On Weakly Extremal Structures in Banach Spaces

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Abstract: This paper deals with the interplay of the geometry of the norm and the weak topology in Banach spaces. Both dual and intrinsic connections between weak forms of rotundity and smoothness are discussed. Weakly locally uniformly rotund spaces, ω-exposed points, smoothness, duality and the interplay of all the above are studied.

Key words: Rotundity, weak topology, higher order smoothness.

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1. Introduction

In introducing weaker forms of strong convexity in Banach spaces it is natural to apply the weak topology instead of the norm topology to describe the size of the extremal sections of the closed unit ball. In this article we study the weak geometry of the norm, that is, the interplay between the weak topology and the geometry of the norm. While we investigate the duality between the smoothness and the convexity, it turns out that also other types of interplay occur (see Theorem 2.5). Our study involves ω-locally uniformly rotundity, ω-exposedness and the G-smoothness of the points $x \in S_X$, as well as the properties of the duality mapping $J \colon S_X \longrightarrow \mathcal{P}(S_{X^*})$. The concept of '$\omega$-exposedness' is natural in the sense that in a reflexive space X a point $x \in S_X$ is ω-exposed if and only if it is exposed.

Let us mention the main themes appearing in this article. First, we give a characterization for ω-exposed points $x \in S_X$ in terms of duality between the norm-attaining G-smooth functionals of the dual space. This also leads to a sufficient condition for a point $x \in S_X$ to possess the ω-LUR property. We also give a characterization for the reflexivity of a Banach space X in terms of an equivalent bidual ω-LUR renorming of the bidual X^{**}. See e.g. [10, 7] for related research.

Secondly, we discuss the abundance of (ω- or strongly) exposed points in various situations. For strongly exposed points this is a classical topic in the

literature (see [6]) and well-studied especially in connection with the Radon-Nikodym property (see e.g. [3, p.35]). In treating the question whether the ω-exposed points are dense in S_X we apply the topological properties of the duality mapping.

We will use the following notations. Real Banach spaces are denoted by X, Y and Z unless otherwise stated. We denote by $B_X = \{x \in X : ||x|| \leq 1\}$ the closed unit ball and by $S_X = \{x \in X : ||x|| = 1\}$ the unit sphere. In what follows τ is a locally convex topology on X.

For a discussion of basic concepts and results concerning the geometry of the norm we refer to [5] and to the first chapter of [3]. The duality mapping $J \colon S_X \to \mathcal{P}(S_{X^*})$ is defined by $J(x) = \{x^* \in S_{X^*} | x^*(x) = 1\}$. If $A \subset S_X$ then we denote $J(A) = \bigcup_{a \in A} J(a)$. Denote by $\mathrm{NA}(S_{X^*}) = J(S_X)$ the set of norm-attaining functionals of S_{X^*}. Let us recall the following well known results:

THEOREM 1.1. *If X is a separable Banach space, then the set of G-smooth points of S_X is a dense G_δ-set.*

THEOREM 1.2. (BISHOP-PHELPS) *The set $\mathrm{NA}(S_{X^*})$ is dense in S_{X^*}.*

Recall that X is called an Asplund space if for any separable $Y \subset X$ it holds that Y^* is separable. An Asplund space X satisfies that the F-smooth points of S_X are a dense G_δ-set (see [3, Ch.1]). If X satisfies this conclusion for G-smooth points respectively, then X is called weakly Asplund. A point $x \in S_X$ is called *very smooth* if $x \in S_X \subset S_{X^{**}}$ is G-smooth considered in X^{**}.

If $f \in S_{X^*}$ is such that $f^{-1}(1) \cap S_X = \{x\}$ then f is said to *expose* x. We say that $x \in S_X$ is a ω-*exposed* point if there is $f \in S_{X^*}$ such that whenever $(x_n) \subset B_X$ is a sequence with $\lim_{n \to \infty} f(x_n) = 1$ then $x_n \xrightarrow{\omega} x$ as $n \to \infty$. If the same conclusion holds for norm convergence then x is called a *strongly exposed* point. In such cases above f is called a ω- (resp. *strongly*) *exposing functional* for x. We denote the ω- (resp. strongly) exposed points of B_X by $\omega-\exp(B_X)$ (resp. $|| \cdot ||-\exp(B_X)$). A point $x \in S_X$ is called τ-strongly extreme, if for all sequences $(z_n), (y_n) \subset B_X$ such that $\frac{z_n + y_n}{2} \xrightarrow{||\cdot||} x$ as $n \to \infty$ it holds that $z_n - y_n \xrightarrow{\tau} 0$ as $n \to \infty$. We say that $x \in S_X$ is a τ-locally uniformly rotund point, τ-LUR point for short, if for all sequences $(x_n) \subset B_X$ such that $\lim_{n \to \infty} ||x + x_n|| = 2$ it holds that $x_n \xrightarrow{\tau} x$ as $n \to \infty$. If above τ is the norm topology, then we abbreviate by LUR. We say that X is τ-LUR when each $x \in S_X$ is τ-LUR. If T is a topological space then a subset $A \subset T$ is called *comeager* provided that it contains a countable intersection of subsets open and dense in T.

