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Topological Open Problems in the Geometry of Banach Spaces *

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Abstract: We survey a brief account of topological open problems inside the area of renormings of Banach spaces. All of them are related with the Stone's theorem on the paracompactness of metric spaces since it is our scalpel to find out the rigidity condition for the renorming process. All of them have been collected in our recent monograph [21] which is the main source for the present survey article. We also present recent results showing the connection between Stone's theorem and Deville's master lemma for locally uniformly rotund renormings.

Key words: locally uniformly rotund norm, Kadec norm, Fréchet norm, network, measure of non compactness.

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1. Introduction

The paracompactness is a generalization of the concept of compactness and it belongs to the class of concepts related with covering properties of a topological spaces. On the other hand, the concept of full normality can be regarded as belonging to another genealogy of concepts, the separation axioms which include regularity, normality, etc. The Stone's theorem says that those two concepts, belonging to different categories, coincide for Hausdorff topological spaces, see [22, Chapter V]. In particular, the fact that every metrizable space is paracompact is going to be a fundamental one when we are looking for convex renorming properties in a Banach space. Indeed the use of Stone's theorem has been extensively considered in order to build up new techniques to construct equivalent locally uniformly rotund norms on a given normed space X in [21]. The σ-discreteness of the basis for the metric topologies gives the necessary rigidity condition that appears in all the known

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cases of existence of such a renorming property, [8, 19]. It is our aim here to survey a brief account of some of my favourite topological open problems in the area, and in particular the ones connected with my lecture in the Conference of Banach Spaces held at Cáceres, Spain, from 4 to 8 of September 2006. Some of them are related with classical questions asked by different people in conferences, papers and books. Others have been presented as open problems in schools, workshops, conferences and recent papers on the matter remaining with this character up to our knowledge. All of them have been collected in our recent monograph [21] which is the main source for the present survey article. We apologize for any fault assigning authorship to a given question. The intention here is to provide a general picture to deal with good questions in the area rather than to formulate precise evaluation for the first time the problems were proposed. I would like to thank Professor Jesús M.F. Castillo for his kind invitation and nice hospitality during my very short visit to Cáceres for the ICM 2006 Satellite Conference of Banach spaces.

2. Kadec renormings

Let us remember that a norm in a normed space is said to be a Kadec norm if the weak and the norm topologies coincide on the unit sphere, and it is said to be strictly convex when the unit sphere doest not contain any non trivial segments. After the results presented in the monograph [21] it comes up that the study of strictly convex or Kadec rotund norms, [8, 17] should be of great interest in the near future. Precisely, we are proposing the following general problem:

Question 1. The study of non linear maps $\Phi : X \to Y$ transferring a Kadec (resp. a strictly convex) norm from a normed space Y with Kadec (resp. strictly convex) norm to X.

For the case of strict convexity we refer to the recent paper [20] where a linear topological characterization of the property of strictly convex renorming for normed spaces is presented. For dual norms in spaces $C(K)^*$, where K is a scattered compact space, and in particular for the compactification of trees, a recent result has been obtained by R. Smith [30, 31]. The problem for Kadec renormings is completely open. The main reason is that there is no example of a normed with a σ-isolated network for the weak topology without admitting an equivalent Kadec norm, [21]. Indeed, transferring results for normed spaces with a σ-isolated network for the weak topology are obtained in [21, Chapter 3]. Nevertheless, the convexification problem in the core of the matter seems to be very difficult to deal with. Let us introduce here some ideas for the study of the question.

If we have a Kadec norm $\|\cdot\|$ on the normed space X, then the identity map from $(S_X, \sigma(X, X^*))$ to $(X, \|\cdot\|)$ is continuous. If we have a subset C of the normed space X, a normed space $(Y, \|\cdot\|)$ and a map $\phi : (C, \sigma(X, X^*)) \to Y$, then ϕ is called *piecewise continuous* if there is a countable cover $C = \cup_{n=1}^{\infty} C_n$ such that every one of the restrictions $\phi_{|C_n}$ is weak to norm continuous. A norm pointwise limit of a sequence of piecewise continuous maps is called a *σ-continuous map*, [19]. Indeed, we have the following result:

