A NOTE ON A CERTAIN SEQUENCE

BY

B. K. LAHIRI & P. L. GANGULI

Let $U_1 + U_2 + U_3 + \ldots + U_n + \ldots$ be an infinite series of real terms. Let, as usual,

$$S_n \equiv U_1 + U_2 + \ldots + U_n$$

denote the n-th partial sum of the series $\Sigma\, U_n$. The series $\Sigma\, U_n$ converges or diverges according as the sequence $\{S_n\}$ is convergent or divergent.

Let us consider all possible sums like

$$U_{r_1} + U_{r_2} + \ldots + U_{r_n}$$

where $r_1, r_2, \ldots r_n$ are any n distinct positive integers ($n = 1, 2, 3, \ldots$). Two such sums, even though they may have the same value, will be treated as different if there is at least one U_k in one sum which does not appear in the other. All such sums can obviously be arranged in the form of a sequence $\{P_n\}$. It is easy to see that several elements of $\{P_n\}$ may have the same value and that all the elements of the sequence $\{S_n\}$ of partial sums as well as all the terms of the series $\Sigma\, U_n$ appear in the sequence $\{P_n\}$.

The object of the present note is to study the convergence of the sequence $\{P_n\}$, examine its limit points and to discuss incidentally how far the convergence of the series $\Sigma\, U_n$ can be characterised by the sequence $\{P_n\}$.

Let us suppose that the sequence $\{P_n\}$ is convergent. Then the sequence $\{S_n\}$ is also convergent, which implies that the series $\Sigma\, U_n$ is convergent. Hence $U_n \to 0$ *as* $n \to \infty$. But the sequence $\{P_n\}$ contains each U_n. Hence in this case the sequence $\{P_n\}$ must converge to zero. Thus, if the sequence $\{P_n\}$ at all converges, it must converge to zero. Also the convergence of the sequence $\{P_n\}$ implies the convergence of the series $\Sigma\, U_n$ and it further implies that the seiies $\Sigma\, U_n$ has the sum zero. It at once follows that if the series $\Sigma\, U_n$ be convergent with a non-zero sum or if it be divergent then the sequence $\{P_n\}$ must be divergent. Thus the convergence of $\Sigma\, U_n$ does not necessarily imply the convergence of the sequence $\{P_n\}$.

Let the sequence $\{Pn\}$ be convergent. Then, as we have seen, the series $\Sigma\ Un$ is also convergent. We shall show that the series $\Sigma\ Un$ must converge absolutely in such a case.

If possible, let $\Sigma\ Un$ be not absolutely convergent. Then $\Sigma\ Un$ is conditionally convergent. Hence we can rearrange its terms to obtain a new series $\Sigma\ U'n$ which will have a non-zero sum [4]. If the sequence $\{P'n\}$, associated with the series $\Sigma\ U'n$, corresponds to the sequence $\{Pn\}$, then it is clear that $\{P'n\}$ is obtained by merely rearranging the elements of $\{Pn\}$. Considering the series $\Sigma\ U'n$, we conclude that the sequence $\{Pn\}$ is divergent. Thus, if the sequence $\{Pn\}$ be convergent, the series $\Sigma\ Un$ must be absolutely convergent.

If G denote the set of all series for which $\{Pn\}$ is convergent, A the set of all absolutely convergent series and C the set of all convergent series, then it follows that.

$$G \subset A \subset C$$

the inclusion being proper. It will, in fact, be shown that the set G has just one element.

Upper and lower limits of $\{Pn\}$

Let the series $\Sigma\ Un$ be convergent with a non-zero sum S. Then the sequence $\{Pn\}$ is divergent. If all $Un > 0$, then clearly the sequence $\{Pn\}$ is bounded and as such $\{Pn\}$ must oscillate between finite limits. In this case $\overline{\lim}\ Pn = S$ and $\underline{\lim}\ Pn = 0$.

Next, let $\Sigma\ Un$ be conditionally convergent. Then the sequence $\{Pn\}$ is divergent. Also the series of positive terms of $\Sigma\ Un$ and the series of absolute values of negative terms of $\Sigma\ Un$ both diverge to $+\infty$. Hence in such a case $\overline{\lim}\ Pn = +\infty$, $\underline{\lim}\ Pn = -\infty$.

If, however, the series $\Sigma\ Un$ be absolutely convergent with a non-zero sum, then $\{Pn\}$ must oscillate between finite limits.

Next we examine the behaviour of the sequence $\{Pn\}$ in the case when $\Sigma\ Un$ converges to zero. Several cases may arise.

If $\Sigma\ Un = 0$, but the series is not absolutely convergent, then as seen above, $\{Pn\}$ must oscillate between $+\infty$ and $-\infty$.

If the series $\Sigma\ Un$ be absolutely convergent to zero, then the sequence $\{Pn\}$ may or may not converge. Let us considerer the series

$$\frac{1}{2^2} - \frac{1}{2^2} + \frac{1}{2^3} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^4} + \dots$$

which converges to zero absolutely. If α denotes the sum

$$\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$$

then clearly

$$\overline{\lim}\ Pn = \alpha\ ,\ \underline{\lim}\ Pn = -\alpha.$$

It follows that if Σ Un consists of an infinity of positive and an infinity of negative terms and if Σ Un converges to zero absolutely, then $\{Pn\}$ is divergent and oscillates between finite limits.