2. Weak topology and convexity

The most essential concept in this article is ω-exposed point $x \in S_X$, which by its definition is exposed by a ω-exposing functional $f \in S_{X^*}$. Let us begin by characterizing the ω-exposing functionals.

THEOREM 2.1. *Let X be a Banach space and suppose that $x \in S_X$, $f \in S_{X^*}$ are such that $f(x) = 1$. Then the following conditions are equivalent:*

(i) *f ω-exposes x.*

(ii) *For each closed convex set $C \subset B_X$ such that $\sup_{y \in C} f(y) = 1$ it holds that $x \in C$.*

(iii) *f is a G-smooth point in X^*, i.e. there is a unique $\psi \in S_{X^{**}}$ such that $\psi(f) = 1$.*

Before giving the proof we will make some remarks. The above result must be previously known. For example by applying [11, Thm.1,Thm.3] one can deduce the equivalence (i) $\Leftrightarrow$ (iii). However, we will give a more elementary proof below. If above X is reflexive and f exposes x then one can see by the weak compactness of B_X that actually f ω-exposes x.

Proof of Theorem 2.1. The equivalence (i) $\Leftrightarrow$ (ii) follows easily by applying Mazur's theorem to $\operatorname{conv}(\{x_n | n \in \mathbb{N}\})$, where $(x_n) \subset B_X$ is a sequence such that $f(x_n)$ tends to 1 as $n \to \infty$.

Direction (iii) $\Rightarrow$ (ii): Towards this suppose that $C \subset B_X$ is a closed convex subset so that $\sup_{y \in C} f(y) = 1$. Fix a sequence $(y_n) \subset C$ satisfying $f(y_n) \to 1$ as $n \to \infty$. Observe that $x \in S_X \subset S_{X^{**}}$ is the unique norm-one functional supporting f, since f is a G-smooth point of X^*. By applying the Šmulyan lemma to x and (y_n) considered in X^{**} we get that $y_n \xrightarrow{\omega^*} x$ in X^{**} as $n \to \infty$. This means that $h(y_n) \to h(x)$ as $n \to \infty$ for all $h \in X^*$, so that $y_n \xrightarrow{\omega} x$ in X as $n \to \infty$. Thus $x \in \overline{\operatorname{conv}}(\{y_n | n \in \mathbb{N}\}) \subset \overline{\operatorname{conv}}^{\omega}(C) = C$ by Mazur's theorem, since C is a norm-closed convex set.

Conversely, suppose that (i) holds and let $\phi \in B_{X^{**}}$ be an arbitrary point such that $\phi(f) = 1$. We claim that $\phi = x$, which yields that f is a G-smooth point. Let $g \in X^*$ be arbitrary and recall that B_X is ω^*-dense in $B_{X^{**}}$ by Goldstine's theorem. In particular, $\phi \in \overline{B_X}^{\omega^*}$. Thus

$$B_X \cap \{\psi \in B_{X^{**}} : \ |\psi(f) - 1| < \frac{1}{n} \text{ and } |\psi(g) - \phi(g)| < \frac{1}{n}\} \neq \emptyset$$

for all $n \in \mathbb{N}$. Hence we may pick a sequence $(y_n) \subset B_X$ for which $f(y_n) \to 1$ and $g(y_n) \to \phi(g)$ as $n \to \infty$. Since f ω-exposes x we know that $y_n \xrightarrow{\omega} x$ in X as $n \to \infty$. This yields that $\phi(g) = \lim_{n\to\infty} g(y_n) = g(x)$ and that $\phi = x$ as this equality holds for all $g \in X^*$. ∎

PROPOSITION 2.2. *Let X be a Banach space, $x \in S_X$ a F-smooth point and $f \in S_{X^*}$ a G-smooth point such that $f(x) = 1$. Then x is a ω-LUR point.*

Proof. Suppose $(x_n) \subset B_X$ is a sequence such that $||x_n + x|| \to 2$ as $n \to \infty$. By Theorem 2.1 the functional f ω-exposes x. Thus it suffices to show that $f(x_n) \to 1$ as $n \to \infty$.

