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Multifractals and Projections

Fadhila Bahroun, Imen Bhouri

Faculté des Sciences de Monastir, 5000 *Monastir, Tunisie*
e-mail: fadhila_bah@yahoo.fr, bhouri_imen@yahoo.fr

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1. Introduction and preliminaries

Let m be an integer with $0 < m < n$ and $G_{n,m}$ stand for the Grassmann manifold of all m- dimensional linear spaces of $\mathbb{R}^n$. For $V \in G_{n,m}$, we denote $p_V : \mathbb{R}^n \longrightarrow V$ the orthogonal projection onto V, then $\{p_V ; V \in G_{n,m}\}$ is compact in the space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, and the identification of V with p_V induces a compact topology for $G_{n,m}$. Fixing $V_0 \in G_{n,m}$, we can define an orthogonally invariant Radon probability measure $\gamma_{n,m}$ on $G_{n,m}$ by

$$\gamma_{n,m}(A) = v_n \{g \in O(n) \ \ ; \ \ g(V_0) \in A\} \quad \text{for} \quad A \in G_{n,m}$$

where v_n denotes the unique Haar measure on the orthogonal group $O(n)$ of $\mathbb{R}^n$ normalized so that $v_n (O(n)) = 1$. The uniqueness implies that $\gamma_{n,m}$ is independent of V_0 (see [6]). In other words,

$$\gamma_{n,m} = f_{V_0*}v_n \quad \text{with} \quad f_{V_0}(g) = g(V_0) \quad \text{for} \quad g \in O(n)$$

and $f_{V_0*}v_n$ is the image of the measure v_n under the map f_{V_0}.

For a Borel probability measure μ on $\mathbb{R}^n$, supported on the compact set S_μ, and for $V \in G_{n,m}$, we define $\widetilde{\mu}$, the projection of μ onto V by

$$\widetilde{\mu}(E) = \mu\big(p_V^{-1}(E)\big)$$

for all $E \subseteq V$. Thus if $f : V \longrightarrow \mathbb{R}$ is continuous then

$$\int_V f(u)\mathrm{d}\widetilde{\mu}(u) = \int_{\mathbb{R}^n} f\big(p_V(x)\big)\mathrm{d}\mu(x).$$

If ν is a Borel probability measure on S_μ, one defines, for $p > 0$

$$T_{\mu,\nu}(p) = \liminf_{r \longrightarrow 0} \frac{1}{\log r} \log \int_{S_\mu} \mu\big(B(x,r)\big)^p \mathrm{d}\nu(x)$$

the lower generalized p-spectral dimension of μ. It is closely related to the Rényi dimension in its integral version (see [9]) and if ν is a Gibbs measure for the measure μ, i. e there exists a measure ν on S_μ and constants $\underline{K} > 0, \overline{K} > 0$ and $t_q \in \mathbb{R}$ such that for every $x \in S_\mu$ and every $0 < r < \lambda$

$$\underline{K}\mu\big(B(x,r)\big)^q (2r)^{t_q} \leq \nu\big(B(x,r)\big) \leq \overline{K}\mu\big(B(x,r)\big)^q (2r)^{t_q}.$$

$T_{\mu,\nu}$ represents the C_μ function of Olsen's multifractal formalism [8]. This quantity appears as a generalization of the lower p-spectral dimension defined in [5]. For $p \geq 0$,

$$D_p(\mu) = \liminf_{r \to 0} \frac{1}{p \log r} \log \int_{S_\mu} \mu\big(B(x,r)\big)^p \mathrm{d}\mu(x).$$

In particular, in the case $\mu = \nu$ one has

$$T_{\mu,\mu}(p) = p D_p(\mu)$$

In [5], Hunt and Kaloshin proved the following statement:

Theorem (HK). *Let $0 < p \leq 1$. If $D_p(\mu) \leq m$, one has*

$$D_p(\widetilde{\mu}) = D_p(\mu) \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G_{n,m}.$$

In this paper, we investigate the relationship between $T_{\mu,\nu}(p)$ and $T_{\widetilde{\mu},\widetilde{\nu}}(p)$. Which allows us to compare the multifractal spectra of the measure μ and that of its projections.

2. Projection results

In this section, we show that the generalized p-spectral dimension is preserved under almost every orthogonal projection.