Proposition 1. (See [19, Theorem 1]) *A map ϕ from $(C, \sigma(X, X^*))$ into a normed space $(Y, \|\cdot\|)$ is σ-continuous if, and only if, for every $\epsilon > 0$ we have $C = \bigcup_{n=1}^{\infty} C_{n,\epsilon}$ in such a way that for every $n \in \mathbb{N}$ and every $x \in C_{n,\epsilon}$ there is a weak neighbourhood U of x with*

$$\operatorname{osc}\left(\phi_{|U \cap C_{n,\epsilon}}\right) : \sup\left\{\|\phi(x) - \phi(y))\| \, : \, x, y \in U \cap C_{n,\epsilon}\right\} < \epsilon \, .$$

In a normed space $(X, \|\cdot\|)$ with a Kadec norm the identity map in X from the weak to the norm topologies is σ-continuous, the norm topology has a network $\mathcal{N}$ that can be written as a countable union of subfamilies, $\mathcal{N} = \cup_{n=1}^{\infty} \mathcal{N}_n$, where every one of the subfamilies $\mathcal{N}_n$ is a discrete family in its union $\cup\{N : N \in \mathcal{N}_n\}$ endowed with the weak topology, i.e., it is an isolated subfamily for the weak topology. No example is known of a Banach space with this kind of network, which is called a descriptive Banach space, and without equivalent Kadec norm, see [21, Chapter 3].

In the classical theory of Banach spaces, not only were normed spaces considered, but also those spaces on which a metric is defined which is compatible with the vector space operations. The uniform structure of a metrizable topological vector spaces is described with the following notion (see [14, p. 163]):

Definition 1. An F-norm in a vector space X is a function $|||\cdot||| : X \to \mathbb{R}^+$ such that:

(1) $x = 0$ if $|||x||| = 0$,

(2) $|||\lambda x||| \le |||x|||$ if $|\lambda| \le 1$,

(3) $|||x + y||| \le |||x||| + |||y|||$,

(4) $\lim_n |||\lambda x_n||| = 0$ if $\lim_n |||x_n||| = 0$,

(5) $\lim_n |||\lambda_n x||| = 0$ if $\lim_n \lambda_n = 0$.

We have been able to obtain the following theorem that gives a characterization of Kadec renormability by means of F-norms:

Theorem 1. (See [25]) *Let $(X, \|\cdot\|)$ be a normed space with a norming subspace F in X^*. The following conditions are equivalent:*

(i) *There is an equivalent $\sigma(X,F)$-lower semicontinuous and $\sigma(X,F)$-Kadec F-norm $|||\cdot|||$, i.e., an F-norm such that the $\sigma(X,F)$ and norm topologies coincide on the unit sphere*

$$\{x \in X : |||x||| = 1\}.$$

(ii) *The identity map from the unit sphere $(S_X, \sigma(X,F))$ into the normed space $(X, \|\cdot\|)$ is σ-continuous.*

(iii) *The norm topology has a network $\mathcal{N}$ that can be written as a countable union of subfamilies, $\mathcal{N} = \cup_{n=1}^{\infty} \mathcal{N}_n$, where every one of the subfamilies $\mathcal{N}_n$ is a discrete family in its union $\cup\{N : N \in \mathcal{N}_n\}$ endowed with the $\sigma(X,F)$ topology.*

Let us remember that a quasi-norm in a vector space X is a function $q : X \to \mathbb{R}^+$ such that:

(1) $x = 0$ if $q(x) = 0$,

(2) $q(\alpha x) = |\alpha| q(x)$ for all $\alpha \in \mathbb{R}$ and $x \in X$,

(3) $q(x+y) \leq k(q(x) + q(y))$ for some $k \geq 1$ and all $x, y \in X$.