If Σ Un, consisting only of a finite number of positive (or negative) terms, converges to zero, then also $\{Pn\}$ is divergent and oscillates between finite limits.

If all but a finite number of Un be zero, then also the sequence $\{Pn\}$ cannot be convergent.

The following results can now be stated.

THEOREM 1

The sequence $\{Pn\}$ is convergent if and only if each Un = O.

This incidentally shows that it is not possible to characterise the convergence of a series Σ Un by the convergence of the corresponding sequence $\{Pn\}$ except in the trivial case when each Un is zero.

Also it follows that the set G has just one element.

THEOREM 2

For any series Σ Un with U$n \to$ O,

$$\underline{\lim}\ Pn \leqslant O \leqslant \overline{\lim}\ Pn$$

THEOREM 3

A necessary and sufficient condition that a series Σ Un may be absolutely convergent is that $\overline{\lim}$ Pn and $\underline{\lim}$ Pn are both finite.

THEOREM 4

For any absolutely convergent series Σ Un having a sum S, we have

$$S = \overline{\lim}\ Pn + \underline{\lim}\ Pn.$$

Limit points of $\{Pn\}$.

Let Σ Un be a convergent series of positive terms with a sum S. Then the sequence $\{Pn\}$ is divergent and $\overline{\lim}$ Pn = S and $\underline{\lim}$ Pn = O. Let us further assume in this case that

$$Un \leqslant Rn = U_{n+1} + U_{n+2} + \ \ldots \text{to } \infty$$

and

$$U_n \geqslant U_{n+1}$$

$$\text{for } n = 1, 2, 3, \ldots$$

Then Kakeya [3] has proved that the set 6 of all numbers of the form

$$\alpha = \sum_{1}^{\infty} U'_n$$

where

$$U'n = O \text{ or } Un \text{ , } n = 1, 2, 3, \ldots$$

consists of the whole interval $O \leqslant x \leqslant S$. It follows that, under the above conditions, every point of the stretch $O \leqslant x \leqslant S$ is a limit point of the sequence $\{Pn\}$.

If the series Σ U*n*, consisting of an infinity of positive as well as negative terms, be convergent with sum S, then S is certainly a limit point of $\{Pn\}$ but S cannot coincide with $\overline{\lim}$ P*n* or $\underline{\lim}$ P*n*.

Let us suppose that the series Σ U*n* is conditionally convergent. Then it has been seen that.

$$\overline{\lim}\ Pn = +\infty \text{ and } \underline{\lim}\ Pn = -\infty$$

Now, it is known that the terms of a conditionally convergent series can be rearranged to make the resulting series converge to any preassigned sum. It, therefore, follows that in such a case every real number is a limit point of $\{Pn\}$.

It has been shown by the authors [2] that an oscillatory series (whose *n*-th term tends to zero) can, by a suitable rearrangement of its terms, be converted into a conditionally convergent series. It therefore, follows that the above results in respect of a conditionally convergent series hold equally well for such oscillatory series also. As a matter of fact the above results hold for any series Σ U*n* having the following properties (1) $Un \to O$ as $n \to \infty$ (2) the sum of positive terms of Σ U*n* is $+\infty$ (3) the sum of negative terms of Σ U*n* is $-\infty$.

If Σ U*n* be an unrestricted convergent series, then it is not a fact that every point in the interval ($\underline{\lim}$ P*n*, $\overline{\lim}$ P*n*) is a limit point of the sequence $\{Pn\}$. This can be seen by considering series like.

$$10 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots \text{ to } \infty$$

and

$$10 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots \text{ to } \infty$$

If Σ U*n* is a divergent series of positive terms where $Um \to O$ then it is known [1] that, given any positive number s, there exists a subseries of Σ U*n* which converges to s, It follows that, in such a case, every positive number is a limit point of $\{Pn\}$.

Lastly we prove the following result.

THEOREM 5.

Let Σ U*n* be any series (convergent or divergent) where $Un \to O$ as $n \to \infty$. Then the corresponding sequence $\{Pn\}$ is dense-in-itself (in the sense that every element of the sequence $\{Pn\}$ is a limit point of $\{Pn\}$).

For, let $\alpha \equiv Ur_1 + Ur_2 + \ldots + Ur_n$ be any element of $\{Pn\}$. Then there exists a positive integer m such that *no* Un with $n \geqslant m$ appears in α. Now the numbers.

$$\alpha, \alpha + Um, \alpha + Um +_1, \ldots, \alpha + Un, \ldots$$

all belong to $\{Pn\}$ and they converge to α. Hence α is a limit point of the sequence $\{Pn\}$.

Department of Pure Mathematics Calcutta University.

REFERENCES

1. BANERJEE, C. R. & LAHIRI, B. K., «Some theorems on subseries of divergent series», *Ind. Jour. of Mech. and Math.*, Vol. II. No. 2, 1964, P. 24.

2. GANGULI, P. L., & LAHIRI, B. K., *A Note on Oscillatory series, to appear.*

3. KAKEYA, S., «On the partial sums of an infinite series». *Science reports of the Tohoku Imperial University* (1), 3 (1914), P. 159.

4. KNOPP, K., «Theory and application of infinite series», 1947, P. 318.