Theorem 2.1. *Let p be a real number.*

1. *If $0 < p \leq 1$ and $T_{\mu,\nu}(p) \leq pm$, then*

$$T_{\widetilde{\mu},\widetilde{\nu}}(p) = T_{\mu,\nu}(p) \quad \text{for } \gamma_{n,m}\text{-almost all } V \in G_{n,m}.$$

2. *If* $p > 1$, *one has*

$$\inf\big(T_{\mu,\nu}(p), m\big) \le T_{\widetilde{\mu},\widetilde{\nu}}(p) \le T_{\mu,\nu}(p) \quad \textit{for } \gamma_{n,m}\textit{-almost all } V \in G_{n,m}.$$

Remark. The first assertion of this theorem is a generalization of that of Hunt and Kaloshin. In fact, in the case $\mu = \nu$, assertion 1 is the main theorem of Hunt and Kaloshin. Assertion 2 extends the result of Hunt and Kaloshin to the case $p > 1$ untreated in their work. Indeed, considering the relation $D_p(\mu) = \frac{1}{p}T_{\mu,\mu}(p)$, the equality $D_p(\mu) = D_p(\widetilde{\mu})$ remain valid for $\gamma_{n,m}$-almost all $V \in G_{n,m}$, if $p > 1$ and $D_p(\mu) < \frac{m}{p}$.

Proof of Theorem 2.1. The first assertion is proved in the same way as Theorem (HK). The second property results from the following lemma.

LEMMA 2.2. *If* $p > 0$, *then*

1. $T_{\mu,\nu}(p) = \inf\{s \ge 0 : I_{s,p}(\mu,\nu) = \infty\}$,
2. $T_{\mu,\nu}(p) = \sup\{s \ge 0 : I_{s,p}(\mu,\nu) < \infty\}$,

where

$$I_{s,p}(\mu,\nu) = \int_{S_\mu}\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}\mu(y)}{|x-y|^{s/p}}\right)^p \mathrm{d}\nu(x).$$

Proof. Let s be a number such that $s < T_{\mu,\nu}(p)$. One has

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{\mathrm{d}\mu(y)}{|x-y|^{s/p}} &\le 1 + \sum_{n\ge 0}\int_{2^{-n-1}<|x-y|\le 2^{-n}}\frac{\mathrm{d}\mu(y)}{|x-y|^{s/p}}\\ &\le 1 + \sum_{n\ge 0} 2^{(n+1)s/p}\mu\big(B(x,2^{-n})\big)\end{aligned}$$

thus

$$I_{s,p}(\mu,\nu) \le \int_{S_\mu}\left(1 + \sum_{n\ge 0} 2^{(n+1)s/p}\mu\big(B(x,2^{-n})\big)\right)^p \mathrm{d}\nu(x).$$

In the case $0 < p \le 1$, we have

$$I_{s,p}(\mu,\nu) \le 1 + \sum_{n\ge 0} 2^{(n+1)s}\int_{S_\mu}\mu\big(B(x,2^{-n})\big)^p \mathrm{d}\nu(x).$$

Then for a suitable choice of $\delta > 0$ such that $s + \delta < T_{\mu,\nu}(p)$, one has

$$I_{s,p}(\mu,\nu) \leq 1 + 2^s \sum_{n \geq 0} 2^{-n\delta} < \infty.$$

Now consider the case $p > 1$, remember that for every $a, b \geq 0$,

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

It results that

$$\left(\int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{|x-y|^{s/p}}\right)^p \leq 2^{p-1} + 2^{p-1} \left(\sum_{n \geq 0} 2^{(n+1)s/p} \mu\big(B(x, 2^{-n})\big)\right)^p$$

For $\alpha > 0$, Hölder inequality implies

$$\left(\int_{\mathbb{R}^n} \frac{\mathrm{d}\mu(y)}{|x-y|^{s/p}}\right)^p \\ \leq 2^{p-1} + 2^{p-1} \left(\sum_{n \geq 0} 2^{(n+1)s+n\alpha p} \mu\big(B(x, 2^{-n})\big)^p\right) \left(\sum_{n \geq 0} 2^{\frac{-n\alpha p}{p-1}}\right)^{p-1}$$

Further, if $\delta > 0$ such that $s + \delta < T_{\mu,\nu}(p)$ we obtain

$$I_{s,p}(\mu,\nu) \leq 2^{p-1} + C \sum_{n \geq 0} 2^{n(\alpha p - \delta)}$$

where C is constant depending only of p. Statement 1 follows by considering $\alpha < \delta/p$. To establish statement 2, fix $p > 0$. For $\varepsilon > 0$, one has

$$I_{s,p}(\mu,\nu) \geq \varepsilon^{-s} \int_{S_\mu} (\mu\big(B(x,\varepsilon)\big)^p \mathrm{d}\nu(x)$$

