A NOTE ON ENUMERATION OF PERMUTATIONS OF (1,...,n) BY NUMBER OF MAXIMA

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A permutation $(a_1, \ldots, a_n)$ of $(1, \ldots, n)$ will be called a *circle (line) n.permutation* iff a_1 and a_n are (are not) considered adjacent. a_i is said to be a maximum of such a permutation iff a_i is greater than the member(s) adjacent to it. For example the permutation 41325 has 2 [3] maxima as a circle (line) permutation.

If we define $M(n, k)$ $(M^*(n, k))$, $n \geqslant 1$ to be the number of circle (line) n-permutations having exactly k maxima and further set $M(1. 0) = 1$, $M(n, 0) = 0$ for $n \geqslant 2$, $M^*(0, 0) = 1$, $M^*(n, 0) = 0$ for $n \geqslant 1$ we have, as was shown in [2], the recurrence relation.

$$(1)\ M(n+1, k) = \frac{n+1}{n} [2kM(n, k) + (n - 2k + 2)\, M(n, k-1)]$$

$$\text{for } n \geqslant 1,\ k \geqslant 1.$$

A generating function

$$\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} M(n, k) \frac{x^n}{n'} y^k = x \frac{1 - \sqrt{1-y}\ \mathrm{Tanh}\ x \sqrt{1-y}}{1 - (\mathrm{Tanh}\ x \sqrt{1-y})/\sqrt{1-y}}$$

and the explicit formula

$$(2)\ M(n, k) = (-1)^{n+k} n 2^{n-1} \sum_{r=k-1}^{[\frac{n}{2}]-1} \sum_{t=n-2r-1}^{n-1}$$

$$\frac{(-1)^t t!}{2^t} \binom{r}{k-1} \binom{t-1}{n-2r-2} S(n-1, t)$$

for

$$n \geqslant 2, k \geqslant 0$$

where

$$S(n-1, t) = \frac{(-1)^t}{t!} \sum_{s=1}^{t} (-1)^s \binom{t}{s} s^{n-1}$$

is a Stirling number of the second kind were also obtained.

The following result shows the simple relation between $M^*(n, k)$ and $M(n, k)$. Theorem: $M(n+1, k) = (n+1) M^*(n, k)$, $n \geqslant 0$, $k \geqslant 0$.

Proof: By definition the equality holds for $n = 0$. Suppose then that $n \geqslant 1$. If $(a_1, \ldots, a_{i-1}, n+1, a_{i+1}, \ldots, a_{n+i})$ is a line $(n + 1)$ permutation with k maxima then $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1})$ is a line n-permutation with k or $k-1$ maxima. Conversely if $(a_1, \ldots, a_n)$ is a line n-permutation with k maxima then $n + 1$ may be inserted in exactly $2k$ positions (since it must be adjacent to one of the maxima) to form a line $(n + 1)$-permutation with k maxima; if $(a_1, \ldots, a_n)$ is a line n-permutation with $k-1$ maxima then $n + 1$ may be inserted in exactly $n + 1 - 2(k-1)$ positions to form a line $(n + 1)$-permutation with k maxima. Since no two of the permutations obtained in this manner can be identical we have

$$M^*(n + 1, k) = 2k\, M^*(n, k) + (n - 2k + 3)\, M^*(n, k - 1) \quad \text{for} \quad n \geqslant 0, k \geqslant 1 \qquad [3]$$

If we rewrite this as

$$M^*(n, k) = 2k M^*(n-1, k) + (n-2k+2) M^*(n-1, k-1) \text{ for } n \geqslant 1, k \geqslant 1$$

and compare with [1] we see that the equality of the theorem holds.

This theorem along with [3] could be used in place of the argument in (2) to obtain [1].

The theorem is equivalent to the existence of a one-to-one correspondence between circle $(n + 1)$-permutations with k maxima and of the form $(a_1, \ldots, a_n, n + 1)$ and all line n-permutations with k maxima; such a correspondence is given by

$$(a_1, \ldots, a_n, n+1) \Longleftrightarrow (n+1-a_1, \ldots, n+1-a_n) \text{ for } n \geqslant 1 \qquad [4]$$

To show [4] actually is a correspondence of the type described we only need observe that $(a_1, \ldots, a_n, n+1)$ has k maxima in the circle sense if and only if $(a_1, \ldots, a_n)$ has k members each of which is less than its adjacent member(s) in the line sense and this in turn holds if and only if $(n+1-a_1, \ldots, n+1-a_n)$ has k maxima in the line sense.

In [1] $A(n)$, $n \geqslant 3$, was defined to be the number of alternating

n-permutations $(a_1, \ldots, a_n)$ with $a_1 < a_2$ where an n-permutation $(a_1, \ldots, a_n)$ was said to be alternating iff $a_i < a_{i+1} \Leftrightarrow a_{i+1} > a_{i+2}$, $i = 1, \ldots, n-2$. Both A(1) and A(2) were defined to be 1. We have the following relations between A(n) and M*(n, k).

Theorem: $A(2n-1) = M^*(2n-1, n)$ and

$$A(2n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{2j} M^*(2j+1, j+1) \text{ for } n \geqslant 1.$$

Proof: The first equality may be proven by noting that a $(2n-1)$-permutation $(a_1, \ldots, a_{2n-1})$ is alternating iff a_{2i} $i = 1, \ldots, n-1$ are maxima which in turn holds iff $(2n-a_1, \ldots, 2n-a_{2n-1})$ is a line $(2n-1)$ permutation with n maxima.

If the Euler and Bernoulli numers E_{2n} and B_{2n+1}, $n \geqslant 0$ are defined as in (1) then since in (1) A($2n$) was shown to be $(-1)^n E_{2n}$ and A($2n-1$) to be

$$(-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{2n} B_2n-1$$

for $n \geqslant 1$ we have from the well-known identity

$$E_{2n} = \sum_{j=1}^{n+1} \binom{2n}{2j-2} \frac{2^{2j}(2^{2j}-1)}{2j} B_{2j-1}, \; n \geqslant 0$$

$$A(2n) = \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{2j} A(2j+1) =$$

$$= \sum_{j=0}^{n} (-1)^{n-j} \binom{2n}{2j} M^*(2j+1, j+1), \; n \geqslant 1$$

REFERENCES

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