

ON SYMMETRIC PARTIAL DERIVATIVES AND SYMMETRIC DIFFERENTIABILITY

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ABSTRACT

The symmetric partial derivatives and the symmetric differentiability of a function of two real variables are introduced as generalizations of the concepts of ordinary partial derivatives and ordinary differentiability. A relationship between the former and the latter concepts is established. Several properties of the symmetric partial derivatives and the symmetric differentiability are proved.

1. INTRODUCTION

Differentiability is one of the most fundamental and widely used concepts in the theory of functions of real variables. Many simple continuous functions of a real variables such as

$$(i) f(x) = |x|, f(0) = 0; (ii) f(x) = x \sin \frac{1}{x}, x \neq 0, f(0) = 0;$$

do not have derivatives at the origin in the ordinary sense. In order to extend the class of ordinary differentiable functions, Riemann, Schwarz, Peano, Dini and de la Vallée-Poussin (1) had extended the concept of the ordinary derivative of a function of a real variable in a number of different ways for various purposes. These authors had also emphasized the importance of these generalizations. In particular, Zygmund (1) shown the significant applications of the generalized derivatives in studying the theory of trigonometric series.

One of the most common generalizations of ordinary derivatives is the symmetric derivatives. Although the functions given in example (i) and (ii) do not have ordinary derivatives at the point $x = 0$, but do

have the symmetric derivatives at that point. In fact, these examples serve as the converse of a theorem which states that if the ordinary derivative of a function exists at a point, then the symmetric derivative of that function exists at the point and they are equal in value. The interesting and contrasting feature of the ordinary and symmetric derivatives is that the existence of the former does guarantee the continuity, but that of the latter does not imply the continuity. Rather, a discontinuous function may have the symmetric derivative. It is thus evident that the class of ordinary differentiable functions is a proper subclass of that of the symmetric differentiable functions.

In the past and recent years, Charzynski (2), Khintchine (3), Mazurkiewicz (4), Szpilrajn (5), Tolstov (6), Mukhopadhyay (7-10), and many others have made an extensive study on various properties of the symmetric derivative of a function of a single real variable. Recently, Bruckner and Leonard (11) has made an informative survey on the developments of the various derivatives of a function of a single real variable. In spite of these developments, it seems that there is no indication in the vast literature about any work on the generalization of the ordinary partial derivatives and the differentiability of a function of several real variables. It is however well known that functions of several variables are found to occur frequently and have numerous applications. In view of these facts, an extensive attention has been given to the development of the theory of functions of several real variables based on the concepts of ordinary partial derivatives and differentiability. In the theory of functions of several variables, many fundamental results differ significantly from those in the theory of functions of one variable.

Motivated by the above consideration and the recent work of Mukhopadhyay (7-10), an attempt is made to make a generalization of the ordinary partial derivatives and the differentiability of a function of two real variables (12). The symmetric partial derivatives and the symmetric differentiability of a function of two variables are introduced. A relationship between the former and the latter concepts is established. Several properties of the symmetric partial derivatives and the symmetric differentiability are proved.

2. SYMMETRIC PARTIAL DERIVATIVES

Definitions: Suppose $f(x, y)$ is a real function of two real variables x, y defined in an open (rectangular or circular) domain R . The *first symmetric partial derivative* of $f(x, y)$ with respect to x at a point $(x, y) \in R$ is denoted by $sf_x(x, y)$ and defined by

$$sf_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x - h, y)}{2h}, \quad [2.1]$$

provided the limit exists as a finite quantity for a fixed y .

Similarly, the first symmetric partial derivative of $f(x, y)$ with respect to y , keeping x fixed, is defined by

$$sf_y(x, y) = \lim_{k \rightarrow 0} \frac{f(x, y + k) - f(x, y - k)}{2k}, \quad [2.2]$$

provided the limit exists as a finite number.

It is easy to show the function $f(x, y) = |x - y|$ has no first ordinary partial derivatives at the origin but it possesses the first symmetric derivatives at the point $(0, 0)$.

Theorem 2.1. If the first order ordinary partial derivatives $f_x(x, y)$, $f_y(x, y)$ of a function $f(x, y)$ defined in an open domain R exist at a point $(x, y) \in R$, then the first symmetric derivatives $sf_x(x, y)$ and $sf_y(x, y)$ exist at that point, and $f_x(x, y) = sf_x(x, y)$; $f_y(x, y) = sf_y(x, y)$.

The proof is almost trivial and hence may be omitted.

The converse of this theorem is not necessarily true as shown by the above example.

3. SYMMETRIC DIFFERENTIABILITY

Definition: A function $f(x, y)$ defined in a neighborhood of a point (x, y) is said to be symmetrically differentiable at a point (x, y) iff

$$\begin{aligned} f(x + h, y + k) - f(x - h, y - k) = \\ = 2hA + 2kB + 2h\varepsilon_1 + 2k\varepsilon_2, \end{aligned} \quad [3.1]$$

where $(x \pm h, y \pm k)$ belong to the neighborhood of (x, y) , A, B are independent of h or k and $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$.