By the Hahn-Banach Theorem one can find a sequence of functionals $(g_n) \subset S_{X^*}$ such that $g_n\left(\frac{x+x_n}{2}\right) = \left|\left|\frac{x+x_n}{2}\right|\right|$ for each $n \in \mathbb{N}$. Clearly $g_n(x) \to 1$ and $g_n(x_n) \to 1$ as $n \to \infty$. Hence the F-smoothness of x together with the Šmulyan Lemma yields that $g_n \xrightarrow{||\cdot||} f$ as $n \to \infty$. Thus we obtain that $f(x_n) \to 1$ as $n \to \infty$. ∎

PROPOSITION 2.3. *Let X be a Banach space and suppose that $x^* \in S_{X^*}$ is a very smooth point. Then there exists $x \in S_X \subset S_{X^{**}}$ such that $x^* \in S_{X^*} \subset S_{X^{***}}$ ω-exposes x in X^{**}.*

Proof. Since by the definition x^* is G-smooth in X^{***} it suffices to show that there exists $x \in S_X$ such that $x^*(x) = 1$. Indeed, once this is established we may apply Theorem 2.1 to obtain the claim.

Let $x^{**} \in S_{X^{**}}$ be such that $x^{**}(x^*) = 1$. By Goldstein's theorem $B_X \subset B_{X^{**}}$ is ω^*-dense. Pick a sequence $(x_n) \subset S_X$ such that $x^*(x_n) \to 1$ as $n \to \infty$. Observe that according to Theorem 2.1 the functional x^* considered in $S_{X^{***}}$ ω-exposes x^{**} in X^{**}. Hence $x_n \xrightarrow{\omega} x^{**}$ in X^{**} as $n \to \infty$. Since $X \subset X^{**}$ is ω-closed by Mazur's theorem, we obtain that $x^{**} \in S_X \subset S_{X^{**}}$. ∎

It is a natural idea to characterize reflexivity of Banach spaces in terms of suitable equivalent renormings (see e.g. [2]).

THEOREM 2.4. *The following conditions are equivalent:*
(1) *X is reflexive.*
(2) *X admits an equivalent renorming such that X^{**} is ω-LUR.*
(3) *X admits an equivalent renorming such that $\Lambda \subset S_{X^{**}}$ given by*

$$\Lambda = \left\{ \phi \in S_{X^{**}} | \forall \, (\phi_n)_{n\in\mathbb{N}} \subset S_{X^{**}} \; \sup_n ||\phi + \phi_n|| = 2 \; \Rightarrow \; \phi \in [(\phi_n)_{n\in\mathbb{N}}] \right\}$$

satisfies that $[\Lambda] = X^{**}$.

Proof. If X is reflexive then it is weakly compactly generated and hence admits an equivalent LUR norm, see e.g. [4, p.1784]. Thus, by using reflexivity again we obtain that $S_{X^{**}}$ is LUR.

Direction $(2) \Rightarrow (3)$ follows by using Mazur's theorem that for convex sets weak and norm closure coincide. Since reflexivity is an isomorphic property we may assume without loss of generality in proving direction $(3) \Rightarrow (1)$ that X already satisfies $[\Lambda] = X^{**}$. Fix $\phi \in \Lambda$. Select a sequence $(f_n) \subset S_{X^*}$ such that $\phi(f_n) \to 1$ as $n \to \infty$. Pick a sequence $(x_n)_{n \in \mathbb{N}} \subset S_X$ such that $f_n(x_n) \to 1$ as $n \to \infty$. This means that

$$\|\phi + x_n\|_{X^{**}} \geq (\phi + x_n)(f_n) \longrightarrow 2 \text{ as } n \to \infty.$$

Hence by the definition of Λ we obtain that $\phi \in [(x_n)]$. Since $[\Lambda] = X^{**}$, this yields that $[X] = X^{**}$ and hence $X = X^{**}$ as $X \subset X^{**}$ is a closed subspace. ∎

It turns out below that a smoothness property (namely Asplund) together with a weak convexity property (namely ω-strongly extreme) yields in fact a stronger convexity property (namely the ω-convergence), which is analogous to the "ω-exposed situation".

