane OXY and has coordinates x and y. Projection p_1 is the isometric projection of a point p_1, called the associate point of (p), and lies on parallel to (oz) drawn through p_0. The distance p_0p_1 is equal to z. Projection p_2 is the axonometric projection of a point p'' and lies on a parallel to (ot) drawn through p_0. The distance p_0p_2 es equal to t.

Evidently, if the projections p_1 and p_2 are given, we can determine all the coordinates of (p). To do this, through p_1 pass a parallel to (oz) and through p_2, parallel to (ot). The intersection of the two lines is projection p_0. This projection gives the coordinates x and y. Coordinates z and t are, respectively, p_0p_1 and p_0p_2. (Figure 12).

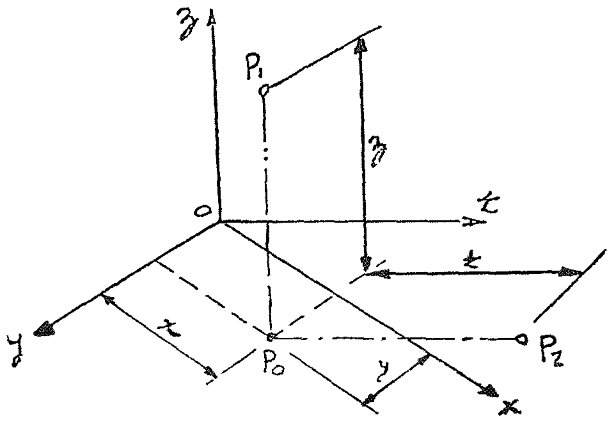

Figure 12

In the presentation of each method, it is shown how to determine the projections of a single point and this, evidently, is sufficient for the establishment of the method. One may proceed with representation of a line (two points), plane (three points), and 3-D space (four points) and projective and metrical relations among these elements.

Let us now investigate the homological transformations in the plane in four-dimensional space using the Mongean-type method for its representation. Consider the projections of three points (A), (B), and (C) of the four-dimensional space. They determine a plane, and we can obtain the projections of the points belonging to (AB), (BC), and (AC) which are equidistant to two spaces of the system of reference.

The points which are of interest in this case are (S), (K), (O), (S′), (K′), (O′), (L), (M), and (N) obtained as follows:

Point (S): coincident projections $S_1S_3 = A_1B_1 \times A_3B_3$
Point (K): coincident projections $K_2K_3 = A_2B_2 \times A_3B_3$
Point (O): coincident projections $O_1O_2 = A_1B_1 \times A_2B_2$

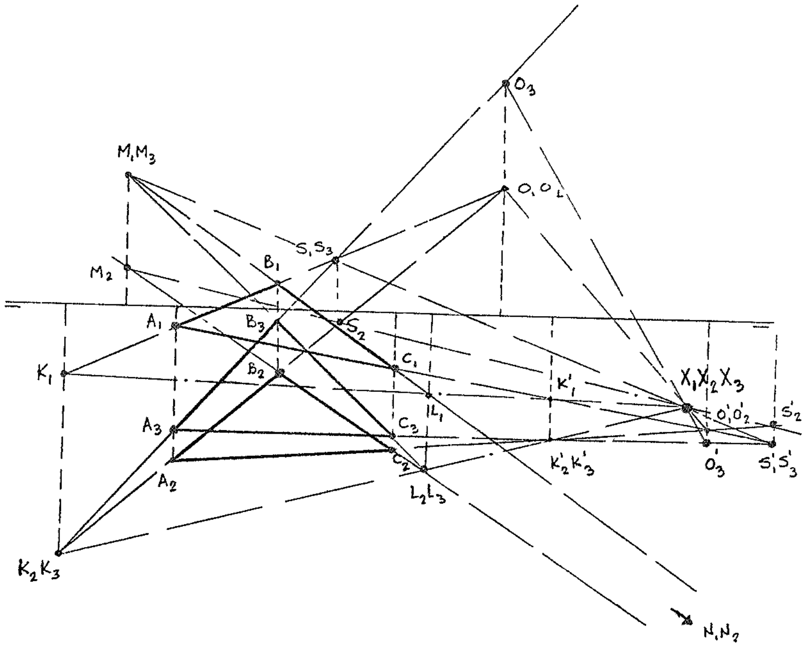

Figure 13

Point (S′): coincident projections $S'_1S'_3 = A_1C_1 \times A_3C_3$.
Point (K′): coincident projections $K'_2K'_3 = A_2C_2 \times A_3C_3$
Point (O′): coincident projections $O'_1O'_2 = A_1C_1 \times A_2C_2$

Projection S_2 of (S) on A_2B_2
Projection K_1 of (K) on A_1B_1
Projection O_3 of (O) on A_3B_3