Let $s > T_{\mu,\nu}(p)$ and $\delta > 0$ such that $s - \delta > T_{\mu,\nu}(p)$, then for small ε we have $I_{s,p}(\mu,\nu) \geq \varepsilon^{-\delta}$, hence $I_{s,p}(\mu,\nu)$ is infinite. ∎

In order to prove the second statement of Theorem 2.1, it is sufficient to prove this implication

$$T_{\mu,\nu}(p) \leq m \implies T_{\mu,\nu}(p) \leq T_{\widetilde{\mu},\widetilde{\nu}}(p). \tag{1}$$

Let $s < T_{\mu,\nu}(p)$. One has

$$\int_{G_{n,m}} I_{s,p}(\widetilde{\mu},\widetilde{\nu})\mathrm{d}\gamma_{n,m}(v) = \int_{S_\mu}\int_{G_{n,m}}\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}\mu(y)}{|p_v(x-y)|^{s/p}}\right)^p \mathrm{d}\gamma_{n,m}(v)\mathrm{d}\nu(x).$$

By Minkowski inequality [10],

$$\int_{G_{n,m}}\left(\int_{\mathbb{R}^n}\frac{\mathrm{d}\mu(y)}{|p_v(x-y)|^{s/p}}\right)^p \mathrm{d}\gamma_{n,m}(v) \leq \left(\int_{\mathbb{R}^n}\left(\int_{G_{n,m}}\frac{\mathrm{d}\gamma_{n,m}(v)}{|p_v(x-y)|^{s}}\right)^{1/p}\mathrm{d}\mu(y)\right)^p.$$

Since $s < m$,

$$\int_{G_{n,m}}\frac{\mathrm{d}\gamma_{n,m}(v)}{|p_v(x-y)|^{s}} = \frac{C}{|x-y|^s},$$

(See [7], Corollary 3.12) where C is a constant depending only on m, n and s. Hence

$$\int_{G_{n,m}} I_{s,p}(\widetilde{\mu},\widetilde{\nu})\mathrm{d}\gamma_{n,m}(v) \leq C I_{s,p}(\mu,\nu),$$

which shows that $I_{s,p}(\widetilde{\mu},\widetilde{\nu})$ is finite for $\gamma_{n,m}$-almost all $v \in G_{n,m}$ and implication (1) follows from Lemma 2.2.

3. Application

In the following section, we compare the multifractal spectrum of a measure μ and its projections $\widetilde{\mu}$ more precisely. In fact, Theorem 2.1 allows us to establish a relationship between the Hausdorff dimension of the singularity spectrum of $\widetilde{\mu}$ and the Legendre transform of the generalized prepacking dimensions Λ_μ introduced by Olsen [9]. Before detailing our results, let us recall the multifractal formalism introduced by Olsen.

For $E \subset \mathbb{R}^n$, $q,t \in \mathbb{R}$ and $\delta \geq 0$, we denote

$$\overline{P}_\mu^{q,t}(E) = \lim_{\delta\to 0}\sup\sum_i \mu\big(B(x_i,r_i)\big)^q(2r_i)^t$$

where the supremum is taken over all centered δ-packing of E

$$P_\mu^{q,t}(E) = \inf_{E\subseteq\bigcup_i E_i}\sum_i \overline{P}_\mu^{q,t}(E_i)$$

It is the multifractal generalization of packing measure. The premeasures $\overline{P}_\mu^{q,t}$ assign in the usual way a dimension to each subset E of $\mathbb{R}^n$

$$\overline{P}_\mu^{q,t}(E) = \begin{cases} \infty & \text{for} \quad t < \Delta_\mu^q(E), \\ 0 & \text{for} \quad \Delta_\mu^q(E) < t. \end{cases}$$

The number $\Delta_\mu^q(E)$ is an extension of the prepacking dimension $\Delta(E)$ of E. Then we are able to define the function $\Lambda_\mu(q) = \Delta_\mu^q(S_\mu)$. Remark that Λ_μ is convex and decreasing (see [8]). This function is related to the multifractal spectrum of the measure μ. More precisely, if $f^*(x) = \inf_y \big(xy + f(y)\big)$ denotes the Legendre transform of the function f, and if

$$\overline{X}_\mu(\alpha) = \left\{ x \in S_\mu : \limsup_{r \to 0} \frac{\log \mu\big(B(x,r)\big)}{\log\ r} \le \alpha \right\}.$$

is the set of singularity, it has been proved in [8] and [1] a lower bound estimate of the singularity spectrum using the Legendre transform of the function Λ_μ. Let recall that the singularity spectrum or the multifractal spectrum of a measure μ is the Hausdorff dimension of the set $\overline{X}_\mu(\alpha)$. Theorem 2.1 allows us to prove a similar inequality for $\tilde{\mu}$ and to compare the multifractal spectrum of μ and that of its projection.