THEOREM 2.5. *Let X be an Asplund Banach space and let $x \in S_X$ be a ω-strongly extreme point. Suppose that $\{x_n | n \in \mathbb{N}\} \subset B_X$ is a set such that $x \in \overline{\text{conv}}(\{x_n | n \in \mathbb{N}\})$. Then there is a sequence $(x_{n_k})_k \subset \{x_n | n \in \mathbb{N}\}$ such that $x_{n_k} \xrightarrow{\omega} x$ as $k \to \infty$.*

Proof. Consider convex combinations

$$y_m = \sum_{n \in J_m} a_n^{(m)} x_n \in \text{conv}(\{x_n | n \in \mathbb{N}\})$$

such that $y_m \to x$ as $m \to \infty$. Above $J_m \subset \mathbb{N}$ is finite and $a_n^{(m)} \geq 0$ are the corresponding convex weights for $m \in \mathbb{N}$.

One shows easily that if $x_n', x_n'' \in B_X$, $n \in \mathbb{N}$, satisfy $\lambda_n x_n' + (1-\lambda_n)x_n'' \to x$ with $\lambda_n \to \lambda \in (0, 1]$ as $n \to \infty$, then $x_n' \xrightarrow{\omega} x$ as $n \to \infty$.

Fix $f \in X^*$, $\epsilon > 0$ and put

$$K_m = \{n \in J_m | \ f(x_n) < f(x) - \epsilon\} \quad \text{for } m \in \mathbb{N}.$$

We claim that $\lambda_m = \sum_{n \in K_m} a_n^{(m)} \to 0$ as $m \to \infty$. Indeed, assume to the contrary that this is not the case. Then, by passing to a subsequence we may

assume without loss of generality that $\lambda_n \to \lambda \in (0,1]$ as $n \to \infty$. Write $y_m = \lambda_m y'_m + (1 - \lambda_m) y''_m$ for $m \in \mathbb{N}$, where

$$y'_m = \frac{\sum_{n \in J_m} a_n^{(m)} x_n}{\lambda_m} \quad \text{and} \quad y''_m = \frac{\sum_{n \in \mathbb{N} \setminus J_m} a_n^{(m)} x_n}{(1 - \lambda_m)}.$$

By the definition of the sequence (y'_m) we have $f(y'_m) \leq f(x) - \epsilon$ for $m \in \mathbb{N}$ but this contradicts the remark that $y'_m \xrightarrow{\omega} x$ as $m \to \infty$.

Thus $\sum_{n \in K_m} a_n^{(m)} \to 0$ as $m \to \infty$ and a similar argument for $L_m = \{n \in J_m | f(x_n) > f(x) + \epsilon\}$, $m \in \mathbb{N}$, gives that $\sum_{n \in L_m} a_n^{(m)} \to 0$ as $m \to \infty$. These observations yield that

$$\lim_{m \to \infty} \sum_{\substack{n \in J_m: \\ |f(x) - f(x_n)| < \epsilon}} a_n^{(m)} = 1$$

for any $\epsilon > 0$.

Since f was arbitrary, we obtain inductively that

$$\lim_{m \to \infty} \sum_{\substack{n \in J_m: \\ x_n \in U}} a_n^{(m)} = 1,$$

where $U = \bigcap_k g_k^{-1}([g_k(x) - \epsilon, g_k(x) + \epsilon])$ and $g_1, \ldots, g_k \in X^*$, $k \in \mathbb{N}$. Recall that x has a weak neighbourhood basis consisting of such sets U. In particular

$$x \in \overline{\{x_n | n \in \mathbb{N}\}}^{\omega}. \tag{2.1}$$

Let $Y = \overline{\text{span}}(\{x_n | n \in \mathbb{N}\})$. Since X is an Asplund space and Y is separable we obtain that Y^* is separable. Fix a sequence $(h_n)_n \subset Y^*$, which is dense in Y^*. Note that (B_Y, ω) is metrizable by the metric $d(x,y) = \sum_{n \in \mathbb{N}} 2^{-n} |h_n(x - y)|$, $x, y \in B_Y$. Hence there is a sequence $(x_{n_k})_k \subset \{x_n | n \in \mathbb{N}\}$ such that $x_{n_k} \xrightarrow{\omega} x$ as $k \to \infty$ in Y. By the Hahn-Banach extension of Y^* to X^* it is straight-forward to see that $x_{n_k} \xrightarrow{\omega} x$ as $k \to \infty$ also in X. ∎

2.1. DENSITY OF ω-EXPOSED POINTS. Recall the following result due to Lindenstrauss and Phelps [6, Cor.2.1.]:

If C is convex body in an infinite dimensional separable reflexive Banach space, then the extreme points of C are not isolated in the norm topology.

