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Canonical Symplectic Structures on the r-th Order Tangent Bundle of a Symplectic Manifold

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0. Introduction

Manifolds and maps are assumed to be smooth (i.e., of class C^∞). Moreover, manifolds are assumed to be finite dimensional and without boundaries.

Symplectic structures are involved in the Hamilton equation of motion, [9]. That is why, constructions on symplectic manifolds or into symplectic manifolds have been studied in many papers, e.g. [1]-[3], [5], [8], [11], etc. For example, in [5], J. Janyška described all symplectic structures on the tangent bundle of a Riemannian manifold. In [8], we described all 2-forms (even all symplectic structures) on the tangent bundle of a symplectic manifold.

In the present paper we generalize by induction the result from [8] on the r-th order tangent bundle $T^rM = J^r_0(\mathbf{R}, M)$. We study how to construct canonically a 2-form (even a symplectic structure) $\Lambda(\omega)$ on T^rM for a given symplectic $2m$-manifold (M,ω). This problem arises in the context of respective not necessarily regular $\mathcal{M}f_{2m}$-natural operators Λ in the sense of [6], where $\mathcal{M}f_{2m}$ is the category of $2m$-dimensional manifolds and their local diffeomorphisms. Since homotheties are not symplectomorphisms, it is difficult to apply the homogeneous function theorem and the problem of the operators in question is more difficult than the one for natural operators defined on all 2-forms.

We recall that a not necessarily regular $\mathcal{M}f_{2m}$-natural operator in question is a family of functions

$$\Lambda : SYMP(M) \to \Omega^2(T^rM)$$

from the set $SYMP(M)$ of symplectic forms on M into the set $\Omega^2(T^rM)$ of 2-forms on T^rM for any $2m$-dimensional manifold M satisfying the naturality condition

$$\Lambda(\varphi^*\omega) = (T^r\varphi)^*\Lambda(\omega)$$

for any symplectic form ω on a $2m$-manifold N and any local diffeomorphism $\varphi : M \to N$ between $2m$-manifolds.

We describe completely all natural operators Λ in question.

The (k)-lift of a 2-form ω on M to T^rM is

$$\omega^{(k)} = pr_k \circ T^r\omega \circ (\eta \times_{id_{T^rM}} \eta) : TT^rM \times_{T^rM} TT^rM \to \mathbf{R} ,$$

where $\eta : TT^rM \to T^rTM$ is the usual flow equivalence,

$$T^r\omega : T^r(TM \times_M TM) = T^rTM \times_{T^rM} T^rTM \to T^r\mathbf{R} = J^r_0(\mathbf{R}, \mathbf{R})$$

is the value of the r-tangent bundle functor T^r at $\omega : TM \times_M TM \to \mathbf{R}$, and $pr_k : J^r_0(\mathbf{R}, \mathbf{R}) \to \mathbf{R}$ is the usual projection

$$pr_k(j^r_0\gamma) = \frac{d^k\gamma}{dt^k}(0) .$$

The lift $\omega^{(k)}$ is a 2-form on T^rM and it is proportional to the respective lift (of ω to T^rM) in the sense of A. Morimoto [10].

Given a symplectic form ω on M we have a 1-form $\omega^o = \tilde{\omega}^*\lambda$ on TM, where $\tilde{\omega} : TM \to T^*M$ is the obvious isomorphism induced by the non-degenerate 2-form ω and λ is the Liouville 1-form on T^*M. Then for $k = 0, \dots, r-1$ we have 1-forms $(\omega^o)^{(k)}$ on $T^{r-1}TM$, the k-lifts defined quite similar as the k-lifts of 2-forms. We have the obvious inclusion $j : T^rM \to T^{r-1}TM$. Then for $k = 0, \dots, r-1$ we have 1-forms $\omega^{[k]} = j^*(\omega^o)^{(k)}$ on T^rM.

The main result of the present note is the following classification theorem.

THEOREM 1. *Let Λ be a not necessarily regular $\mathcal{M}f_{2m}$-natural operator in question. Then there exist (uniquely determined by Λ) real numbers α_k and real numbers $c_{p,q}$ such that*

$$\Lambda(\omega) = \sum_{k=0}^{r} \alpha_k \omega^{(k)} + \sum_{0 \le p < q \le r-1} c_{p,q}\, \omega^{[p]} \wedge \omega^{[q]} \tag{1}$$

for any symplectic structure ω on M.