Example: Consider the function f defined by

$$\begin{aligned} f(x, y) &= \frac{xy}{\sqrt{x^2 + y^2}}, \quad (x, y) \neq (0, 0) \\ &= 0, \quad (x, y) = (0, 0) \end{aligned}$$

This function is not differentiable at the origin in the ordinary sense. However, it is symmetrically differentiable at the origin.

Theorem 3.1. If $f(x, y)$ is defined and differentiable in an open domain R in the ordinary sense, then it is symmetrically differentiable in R .

Proof: Let $(x, y), (x \pm h, y \pm k)$ be any points of R . In view of the given hypothesis, we have

$$f(x + h, y + k) - f(x, y) = Ah + Bk + h\varepsilon_1 + k\varepsilon_2, \quad [3.2]$$

where A, B are independent of h or k and $\varepsilon_1, \varepsilon_2$ tend to the limit zero as $(h, k) \rightarrow (0, 0)$. Further,

$$\begin{aligned} f(x + h, y + k) - f(x - h, y - k) &= f(x + h, y + k) - \\ &- f(x, y) + f(x, y) - f(x - h, y - k) \end{aligned}$$

which is, by [3.2],

$$= 2h A + 2k B + 2h \varepsilon_1 + 2k \varepsilon_2.$$

This means that $f(x, y)$ is symmetrically differentiable at any point of R . This proves the theorem.

The converse is not necessarily true and has been shown by the above example.

Theorem 3.2. If $f(x, y)$ is symmetrically differentiable at a point (x, y) , then the first symmetric partial derivatives exist at (x, y) and $A = sf_x(x, y)$, $B = sf_y(x, y)$.

Proof: Substituting $k = 0$ and taking $h \rightarrow 0$ in expression [3.1], it follows at once $A = sf_x(x, y)$. Similarly, putting $h = 0$ and $k \rightarrow 0$ in [3.1], it turns out that $B = sf_y(x, y)$.

Theorem 3.3. If $sf_x(x, y)$ and $sf_y(x, y)$ exist and continuous at all points of an open domain R , the function $f(x, y)$ has a symmetric derivative at a point in R .

Proof: Suppose h, k are small enough so that $(x \pm h, y \pm k)$ lie within a small neighborhood of (x, y) . Then we have $f(x + h, y + k) - f(x - h, y - k) = f(x + h, y + k) - f(x - h, y + k) + f(x - h, y + k) - f(x - h, y - k) = 2h sf_x(x + \theta_1 h, y + k) + 2k sf_y(x - h, y + \theta_2 k)$, [3.2]

where $-1 < \theta_1, \theta_2 < 1$.

In view of the continuity of the partial derivatives, it follows that

$$sf_x(x + \theta_1 h, y + k) = sf_x(x, y) + \varepsilon_1(h, k)$$

$$sf_y(x - h, y + \theta_2 k) = sf_y(x, y) + \varepsilon_2(h, k)$$

where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$. Thus

$$\begin{aligned} f(x + h, y + k) - f(x - h, y - k) &= 2h sf_x(x, y) + 2k sf_y(x, y) \\ &\quad + 2h \varepsilon_1(h, k) + 2k \varepsilon_2(h, k) \end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \rightarrow 0$ as $(h, k) \rightarrow (0, 0)$. This means that $f(x, y)$ has symmetrically differentiable at (x, y) .

4. HIGHER ORDER SYMMETRIC PARTIAL DERIVATIVES

With the aid of the definition of the first symmetric partial derivatives, we have the second order partial derivative,

$$sf_{yx}(a, b) = \lim_{k \rightarrow 0} \frac{sf_x(a, b + k) - sf_x(a, b - k)}{2k},$$

provided the limit exists,

$$\begin{aligned}
 &= \lim_{k \rightarrow 0} \frac{1}{2k} \left[\lim_{h \rightarrow 0} \frac{f(a+h, b+k) - f(a-h, b+k)}{2h} \right. \\
 &\quad \left. - \lim_{h \rightarrow 0} \frac{f(a+h, b-k) - f(a-h, b-k)}{2h} \right] \\
 &= \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} \frac{\Delta^2 f(h, k)}{4kh}, \quad [4.1]
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta^2 f(h, k) &= f(a+h, b+k) - f(a-h, b+k) - \\
 &\quad - f(a+h, b-k) + f(a-h, b-k), \quad [4.2]
 \end{aligned}$$

Similarly, we may obtain

$$sf_{xy}(a, b) = \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \frac{\Delta^2 f(h, k)}{4hk}, \quad [4.3]$$

These partial derivatives are referred to as the mixed symmetric partial derivatives of the second order of $f(x, y)$ at (a, b) .

Like the non-commutative property of the ordinary mixed partial derivatives f_{xy} , f_{yx} , there is no a prior reason for the existence of the commutative property of the mixed symmetric partial derivatives. However, there exist sufficient conditions on the function $f(x, y)$ under which

$$sf_{xy}(a, b) = sf_{yx}(a, b), \quad [4.4]$$

We establish a set of sufficient conditions for the validity of [4.4].