Analyzing the F-norm constructed in the former theorem we arrive to the following:

Corollary 1. (See [25]) *Descriptive Banach spaces coincide with the ones with an equivalent quasinorm q, such that the weak and the norm topology coincide on the unit sphere $\{x \in X : q(x) = 1\}$.*

Prior estimates for the quasinorm q above were firstly obtained by M. Raja in [28] with a positively homogeneous function $q : X \to \mathbb{R}^+$ only. The question that remains is the following:

QUESTION 2. Let X be a normed space and F a norming subspace of X^* such that the identity map $Id : (X, \sigma(X, F)) \to (X, \|\cdot\|)$ is σ-continuous. Is it possible to convexify the construction above to get an equivalent $\sigma(X, F)$-Kadec norm $|||\cdot|||$; i.e., the $\sigma(X, F)$ and norm topologies coincide on the unit sphere

$$\{x \in X : |||x||| = 1\}\,?$$

If the former question has a positive answer, then Question 1 for Kadec renormings has a similar answer to the one given in [21] for LUR renormings. Indeed, a σ-continuous map ϕ from the weak to the norm topologies which is also co-σ-continuous for the norms will transfer the Kadec norm in Y to X, see [21].

In relation with descriptive properties let us remind that for a descriptive Banach space the family of weak Borel sets coincides with the norm Borel sets, [7, 24]. Based on a sophisticated construction of S. Todorcevic [33], R. Pol has kindly informed us that it is consistent the existence of a compact scattered space K such that in the function space $C(K)$ each norm open set is an $\mathcal{F}_\sigma$-set with respect to the weak topology but the identity map

$$Id : \big(C(K), \sigma(C(K), C(K)^*)\big) \to (C(K), \|\cdot\|_\infty)$$

is not σ-continuous, [15]. Descriptive Banach spaces are Souslin sets made up with $\sigma(X^{**}, X^*)$ open or closed subsets of the bidual. This class of spaces are called weakly Cech analytic and coincide with the ones that can be represented with a Souslin scheme of Borel subsets in their $\sigma(X^{**}, X^*)$ biduals. The fact that every weakly Cech analytic Banach space is σ-fragmented is the main result in [11]. The reverse implications are open questions considered in [12, 13], and we remind here the following:

QUESTION 3. Is there any gap between the classes of descriptive Banach spaces and that of σ-fragmented Banach spaces?

After the seminal paper of R. Hansell [7] we know that a covering property on the weak topology of a Banach space, known as hereditarely weakly θ-refinability, is a necessary and sufficient condition for the coincidence of both classes. Indeed, all known examples of normed spaces which are not weakly θ-refinable are not σ-fragmentable by the norm, [2, 3]. For spaces of continuous functions on trees R. Haydon has proved that there is no gap between σ-fragmented and the pointwise Kadec renormability property of the space, [8]. We can consider a particular case of the former question as follows. For this question R. Hansell conjectures a positive answer:

Question 4. Let X be a weakly Cech analytic Banach space where every norm open set is a countable union of sets which are differences of closed sets for the weak topology. Doest it follow that the identity map

$$Id : (X, \sigma(X, X^*)) \to (X, \|\cdot\|)$$

is σ-continuous?

In the particular case of a Banach space X with the Radon Nikodym property it is still an open problem to decide if X has even an equivalent rotund norm. In that case the LUR renormability reduces to the question of Kadec renormability by our results in [18]. So we summarize here:

Question 5. If the Banach space X has the Radon Nikodym property (i.e., every bounded closed convex subset of X has slices of arbitrarily small diameter), does it follow that X has an equivalent Kadec norm? Does it have an equivalent rotund norm?

Let us remark here that a result of D. Yost and A. Plicko [27] shows that the Radon Nikodym property doest not imply the separable complementation property. Thus it is not possible any approach to the former question based on the projectional resolution of the identity which works for the dual case, [4].

3. Fréchet smooth norms and locally uniformly rotund renormings

In a Hilbert space $(H, <\cdot>)$ the parallelogram law says that

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

for every $x, y \in H$. It is in the core of the basic results on the geometry of Hilbert spaces such as the projection theorem, the Lax-Milgram theorem or the variational approach for minimizing quadratic forms on H. A norm $\|\cdot\|$ in a real vector space X is said to be locally uniformly rotund (LUR) if, asymptotically, it has a local behavior similar to the parallelogram law:

$$\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x+x_n\|^2\right) = 0 \qquad \Rightarrow \qquad \lim_n \|x - x_n\| = 0$$

for any sequence (x_n) and x in X. In this case for every point of the unit sphere $x \in S_X$, if we select $f_x \in S_{X^*}$ such that $f_x(x) = 1$, then we have that the slices

$$S(f_x, 1-\delta) := \{y \in B_X : f_x(y) > 1-\delta\}, \qquad \delta \in (0,1),$$