This result is the starting point for the studies in this section.

Let us first consider a natural property of the duality mapping $J \colon S_X \to \mathcal{P}(S_{X^*})$. The following fact is an elementary topological statement about J and it is proved here for convenience.

PROPOSITION 2.6. *The following conditions are equivalent for a Banach space X:*

(1) *For each relatively open non-empty $U \subset S_X$ the set $J(U)$ contains an interior point relative to* $\mathrm{NA}(S_{X^*})$.

(2) *For each relatively dense $A \subset \mathrm{NA}(S_{X^*})$ the subset $\{x \in S_X \mid J(x) \cap A \neq \emptyset\} \subset S_X$ is dense.*

Proof. Suppose that (1) holds. Let $A \subset \mathrm{NA}(S_{X^*})$ be a dense subset. Assume to the contrary that $\{x \in S_X \mid J(x) \cap A \neq \emptyset\} \subset S_X$ is not dense. Thus there is a non-empty set $U \subset \{x \in S_X \mid J(x) \cap A = \emptyset\}$, which is open in S_X. Hence $J(U) \cap A = \emptyset$ where $J(U) \subset S_{X^*}$ is open according to condition (1). This contradicts the assumption that $A \subset \mathrm{NA}(S_{X^*})$ is dense and consequently condition (2) holds.

Suppose that (2) holds and $U \subset S_X$ is a non-empty open set. Assume to the contrary that $J(U)$ does not contain an interior point relative to $\mathrm{NA}(S_{X^*})$. Then $\mathrm{NA}(S_{X^*}) \setminus J(U)$ is dense in $\mathrm{NA}(S_{X^*})$. Hence condition (2) states that $S_X \setminus U = J^{-1}(\mathrm{NA}(S_{X^*}) \setminus J(U))$ is dense in S_X. This contradicts the assumption that U is open and hence condition (1) holds. ∎

When X satisfies the above equivalent conditions of Proposition 2.6, we say that X satisfies $(*)$ for the sake of brevity. The following result describes this condition.

PROPOSITION 2.7. *Suppose that X^* is an Asplund space. Then X satisfies condition $(*)$ if and only if the set of strongly exposed points of B_X is dense in S_X.*

Proof. The "if" case. Fix non-empty $A \subset \mathrm{NA}(S_{X^*})$. Suppose $S_X \setminus \{x \in S_X \mid J(x) \cap A \neq \emptyset\}$ contains an interior point relative to S_X. We aim to show that in such case $A \subset S_{X^*}$ is not dense. Indeed, by the density assumption regarding the strongly exposed points we obtain that there is $x \in ||\cdot||-\exp(B_X)$, which is not in the closure of $\{x \in S_X \mid J(x) \cap A \neq \emptyset\}$. If $f \in S_{X^*}$ is a strongly exposing functional for x then there is $\epsilon > 0$ such that

$f(\{x \in S_X | \ J(x) \cap A \neq \emptyset\}) \subset [-1, 1 - \epsilon]$. This gives that $||f - g|| \geq \epsilon$ for all $g \in A$, so that $A \subset S_{X^*}$ is not dense.

The "only if" case. Let us apply the fact that X^* is Asplund. Denote by F the set of F-smooth points $x^* \in S_{X^*}$. Recall that $F \subset S_{X^*}$ is dense since X^* is Asplund. Condition $(*)$ of X gives that $\{x \in S_X | \ J(x) \cap \mathrm{F} \neq \emptyset\}$ is dense in S_X. By applying the Šmulyan Lemma it is easy to see that each F-smooth functional $x^* \in S_{X^*}$ is norm-attaining and in fact a strongly exposing functional. ∎

The following main result is a version of the above-mentioned result by Lindenstrauss and Phelps.