The rather inductive proof of Theorem 1 will occupy Sections 1–4 of the paper.

Using Darboux coordinates, one can verify that the formula (1) defines a non-degenerate 2-form (i.e., a presymplectic form) if and only if $\alpha_r \neq 0$.

The formula (1) gives a closed 2-form if and only if all $c_{p,q} = 0$. Thus we have

COROLLARY 1. *All canonical symplectic structures $\Lambda(\omega)$ on T^rM over a symplectic manifold (M, ω) is of the form*

$$\Lambda(\omega) = \sum_{k=0}^{r} \alpha_k \omega^{(k)}$$

for all real numbers α_k with $\alpha_r \neq 0$.

The usual coordinates on $\mathbf{R}^{2m}$ will be denoted by $x^i, x^{\bar{i}}$, where $i = 1, \dots, m$ and $\bar{i} = m + i$. From now on $\eta^o = j_0^r(t, 0, \dots, 0) \in T^r\mathbf{R}^{2m}$. The standard symplectic form on $\mathbf{R}^{2m}$ will be denoted by ω^o.

1. On density of some orbit

We start with the proof of the following fact.

PROPOSITION 1. *The orbit of η^o with respect to local ω^o-symplectomorphisms preserving 0 is dense in $T_0^r\mathbf{R}^{2m}$.*

Proof. The ω^o-symplectomorphism

$$(x^1, x^2 + x^1, x^3, \dots, x^m, x^{\bar{1}} - x^{\bar{2}}, x^{\bar{2}}, \dots, x^{\bar{m}})$$

sends η^o into $j_0^r(t, t, 0, \dots, 0, 0, \dots, 0)$, i.e., $j_0^r(t, t, 0, \dots, 0, 0, \dots, 0)$ is in the orbit.

Then using the ω^o-symplectomorphism

$$(x^1, x^2, x^3 + x^2, x^3, \dots, x^m, x^{\bar{1}}, x^{\bar{2}} - x^{\bar{3}}, x^{\bar{3}}, \dots, x^{\bar{m}})$$

we deduce that $j_0^r(t, t, t, 0, \dots, 0, 0, \dots, 0)$ is in the orbit.

Repeating this process $(m-1)$-times we deduce that $j_0^r(t, \dots, t, 0, \dots, 0)$ (m-times of t) is in the orbit.

Then using ω^o-symplectomorphisms $(x^i, x^{\bar{i}} + \gamma^{\bar{i}}(x^i))$ we deduce that

$$j_0^r\big(t, \dots, t, \gamma^{\bar{1}}(t), \dots, \gamma^{\bar{m}}(t)\big)$$

is in the orbit for any $\gamma^{\bar{i}} : \mathbf{R} \to \mathbf{R}$ with $(\gamma^{\bar{i}})'(0) \neq 0$ and $\gamma^{\bar{i}}(0) = 0$.

Then using local ω^o-symplectomorphisms

$$\big(x^i - (\gamma^{\bar{i}})^{-1}(x^{\bar{i}}) + \gamma^i((\gamma^{\bar{i}})^{-1}(x^{\bar{i}})), x^{\bar{i}}\,\big)$$

we deduce that

$$j^r_0\big(\gamma^1(t), \dots, \gamma^m(t), \gamma^{\bar{1}}(t), \dots, \gamma^{\bar{m}}(t)\big)$$

is in the orbit for any $\gamma^{\bar{i}} : \mathbf{R} \to \mathbf{R}$ with $(\gamma^{\bar{i}})'(0) \neq 0$ and $\gamma^{\bar{i}}(0) = 0$ and for any $\gamma^i : \mathbf{R} \to \mathbf{R}$ with $\gamma^i(0) = 0$. ∎

2. A reducibility lemma

We will use the following reducibility lemma.

Lemma 1. *Let $\Lambda : SYMP(M) \to \Omega^2(T^r M)$ be a not necessarily regular $\mathcal{M}f_{2m}$-natural operator transforming symplectic structures on $2m$-manifolds M into 2-forms on $T^r M$. If $\Lambda(\omega^o)_{\eta^o} = 0$ then $\Lambda = 0$.*

Proof. The lemma is a consequence of the naturality of Λ, the Darboux theorem and Proposition 1. ∎

The reducibility lemma shows that Λ is uniquely determined by $D(\eta^o) = \Lambda(\omega^o)_{\eta^o}$. So it is sufficient to study 2-forms

$$D : \mathbf{R}\eta^o \to \bigwedge^2 T^* T^r \mathbf{R}^{2m}$$

defined on $\mathbf{R}\eta^o = \{\tau\eta^o \in T^r_0\mathbf{R}^{2m} \mid \tau \in \mathbf{R}\} \subset T^r_0\mathbf{R}^{2m}$ such that D is invariant with respect to 0-preserving ω^o-symplectomorphisms ψ with $T^r\psi(\mathbf{R}\eta^o) \subset \mathbf{R}\eta^o$ and $D(j^r_0(0))$ is invariant with respect to all 0-preserving ω^o-symplectomorphisms.