form a neighbourhood basis at x for the norm topology induced on the unit ball B_X. Indeed,

$$\lim_n \left(2\|x\|^2 + 2\|x_n\|^2 - \|x+x_n\|^2\right) = 0$$

whenever $\lim_n f_x(x_n) = 1$ and $x_n \in B_X$ for every $n \in \mathbb{N}$, since

$$\lim_n \|x+x_n\|^2 \geq \lim_n f_x(x+x_n)^2 = 4.$$

Thus, the identity map from $(S_X, \sigma(X, X^*))$ to $(X, \|\cdot\|)$ is continuous in a very special way that we call *slice continuity*.

Let us see that we have a similar phenomena for a Fréchet smooth norm if we allow countable splittings and sequence limits, i.e., if we consider the class of mappings called σ-slicely continuous, togheter with a jump to the dual space X^* through the duality mapping.

For a given subset C of the normed space X, a norming subspace $F \subset X^*$, and a normed space $(Y, \|\cdot\|)$, a map $\phi : (C, \sigma(X,F)) \to Y$ is called *slicely continuous* if for every $x \in C$ and every $\epsilon > 0$ there is a $\sigma(X,F)$-open half space H with $x \in H$ and such that $\operatorname{osc}\left(\phi_{|H\cap C}\right) < \epsilon$; ϕ is called *piecewise slicely continuous* if there is a countable cover $C = \cup_{n=1}^{\infty} C_n$ such that every one of the retrictions $\phi_{|C_n}$ is slicely continuous. A norm pointwise limit of a sequence of piecewise slicely continuous maps is called a *σ-slicely continuous map*, [19]. Indeed, we have the following result:

Proposition 2. (See [19, Theorem 2]) *Let C be a subset of the normed space X and $F \subset X^*$ a norming subspace for X. A map ϕ from $(C, \sigma(X,F))$ into a normed space $(Y, \|\cdot\|)$ is σ- slicely continuous if, and only if, for every $\epsilon > 0$ we have $C = \bigcup_{n=1}^{\infty} C_{n,\epsilon}$ in such a way that for every $n \in \mathbb{N}$ and every $x \in C_{n,\epsilon}$ there is a $\sigma(X,F)$ open half space H with $x \in H$ and*

$$\operatorname{osc}\left(\phi_{|_{H\cap C_{n,\epsilon}}}\right) : \sup\left\{\|\phi(x) - \phi(y))\| : x, y \in H \cap C_{n,\epsilon}\right\} < \epsilon.$$

A main result here is the following characterization of normed spaces admitting an equivalent LUR norm:

Theorem 2. (See [21]) *A normed space X with a norming subspace $F \subset X^*$ admits an equivalent $\sigma(X,F)$-lower semicontinuous and LUR norm if, and only if, the identity map from S_X into X is σ-slicely continuous from the $\sigma(X,F)$ to the norm topology.*

Based on this result we build up in [21] the necessary tools for the study of Question 1 above for the locally uniformly rotund renormnings. Among the results presented there is a careful analysis of the class of σ-slicely continuous map. It is proved, for instance, that this class is stable by sums and products (when X is an algebra too) leading us to the following inversion theorem:

Theorem 3. (See [21, Theorem 5.11]) *Let $(X, \|\cdot\|)$ be a normed space and $F \subset X^*$ a norming subspace for it. Let $\Phi : (X, \sigma(X,F)) \to X$ be a σ-slicely continuous map and let us consider the set*

$$S_\Phi := \left\{ x \in X \ : \ 0 \in \overline{\{(Id - \Phi)^n(x) \ : \ n \in \mathbb{N}\}}^{\sigma(X,X^*)} \right\}.$$

Then the identity map

$$Id : (S_\Phi, \sigma(X,F)) \to (X, \|\cdot\|)$$

is σ-slicely continuous and consequently the normed space X admits an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund equivalent norm on every point of S_Φ.