Theorem 4.1. If (i) $f(x, y)$ is a continuous function of x alone for all y in the neighborhood of the point (a, b) , (ii) sf_x , sf_y and sf_{yx} exist in the neighborhood of (a, b) ; then sf_{xy} exists at (a, b) and $sf_{xy}(a, b) = sf_{yx}(a, b)$.

Proof: Consider the function

$$\Phi(x) = f(x, b+k) - f(x, b-k), \quad a-h \leq x \leq a+h,$$

which satisfies the requirements of the Mean value theorem in the neighborhood of (a, b) . Then

$$\begin{aligned}
 \Phi(a+h) - \Phi(a-h) &= 2h {}_s\Phi'(a + \theta_1 h), \quad -1 < \theta_1 < 1 \\
 &= 2h [sf_x(a + \theta_1 h, b+k) - sf_x(a + \theta_1 h, b-k)]
 \end{aligned}$$

Since sf_{yx} exist in the neighborhood of (a, b) , we can apply the Mean Value theorem to deduce

$$\Phi(a+h) - \Phi(a-h) = 4hk {}_s f_{yx}(a + \theta_1 h, b + \theta_2 k).$$

In view of the assumption (iii), it turns out that

$$\Phi(a + h) - \Phi(a - h) = 4hk \{sf_{yx}(a, b) + \varepsilon(h, h)\},$$

where

$$\varepsilon(h, k) \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0).$$

It follows from [4.2] that

$$\lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \frac{\Delta^2 f(h, h)}{4hk} = \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} \{sf_{yx}(a, b) + \varepsilon\}$$

Thus,

$$sf_{xy}(a, b) = sf_{yx}(a, b).$$

Thus $sf_{xy}(a, b)$ exists and is equal to $sf_{yx}(a, b)$.

Theorem 4.2. If (i) $f(x, y)$ is a continuous function of x , for all y and is also continuous function of y , for all x in the neighborhood of the point (a, b) , (ii) sf_x, sf_y exist in the neighborhood of (a, b) and are symmetrically differentiable at (a, b) ; and (iii) sf_{xx} is a continuous function of y alone, for every x and sf_{yy} is also continuous in x alone, for every y in the neighborhood of (a, b) ; then sf_{xy} and sf_{yx} exist at (a, b) and are equal.

Proof: We have

$$\begin{aligned} \Delta^2 f(h, h) &= \{f(a + h, b + h) - f(a + h, b - h)\} - \{f(a - h, b + h) - \\ &\quad - f(a - h, b - h)\} \\ &= \Phi(a + h) - \Phi(a - h), \end{aligned}$$

where

$$\Phi(x) = f(x, b + h) - f(x, b - h), a - h \leq x \leq a + h.$$

In view of the Mean value theorem, it turns out that

$$\begin{aligned} \Phi(a + h) - \Phi(a - h) &= 2h {}_s\Phi'(a + \theta_1 h), -1 < \theta_1 < 1 \\ &= 2h [sf_x(a + \theta_1 h, b + h) - sf_x(a + \theta_1 h, b - h)] \\ &= 2h [\{sf_x(a + \theta_1 h, b + h) - sf_x(a - \theta_1 h, b - h)\} \\ &\quad - \{sf_x(a + \theta_1 h, b - h) - sf_x(a - \theta_1 h, b - h)\}], \end{aligned} \quad [4.5]$$

By virtue of assumption (ii), it follows that the right hand side of [4.5] is equal to

$$\begin{aligned} &2h [\{2\theta_1 h sf_{xx}(a, b) + 2h sf_{yx}(a, b) + 2k \varepsilon_1\} \\ &\quad - \{2\theta_1 h sf_{xx}(a, b - h) + 2h \varepsilon_2\}] \end{aligned}$$

which is, by continuity of sf_{xx} ,

$$= 4h^2 sf_{yx}(a, b) + 4h^2 (\varepsilon_1 - \varepsilon_2), \quad [4.6]$$

where $\varepsilon_1, \varepsilon_2$ tend to zero as $h \rightarrow 0$.

Similarly, we write

$$\Delta^2 f(h, h) = \Psi(b + h) - \Psi(b - h),$$

where

$$\Psi(y) = f(a + h, y) - f(a - h, y), \quad b - h \leq y \leq b + h.$$

An argument similar to that advanced above gives

$$\Delta^2 f(h, h) = 4h^2 s_{fxy}(a, b) + 4h^2 (\varepsilon'_1 - \varepsilon'_2), \quad [4.7]$$

where $\varepsilon'_1, \varepsilon'_2$ tend to zero as $h \rightarrow 0$.

It is evident from [4.6] and [4.7] that

$$\lim_{h \rightarrow 0} \frac{\Delta^2 f(h, h)}{4h^2} = s_{fxy}(a, b) = s_{fyx}(a, b).$$

This proves the theorem.

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