Projection S'_2 of (S′) on A_2C_2
Projection K'_1 of (K′) on A_1C_1
Projection O'_3 of (O′) on A_3C_3

Point (M): coincident projections $M_1M_3 = B_1C_1 \times B_3C_3$

M_1M_3 belongs to $\overline{S_1S_3,\ S'_1S'_3}$

Point (L): coincident projections $L_2L_3 = B_2C_2 \times B_3C_3$

L_2J_3 belongs to $\overline{K_2K_3,\ K'_2K'_3}$

Point (N): coincident projections $N_1N_2 = B_1C_2 \times B_2C_2$

N_1N_2 belongs to $\overline{O_1O_2, \quad O'_1O'_2}$

Projection M_2 of (M) on B_2C_2
Projection L_1 of (L) on B_1C_1
Projection N_3 of (N) on B_3C_3

An analysis of this orthographic projection shows that:

(1) Considering the projections of the triangle (ABC), two by two, we can consider three systems of homology and operating with any of them obtain the projection of a point (P) of the plane of the triangle when one of the projections of this point is given.

Figure 14 is an example in which we considered the line K_2K_3—$K'_2K'_3$ as the axis of affinity. The projection P_2 of (P) is given.

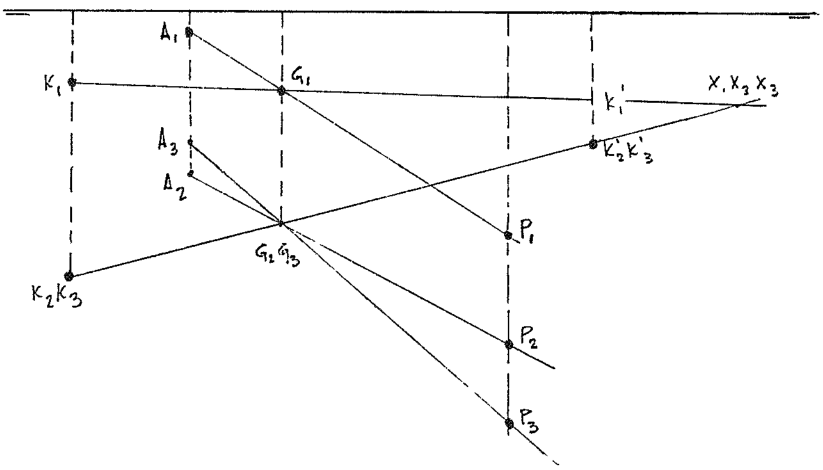

Figure 14

The same example could be solved by using either O_1O_2—$O'_1O'_2$ or S_1S_3—$S_1'S'_3$ as axis of affinity and also by considering either P_2 or P_3 as the given projection of (P).

(2) The geometric locus of the points equidistant two spaces of the system of reference is a space determined by two planes belonging to the reference line and to two distinct points of a line that belongs to the geometric locus. The projections of this line on those spaces are coincident.

In figure 13, the reference line and points (O) and (O′) determine two planes, and their space is the geometric locus of the points equidistant to the space Σ_1, and Σ_2 of the system of reference. The two planes belonging to the reference line and to the points (K) and (K′) determine a space, geometric locus of points equidistant to Σ_2, and Σ_3.*

Finally, the plane belonging to the reference line and to the points (S) and (S′) determines a space, geometric locus of points equidistant to Σ_1, and Σ_3.

The plane (ABC), because it does not belong to any one of those spaces, has on them one line only, which are respectively (OO′), (K,K′) and (SS′). Each one of these lines is, in the plane (ABC), a geometric locus, and two of them evidently have a point in common (two lines of a plane determine a point), which belonging to two loci is equidistant to the three spaces Σ_1, Σ_2, and Σ_3. Let us assume then that this is a point (X), common to (OO′) and (KK′). The third line (SS′) now being considered also has a point in common with (OO′), (Y), and with (KK′) a point of concurrence, (Z). Both points (Y) and (Z), we can observe, are also equidistant to Σ_1, Σ_2, and Σ_3. Under these circumstances, the three points (X), (Y), and (Z) are coincident, because if they were not, we would obtain in the plane (ABX) more than one line belonging to each one of the bisector spaces.

According to figure 15, if the three points (X), (Y), and (Z) were distinct, in addition to line (SS′) equidistant to Σ_1 and Σ_3, we would obtain in the plane (ABC) a second line (XS) also equidistant to Σ_1 and Σ_3. However, we already said that the plane (ABC) does not belong to any of the bisector spaces and has one line only. Consequently, the hypothesis suggested in figure 15 is an impossibility. Indeed, (X), (Y), and (Z) are coincident. Under these conditions, the three lines are concurrent at one point only, and in orthographic projections, due to the conditions of belonging between line and plane, all the projections are concurrent at the same point.