In a Hilbert space $(H, <\cdot>)$ the norm is Fréchet differentiable at every point other than the origin and the duality map is nothing else but the Riesz isomorphism theorem which is slicely continuous on the unit sphere S_H. Indeed, $\|x-y\|^2 = 1 - 2 < x, y > +1$ for every $x, y \in S_H$. Then if we fix $x \in S_H$ we see that for every $y \in \{z \in S_H :< z, x >\geq 1-\delta\}$ we have $\|x-y\| \leq \sqrt{2\delta}$. The duality map δ behaves in that way for a Fréchet differentiable norm too. We follow the arguments showed in [21]:

Proposition 3. *Let $(X, \|\cdot\|)$ be a Banach space with a Fréchet differentiable norm and $\delta : (S_X, \sigma(X,X^*)) \to X^*$ be the duality map; i.e., $\delta(x)(x) = 1 = \|x\|$ for every $x \in S_X$. Then δ is σ-slicely continuous.*

Proof. The Smulyan test says that $\|\cdot\|$ is Fréchet differentiable at $x \in S_X$ if, and only if, $\lim_n \|f_n - g_n\| = 0$ whenever $f_n, g_n \in S_{X^*}$ satisfy $\lim_n(f_n(x)) = \lim_n(g_n(x)) = 1$. If we fix $\epsilon > 0$ and $x \in S_X$ there is $1 > \mu_x^\epsilon > 0$ such that

$\|f - g\| \leq \epsilon/2$ whenever $f, g \in S_{X^*}$ and $f(x) > 1 - \mu_x^\epsilon$ and $g(x) > 1 - \mu_x^\epsilon$. Let us define $S_{n,\epsilon} := \{x \in S_X : \mu_x^\epsilon > 1/n\}$. It is clear that $S_X = \cup_{n=1}^\infty S_{n,\epsilon}$ and that for any $x \in S_{n,\epsilon}$ we have

$$\|\delta(y) - \delta(z)\| \leq \|\delta(y) - \delta(x)\| + \|\delta(x) - \delta(z)\| \leq \epsilon$$

whenever

$$\delta(x)(y) > 1 - 1/n\,, \qquad \delta(x)(z) > 1 - 1/n \qquad \text{and} \qquad y, z \in S_{n,\epsilon}\,,$$

since $\mu_y^\epsilon > 1/n$ and $\mu_z^\epsilon > 1/n$ by the very definition of our sets $S_{n,\epsilon}$. Thus for the open half space $H := \{y \in X : \delta(x)(y) > 1 - 1/n\}$ we have that $x \in H$ and

$$\operatorname{osc}\left(\delta_{|_{H \cap S_{n,\epsilon}}}\right) : \sup\left\{\|\delta(y) - \phi(z))\| \,:\, y, z \in H \cap S_{n,\epsilon}\right\} \leq \epsilon$$

so the proof is over. ∎

We have seen in [21] that for a Banach space X with a Fréchet differentiable norm which has Gâteaux differentiable dual norm the duality map δ provides a σ-slicely continuous and co-σ-continuous map between the unit spheres of X and the dual space X^*, and therefore X admits an equivalent locally uniformly rotund norm. It seems to be possible that the requirement on the dual norm could be relaxed by looking for σ-continuous maps from the dual space X^* into X such that the composition with the duality map would provide enough σ-slicely continuous maps from X into X to approximate the identity map, and to finally get the LUR renormability of the space X itself. So we propose to study the following:

Question 6. Let X be a Banach space with a Fréchet differentiable norm. Is it possible to construct a sequence of σ-continuous maps for the norm topologies $\phi_p : X^* \to X$ such that the sequence $\{\phi_p \circ \delta : p \in \mathbb{N}\}$ will provide a way to approximate the identity map on the unit sphere of X? For instance, in such a way that:

$$0 \in \overline{\{(Id - \phi_p)^n(x) \,:\, n, p \in \mathbb{N}\}}^{\sigma(X, X^*)}$$

for every $x \in S_X$ (see Theorem 3).

A place to begin to look for the sequence $(\phi_n \circ \delta)$ could be modifications of the Toruncyck homeomorphism between the dual X^* and X for every Asplund space X since the density characters of X and X^* coincide. If it is so, the Banach space will be LUR renormable and we will have a positive answer to the old question:

Question 7. Let X be a Banach space with a Fréchet differentiable norm. Does it follow that X admits an equivalent locally uniformly rotund norm?