THEOREM 2.8. *Let X be a Banach space, which satisfies $\dim(X) \geq 2$ and suppose that the following conditions hold:*

(1) $\mathrm{NA}(S_{X^*}) \subset S_{X^*}$ *is comeager.*
(2) X^* *is weakly Asplund.*

Then there does not exist a G-smooth point $x \in S_X$, which is ω-isolated in $\omega - \exp(B_X)$. Moreover, if X satisfies additionally $(*)$, then $\omega - \exp(B_X) \subset S_X$ is dense.

To comment on the assumptions shortly, observe that condition (1) above holds for instance if X has the RNP (see [8] and [1, Thm.8]) and X^* is weakly Asplund for instance if X^* is separable by Theorem 1.1. It follows that the Asplund property of X^* is sufficient for both the conditions (1) and (2) to hold. Observe that we do not require X above to be separable, nor infinite dimensional. On the other hand, the assumption about the G-smoothness of x can not be removed. For example consider ℓ_n^∞ for $n \in \{2, 3, \ldots\}$.

Proof of Theorem 2.8. According to the weak Asplund property of X^* there is a dense G_δ-set of G-smooth points in S_{X^*}. We apply this fact together with assumption (1) as follows. By using the Baire category theorem we obtain that

$$\mathcal{NG} = \{x^* \in \mathrm{NA}(S_{X^*}) | x^* \text{ is G} - \text{smooth}\}$$

is dense in S_{X^*}. Observe that by Theorem 2.1 all $f \in \mathcal{NG}$ are in fact ω-exposing functionals.

Now, assume to the contrary that x is a G-smooth ω-isolated point in $\omega - \exp(B_X)$. Then x is ω-exposed by a unique support functional $f \in \mathcal{NG}$. It is easy to see that since f is a ω-exposing and ω-isolated functional, there

is $\epsilon > 0$ such that $f(\omega-\exp(B_X) \setminus \{x\}) \subset [-1, 1 - \epsilon]$. Consequently, by the uniqueness of f we obtain that $||f - g|| \geq \epsilon$ for all $g \in \mathcal{NG} \setminus \{f\}$. Thus we obtain that the relatively open set $\{h \in S_{X^*} : 0 < ||f - h|| < \epsilon\} \subset S_{X^*}$ is non-empty as $\dim(X) \geq 2$ and it does not intersect the dense subset $\mathcal{NG} \subset S_{X^*}$, which provides a contradiction. Hence the first part of the claim holds.

Finally, let us assume that X satisfies $(*)$. Hence $\{x \in S_X | J(x) \cap \mathcal{NG} \neq \emptyset\} = J^{-1}(\mathcal{NG})$ is dense in S_X. Recall that the points in $J^{-1}(\mathcal{NG})$ are ω-exposed. ∎

REFERENCES

[1] BOURGAIN, J., On dentability and the Bishop-Phelps property, *Israel J. Math.*, **28** (4) (1977), 265 – 271.

[2] HÁJEK, P., JOHANIS, M., Characterization of reflexivity by equivalent renorming, *J. Funct. Anal.*, **211** (1) (2004), 163 – 172.

[3] JOHNSON, W.B., LINDENSTRAUSS, J., "Handbook of the Geometry of Banach Spaces, Vol. 1", North-Holland, Amsterdam, 2001.

[4] JOHNSON, W.B., LINDENSTRAUSS, J., "Handbook of the Geometry of Banach Spaces, Vol. 2", North-Holland, Amsterdam, 2003.

[5] HABALA, P., HÁJEK, P., ZIZLER, V., "Introduction to Banach spaces, I", Matfyzpress, 1996.

[6] LINDENSTRAUSS, J., PHELPS, R.R., Extreme point properties of convex bodies in reflexive Banach spaces, *Israel J. Math.*, **6** (1968), 39 – 48.

[7] NAN, C.X., On weakly exposed points, *Northeast. Math. J.*, **6** (4) (1990), 449 – 454.

[8] PHELPS, R.R., Dentability and extreme points in Banach spaces, *J. Funct. Anal.*, **17** (1974), 78 – 90.

[9] STEGALL, C., The Radon-Nikodým property in conjugate Banach spaces, II, *Trans. Amer. Math. Soc.*, **264** (2) (1981), 507 – 519.

[10] SULLIVAN, F., Geometrical properties determined by the higher duals of a Banach space. *Illinois J. Math.*, **21**,(2) (1977), 315 – 331.

[11] ZHANG, Z.H., ZHANG, C.J., On very rotund Banach space. *Appl. Math. Mech.*, **21** (8) (2000), 965 – 970.