Considerer now this point (X). With the reference line (geometric locus of points with coinciding projections), it determines a plane which is then the geometric locus of all the points in 4-D space equidistant to the three 3-D spaces Σ_1, Σ_2, and Σ_3,

a. Any space that does not belong to this plane has only one line on this geometric locus.

b. A plane that does not belong to a line of this plane has only one point on this geometric locus.

c. A line that does not belong to a point of this plane can only belong to points equidistant to two of the spaces of projection, and in a line we can determine only three points with these characteristics.

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* $\Sigma_1, \Sigma_2, \Sigma_3$ are 3-D spaces of the system of reference for the representative methods in four-dimensional descriptive geometry. See «Four-Dimensional Descriptive Geometry» by C. Ernesto S. Lindgren and Steve M. Slaby, McGrau -Hill Book Co., Neu York, 1968.

(3) In view of the existance of this point (X) and of the possibility of determining the projections of a point by operating individually with the three systems of affinity (in fact this is so because of the existence of the point (X), common to the three axes of affinity), we realize that there is a relationship between the three systems. This can be demonstrated by using the anharmonic ratio and is suggested as an exercise to the reader.

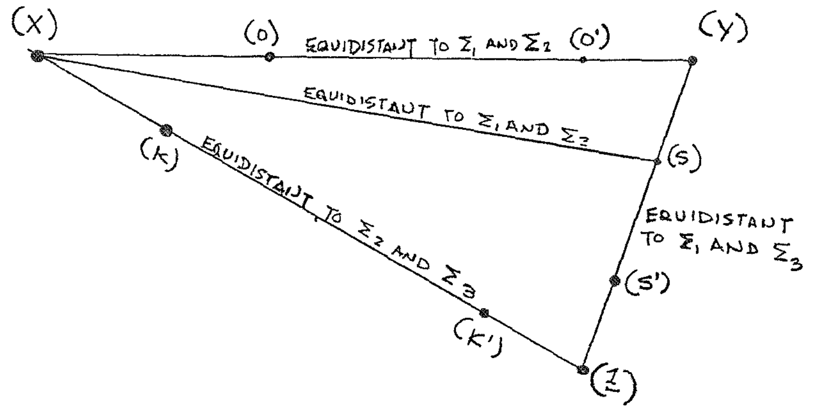

Figure 15

This discussion would be incomplete if we failed to investigate the practical aspects of a study of homological transformations in four-dimensional space. Thus, below we enumerate some of the problems already solved through direct application of those transformations.

1. *The halving of three distributions*:

This problem, proposed as the «Sandwich Theorem» calls for the simultaneous halving of three interrelated distributions. The mathematical demonstration shows the existence of one plane permiting the attainment of a solution. We have investigated the same problem geometrically, showing that the three distributions and the plane that halves them co-exist in a four-dimensional space. A generalization was made to consider the case of more than three distributions.

A computer program was prepared and successfully run at the Laboratory for Computer Graphics and Spatial Analysis, Harvard University. The complete analysis of the problem was made in a paper of the Harvard Papers in Theoretical Geography (Paper # 38 forthcoming, Laboratory for Computer Graphics and Spatial Analysis, Graduate School of Design, Harvard University). Practical applications include regionalization, the study of population distributions, etc.

2. *Graphical representation of a matrix*:

The problem consists of the development of an algorithm for the solution of one of the typical problems in architectural and urban design-the allocation of a set of spaces with locational demands relative to each other. The problem is analogous to the fitting of a new science complex and gynmansium into an existing college campus or to the expansion of a large hospital by the addition fo new service spaces. The process of solution requires the consideration of the dimensionality of the space defined by the elements of the set represented as points. This can be successfully achieved by utiliazing the descriptive method of representation in four-dimensional space.

This problem is throughly analyzed in Paper 33 of the Harvard Papers in Theoretical Geography (Laboratory for Computer Graphics, Graduate School of Design, Harvard University).

3. *Boundaries, fronts, and flows: geometric interpretation*

This problem uses the methodology developed for obtaining the representative of a set in multi-dimensional space. When at each point, whose representative is two-dimensional (points of a plane), a certain number of factors are measured, a multi-dimensional representative can be obtained and its properties analyzed. If two or more representatives are involved, corresponding however to two or more subsets in the two-dimensional representative, it is possible to relate the boundary or boundaries between or among the subsets in the two-dimensional representative through the determination of the intersections between or among the multi-dimensional representatives. In a sense, the multi-dimensional representatives and their intersections are thus related to the Venn diagrams constructed on the two-dimensional representative (plane).

This study is of particular value in the analysis of weather fronts and in the theory of regions.

These three examples are sufficiently general to bring forth the practical side of the seemingly theoretical study of the homological transformations in four-dimensional space.