From now on we consider such D.

3. An expression of D

For $k = 0, \dots, r$ and a map $f : M \to \mathbf{R}$ on a manifold M we have the (k)-lift

$$f^{(k)} = pr_k \circ T^r f : T^r M \to \mathbf{R}$$

of f to $T^r M$, see [4]. The system

$$((x^i)^{(k)}, (x^{\bar{i}})^{(k)})$$

for $i = 1, \dots, m$ and $k = 0, \dots, r$ is the so called induced coordinate system on $T\mathbf{R}^{2m}$, see [4].

LEMMA 2. *There exist real numbers $a_{k,l}$ and $b_{p,q,i}$ such that*

$$
\begin{aligned}
D(\tau\eta^o) = &\sum_{0\le k<l\le r} a_{k,l}\tau^2 d_{\tau\eta^o}(x^{\bar 1})^{(k)} \wedge d_{\tau\eta^o}(x^{\bar 1})^{(l)} \\
&+ \sum_{i=1}^{m}\sum_{p,q=0}^{r} b_{p,q,i} d_{\tau\eta^o}(x^{i})^{(p)} \wedge d_{\tau\eta^o}(x^{\bar i})^{(q)}
\end{aligned}
\tag{2}
$$

for $\tau \in \mathbf{R}$. Moreover, $a_{k,r} = 0$ for $k = 0, \dots, r-1$.

Proof. We express $D(\tau\eta^o)$ as the combination with coefficients being smooth functions in τ of the restrictions of 2-forms from the obvious basis of 2-forms on $T^r\mathbf{R}^{2m}$ corresponding to the coordinates $((x^i)^{(k)}, (x^{\bar i})^{(k)})$. Then the invariance of D with respect to the ω^o-symplectomorphisms $(\tau^i x^i, \frac{1}{\tau^i}x^{\bar i})$ for $\tau^i \neq 0$ gives respective homogeneous conditions on the coefficients of D. Then using the homogeneous function theorem we get the expression (2) of $D(\tau\eta^o)$.

The ω^o-symplectomorphism

$$\psi = (x^1, \dots, x^m, x^{\bar 1} - (x^1)^{r+1}, x^{\bar 2}, \dots, x^{\bar m})$$

preserves η^o, $d_{\eta^o}(x^{\bar 1})^{(k)}$ for $k = 0, \dots, r-1$, $d_{\eta^o}(x^i)^{(k)}$ for $k = 0, \dots, r$ and $i = 1, \dots, m$ and $d_{\eta^o}(x^{\bar j})^{(k)}$ for $j = 2, \dots, m$ and $k = 0, \dots, r$, and it sends $d_{\eta^o}(x^{\bar 1})^{(r)}$ into $d_{\eta^o}(x^{\bar 1})^{(r)} + d_{\eta^o}(x^1)^{(0)}$. Then using the invariance of $D(\eta^o)$ with respect to ψ we get $a_{k,r} = 0$ for $k = 0, \dots, r-1$.

The above facts complete the proof. ∎

4. PROOF OF THEOREM 1

Applying Lemma 2 and replacing D by

$$D - \sum_{0\le p<q\le r-1} c_{p,q}(\omega^{[p]} \wedge \omega^{[q]})_{|\mathbf{R}\eta^o}$$

for respective real numbers $c_{p,q}$, we can assume

$$D(\tau\eta^o) = \sum_{i=1}^{m}\sum_{p,q=0}^{r} e_{p,q,i} d_{\tau\eta^o}(x^{i})^{(p)} \wedge d_{\tau\eta^o}(x^{\bar i})^{(q)} \tag{3}$$

for some real numbers $e_{p,q,i}$.