Indeed, R. Haydon has recently showed that if the dual space X^* has a dual LUR norm then X admits an equivalent LUR norm and these spaces admit C^1-partitions of unity as the one considered by Asplund, [9]. The former problem has been asked by R. Haydon in [8] and it appears in the book by R. Deville, G. Godefroy and V. Zizler, [1, Chapter VIII, Theorem 3.2 and Theorem 3.12] as well as in Zizler's renorming paper [34].

The function $s(x, y) := 2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2$ defines a symmetric on a given normed space $(X, \|\cdot\|)$. When we have $X \subset l^\infty(\Gamma)$ we can consider the pointwise topology $\mathcal{T}_P$ induced from the product space $\mathbb{R}^\Gamma$. The norm of X is said to be pointwise LUR if

$$\lim_n s(x_n, x) = 0 \qquad \Rightarrow \qquad \mathcal{T}_p - \lim_n x_n = x$$

for any sequence (x_n) and x in X. When we use the standard embedding of a Banach space X into $l^\infty(B_{X^*})$ we would have the notion of a weakly LUR norm. In that case the Banach space admits an equivalent LUR norm too as we proved in [16]. Nevertheless the notion of pointwise LUR norm is weaker than that of weakly LUR. Indeed, the space $l^\infty(\mathbb{N})$ admits the equivalent pointwise LUR norm defined by:

$$|||x|||^2 := \|x\|_\infty^2 + \sum_{n=1}^{\infty} 1/2^n |x(n)|^2$$

for every $(x(n)) \in l^\infty(\mathbb{N})$, but it does not have any equivalent Kadec norm, [1].

We have been able to obtain the following characterization for this kind of strictly convex norms:

Theorem 4. (See [6]) *Let X be a subspace of the normed space $l^\infty(\Gamma)$ for some non empty set Γ. Then X admits an equivalent pointwise locally uniformly rotund norm if, and only if, the pointwise topology $\mathcal{T}_p$ on X admits a network*

$$\mathcal{N} = \cup_{n=1}^\infty \mathcal{N}_n$$

where every one of the families $\mathcal{N}_n$ is pointwise-slicely discrete in its union $\cup\{N : N \in \mathcal{N}_n\}$, i.e., for every $x \in \cup\{N : N \in \mathcal{N}_n\}$ there is a pointwise open half space H with $x \in H$ such that H meets only one member of the family $\mathcal{N}_n$.

The method to prove the former theorem is based on the following lemma, [26]. It connects Deville's master lemma, see Lemma 1.1 in [1, Chapter VII], with our approach as developed in [21] and it shows that the concept of slicely discrete family in its union (or slicely isolated family) is a central one to deal with symmetrics of the form

$$s(x,y) := 2|||x|||^2 + 2|||y|||^2 - |||x+y|||^2.$$

Indeed, for every slicely isolated family $\mathcal{B}$ it is possible to construct, using Deville's lemma, an equivalent norm $|||\cdot|||$ such that from the fact that $\lim_n s(x_n, x) = 0$ it follows that eventually x_n and x belong to essentially the same set of the family $\mathcal{B}$.

LEMMA 1. (See [26, Connection Lemma]) *Let $(X, \|\cdot\|)$ be a normed space and F be a norming subspace in X^*. Let $\mathcal{B} := \{B_i : i \in I\}$ be an uniformly bounded and slicely discrete in its union family of subsets of X for the $\sigma(X,F)$-topology. Then there is an equivalent and $\sigma(X,F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{B}}$ on X such that:*

for every sequence $\{x_n : n \in \mathbb{N}\}$ and x in X with $x \in B_{i_0}$ for $i_o \in I$ the condition

$$\lim_n \left(2\,\|x_n\|_{\mathcal{B}}^2 + 2\,\|x\|_{\mathcal{B}}^2 - \|x_n + x\|_{\mathcal{B}}^2\right) = 0$$

implies that

(i) *there is n_0 such that*

$$x_n, (x_n + x)/2 \notin \overline{\mathrm{co}\,\{B_i \,:\, i \neq i_0,\, i \in I\}}^{\sigma(X,F)}$$

for all $n \geq n_0$;

(ii) *for every positive δ there is $n_\delta \in \mathbb{N}$ such that*

$$x_n \in \overline{\mathrm{co}(B_{i_0}) + B(0,\delta)}^{\sigma(X,F)}$$

whenever $n \geq n_\delta$.

The former lemma also applies to prove our basic result:

COROLLARY 2. *In a normed space X with a norming subspace F in X^* we have an equivalent $\sigma(X,F)$-lower semicontinuous and locally uniformly rotund norm whenever there are families*

$$\{\mathcal{B}_n : n = 1, 2, \dots\}$$

which are slicely discrete in their unions for the $\sigma(X,F)$ topology and such that, for every x in X, and every $\epsilon > 0$, there is some positive integer n with the property that $x \in B \in \mathcal{B}_n$ and that $\|\cdot\| - \mathrm{diam}(B) < \epsilon$; i.e., whenever the norm topology has a $\sigma(X,F)$-slicely isolated network

$$\mathcal{B} := \cup\{\mathcal{B}_n : n = 1, 2, \dots\}.$$

Proof. It is enough to consider the norms $\|\cdot\|_{\mathcal{B}_n}$ constructed using the connection lemma for every one of the families $\mathcal{B}_n$ and to define the new norm by

$$|||x|||^2 := \sum_{n=1}^{\infty} c_n \|x\|^2_{\mathcal{B}_n},$$

for every $x \in X$, where the sequence (c_n) is choosen accordingly for the convergence of the series. ∎

As an application of the connection lemma we have for instance the following result (see [26]):

THEOREM 5. *Let K be a separable compact space. Let $\{\mathcal{B}_n : n = 1, 2, \dots\}$ be slicely isolated families for the pointwise topology in $C(K)$ such that for every x in $C(K)$, and every $\epsilon > 0$, there is some positive integer n, with the property that*

$$x \in B \in \mathcal{B}_n \qquad \text{and} \qquad \|\cdot\| - \mathrm{dist}(y, K^B)) < \epsilon$$

for all $y \in B$ and some pointwise compact subset K^B in $C(K)$. Then the Banach space $C(K)$ admits an equivalent and $\mathcal{T}_p$-lower semicontinuous locally uniformly rotund norm. When K^B can be taken a norm compact subset in $C(K)$ the separability assumption on the compact space K can be dropped.

The former result fits in the study of compact spaces $K \subset \mathbb{R}^\Gamma$ formed with Baire one functions on a Polish space Γ, the so called Rosenthal compacta. When K is separable and every element in K has at most countably many discontinuities, then we have obtained in [10] that the Banach space $C(K)$ admits an equivalent $\mathcal{T}_p$-lower semicontinuous locally uniformly rotund norm.

QUESTION 8. If K is a separable Rosenthal compact then does it follow that $\mathcal{C}(K)$ admits an equivalent locally uniformly rotund renorming?

A positive answer is conjectured in [10] and it would yield as an immediate corollary that X^* is LUR renormable whenever X is a separable Banach space with no subspace isomorphic to ℓ_1. Indeed, in this case, we may take Γ to be the unit ball of the dual space X^*, which is compact and metrizable (so certainly Polish) under the weak* topology $\sigma(X^*, X)$ and K to be the unit ball of X^{**} under the weak* topology $\sigma(X^{**}, X^*)$. By the results of [23], the elements of K are then of the first Baire class when we look at them as functions on Γ. Moreover, K is separable, since the unit ball of X (which we are assuming to be separable) is dense in K by Goldstine's theorem. Todorcevic has recently observed that there is a scattered Rosenthal compactification H of a tree space such that $C(K)$ has no LUR renorming, [33]. Since that example H is non-separable and there are elements with uncountable many discontinuities we may also ask the following:

QUESTION 9. Let $K \subset \mathbb{R}^\Gamma$ be a compact subset of Baire one functions on a Polish space Γ such that every function of K has at most countably many discontinuities.

Do the Borel sets for the pointwise and the norm topologies coincide in C(K)? Is the identity map $Id : C_p(K) \to (C(K), \|\cdot\|_\infty)$ σ-continuous? Does the Banach space $C(K)$ admit an equivalent LUR norm?

Let us remark here that in this case the space $C_p(K)$ is σ-fragmented by the norm, see [10].