(∗) Then

$$D\big(j^r_0(0)\big) = \sum_{i=1}^{m}\sum_{p,q=0}^{r} e_{p,q,i} d_{j^r_0(0)}(x^i)^{(p)} \wedge d_{j^r_0(0)}(x^{\bar{i}})^{(q)} \tag{4}$$

for some real numbers $e_{p,q,i}$. Moreover $D\big(j^r_0(0)\big)$ is invariant with respect 0-preserving ω^o-symplectomorphisms.

By the induction on r we prove that

$$D\big(j^r_0(0)\big) = \sum_{k=0}^{r} \alpha_k \omega^{(k)}\big(j^r_0(0)\big) \tag{5}$$

for some reals α_k.

We start with preparations.

By the invariance of $D\big(j^r_0(0)\big)$ with respect to the ω^o-symplectomorphism $(-x^{\bar{i}}, x^i)$ we deduce that $e_{p,q,i} = e_{q,p,i}$.

Then using the invariance of $D\big(j^r_0(0)\big)$ with respect to ω^o-symplectomorphism permuting the coordinates x^i and respectively permuting the coordinates $x^{\bar{i}}$ we get that $e_{p,q,i} = e_{p,q}$ for $i = 1, \dots, m$.

The ω^o-symplectomorphism

$$\psi = (x^1, \dots, x^m, x^{\bar{1}} - (x^1)^{r+1}, x^{\bar{2}}, \dots, x^{\overline{m}})$$

preserves $d_{j^r_0(0)}(x^i)^{(k)}$ for $k = 0, \dots, r$ and $i = 1, \dots, m$ and it preserves $d_{j^r_0(0)}(x^{\bar{j}})^{(k)}$ for $j = 2, \dots, m$ and $k = 0, \dots, r$, and it sends $d_{j^r_0(0)}(x^{\bar{1}})^{(r)}$ into $d_{j^r_0(0)}(x^{\bar{1}})^{(r)} + d_{j^r_0(0)}(x^1)^{(0)}$. Then using the invariance of $D\big(j^r_0(0)\big)$ with respect to ψ we get $e_{p,r,1} = 0$ for $p = 1, \dots, r$.

Then replacing $D\big(j^r_0(0)\big)$ by $D\big(j^r_0(0)\big) - \alpha_r \omega^{(r)}\big(j^r_0(0)\big)$ for some real number α_r, we have

$$D\big(j^r_0(0)\big) = \sum_{i=1}^{m}\sum_{p,q=0}^{r-1} e_{p,q} d_{j^r_0(0)}(x^i)^{(p)} \wedge d_{j^r_0(0)}(x^{\bar{i}})^{(q)} \ . \tag{6}$$

Then the first inductive step (the case $r = 1$) of the proof of formula (5) is clear.

Assume that (5) holds for $r - 1$ instead of r.

By (6),

$$D\big(j^r_0(0)\big)(v_1, v_2) = D\big(j^r_0(0)\big)(\tilde{v}_1, \tilde{v}_2)$$

for any $v_1, v_2, \tilde{v}_1, \tilde{v}_2 \in T_{j_0^r(0)} T^r \mathbf{R}^{2m}$ satisfying $T\pi^r_{r-1}(v_1) = T\pi^r_{r-1}(\tilde{v}_1)$ and $T\pi^r_{r-1}(v_2) = T\pi^r_{r-1}(\tilde{v}_2)$.

The last fact yields that there is a unique 2-form $\tilde{D}\big(j_0^{r-1}(0)\big)$ satisfying the same property $(*)$ as $D\big(j_0^r(0)\big)$ with $r-1$ playing the role of r such that

$$D\big(j_0^r(0)\big)(v,w) = \tilde{D}\big(j_0^{r-1}(0)\big)\big(T\pi^r_{r-1}(v), T\pi^r_{r-1}(w)\big) \tag{7}$$

for any $v, w \in T_{j_0^r(0)} T^r \mathbf{R}^{2m}$.

So, $D\big(j_0^r(0)\big) = \sum_{k=0}^{r-1} \alpha_k \omega^{(k)}_{j_0^r(0)}$ by the inductive assumption.

The proof of formula (5) is complete.

Summing up, we have proved that

$$D = \sum_{k=0}^{r} \alpha_k \omega^{(k)}_{|\mathbf{R}\eta^o} + \sum_{0 \le p < q \le r-1} c_{p,q} (\omega^{[p]} \wedge \omega^{[q]})_{|\mathbf{R}\eta^o}$$

for some reals α_k and $c_{p,q}$.

Now, Theorem 1 is an immediate consequence of Lemma 1.

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