4. Measures of non compactness

For a bounded set B in a metric space X, the Kuratowski index of non-compactness of B is defined by

$$\alpha(B) := \inf\left\{\epsilon > 0 \;:\; B \subseteq \cup_{i=1}^{m} A_i \text{ such that } \operatorname{diam}(A_i) \leq \epsilon \text{ and } m \in \mathbb{N}\right\}.$$

The main results in the recent work [5] provide extensions of Theorem 2 when the Kuratowski index of non-compactness is used instead of the diameter. Indeed, the following result is proved there:

THEOREM 6. (See [5]) *Let X be a normed space and let F be a norming subspace of its dual. Then X admits an equivalent $\sigma(X, F)$-lower semi-continuous LUR norm if, and only if, for every $\epsilon > 0$ we can write*

$$X = \bigcup_{n=1}^{\infty} X_{n,\epsilon}$$

in such a way that for every $x \in X_{n,\epsilon}$, there exists a $\sigma(X,F)$-open half space H containing x with $\alpha\,(H \cap X_{n,\epsilon}) < \epsilon$.

From the topological point of view we arrive to the following corollary turning a discrete condition into a locally finite one.

Corollary 3. (See [5]) *A normed space X admits an equivalent $\sigma(X,F)$-lower semi-continuous LUR norm if, and only if, the norm topology has a network $\mathcal{N} = \cup_{n=1}^{\infty}\mathcal{N}_n$ such that for every $n \in \mathbb{N}$ and for every $x \in \cup\mathcal{N}_n$ there is a $\sigma(X,F)$-open half space H with $x \in H$ such that H meets only a finite number of elements in $\mathcal{N}_n$.*

These results open the door to the study of the situation for different measures of non compactness. Indeed, the index α measures the distance of a set B to a norm compact subset of the normed space X. We can introduce the index β, studied by S. Troyanski in [32], to measure the distance to a uniform Ebelein compacta, i.e., we set for a normed space X and a bounded subset B of it the index:

$$\beta(B) := \inf\{\epsilon > 0 \,:\, B \subseteq K + \epsilon B_X \text{ where } K \text{ is a uniform Eberlein compact}\}.$$

Another one is the distance to a weakly compact subset of the normed space:

$$\chi(B) := \inf\{\epsilon > 0 \,:\, B \subseteq K + \epsilon B_X \text{ where } K \text{ is a weakly compact subset}\},$$

or the separability index measuring the distance to separable subsets:

$$\lambda(B) := \inf\{\epsilon > 0 \,:\, B \subseteq \cup_{i=1}^{\infty} A_i \text{ such that } \operatorname{diam}(A_i) \leq \epsilon\}.$$

We can now formulate the following:

Question 10. Given a normed space X that verifies the conditions of Theorem 6 with the index λ (resp. β, χ) instead of α.

Is it true that the Borel sets for the norm and the $\sigma(X,X^*)$ topology coincide? Is the identity map $Id : (X(\sigma(X,X^*)) \to (X, \|\cdot\|)$ σ-continuous? Doest X admit an equivalent LUR norm? In case of positive answers to any of the above questions what can be said for the network characterizations?

The main result in [32] ensures that for a normed space X with a unit ball B_X such that for every point $x \in S_X$ and every $\epsilon > 0$ there is an open half space H with $x \in H$ and such that $\beta(H \cap B_X) < \epsilon$ it follows that X admits an equivalent locally uniformly rotund norm. The proof involved martingales calculus, nevertheless M. Raja [29] has just provided a geometrical proof of that result completing the answer to the former question for the β index in full generality. Indeed, M. Raja applies the methods of [5] to deal with general measures of non compactness and for the β measure provides the LUR renorming. Thus, it seems to be the right time to study the above question with the index χ at least. With the index λ the question seems more difficult, nevertheless we know examples of compact spaces K such that the space $C(K)$ can be decomposed as in Theorem 6, with the index λ instead of the index α and nothing is known on descriptiveness of the space $C(K)$. For instance, that happens for the not necessarily separable Rosenthal compacta of Question 9 that has been studied in [10]. We certainly know that the space $X[\sigma(X, X^*)]$ is σ-fragmented by its norm, [11], when it satisfies the conditions of Theorem 6 with the index λ instead of α. Thus a negative answer to that question is related with the answer to Question 3 too.

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