Let q be a real number. We consider the following conditions:

(H_1) There exists a probability measure on S_μ such that

$$\nu\big(B(x,r)\big) \le \overline{K}\, \mu\big(B(x,r)\big)^q (2r)^{\Lambda_\mu(q)},$$

(H_2) $\Lambda'_\mu(q)$ exists,

(H_3) $T'_{\mu,\nu}(0)$ exists,

(H_4) $\Lambda^*_\mu\big(-\Lambda'_\mu(q)\big) \le m$ and $T'_{\mu,\nu}(0) \le m$,

(H_5) There exists a probability measure ν in S_μ such that there exist $\overline{K} > 0$ and $\underline{K} > 0$ satisfying

$$\underline{K}\mu\big(B(x,r)\big)^q (2r)^{\Lambda_\mu(q)} \le \nu\big(B(x,r)\big) \le \overline{K}\mu\big(B(x,r)\big)^q (2r)^{\Lambda_\mu(q)}$$

for all $x \in S_\mu$ and $r > 0$ small enough. ν is a Gibbs state at the point q.

Theorem 3.1. *Under the assumptions* (H_1), (H_2), (H_3) *and* (H_4) *one has*

$$\dim \overline{X}_{\tilde{\mu}}\big(T'_{\tilde{\mu},\tilde{\nu}}(0)\big) \ge \Lambda^*_\mu\big(-\Lambda'_\mu(q)\big) \quad \textit{for } \gamma_{n,m}\textit{-almost all } V \in G_{n,m},$$

where dim *denotes the Hausdorff dimension.*

Before giving the proof of this theorem let us comment it.

Commentaries: 1) Under the hypothesis of Theorem 3.1 we have a relationship between the multifractal spectrum of the measure μ and that of its projection $\widetilde{\mu}$. In fact, we have that $\dim \overline{X}_{\widetilde{\mu}}\big(T'_{\widetilde{\mu},\widetilde{\nu}}(0)\big) \geq \dim \overline{X}_{\mu}\big(T'_{\mu,\nu}(0)\big)$ for $\gamma_{n,m}$-almost all $V \in G_{n,m}$, which constitute a natural prolongement of the results of Mattila, Howroyd and Falconer([2], [3] and [6]) about the relationship between the dimension of a set or a measure and those of their projections.
2) Under the hypothesis (H_5), we have $T_{\mu,\nu} = -C_\mu$ where C_μ is another scaling μ-function introduced by L. Olsen in [8]. So, if we replace (H_2) by (H_5) in Theorem 3.1, we have

$$\dim \overline{X}_{\widetilde{\mu}}\big(-C'_{\widetilde{\mu}}(0)\big) \geq \begin{cases} -\Lambda'_{\mu+}(q)q + \Lambda_\mu(q) & \text{for} \quad q \geq 0, \\ -\Lambda'_{\mu-}(q)q + \Lambda_\mu(q) & \text{for} \quad q < 0. \end{cases}$$

Let us prove the Theorem 3.1. One of the ingredients to prove this theorem is the following proposition.

PROPOSITION 3.2. *Let q be a real number. One has*

$$\Lambda'_{\mu-}(q) \leq -T'_{\mu,\nu-}(0) \leq \Lambda'_{\mu+}(q).$$

Remark. If we replace the condition (H_1) by (H_5) we have also

$$\Lambda'_{\mu+}(q) \leq -T'_{\mu,\nu+}(0).$$

Proof of Proposition 3.2. It results from the lower bound

$$\limsup_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log\ r} \leq T'_{\mu,\nu-}(0) \ \nu\text{-a.e. established in [8],}$$

and the lower bound

$$\liminf_{r\to 0} \frac{\log \mu\big(B(x,r)\big)}{\log\ r} \geq -\Lambda'_{\mu+}(q) \ \nu\text{-a.e. established in [1],}$$

that $-T'_{\mu,\nu-}(0) \leq \Lambda'_{\mu+}(q)$. To prove the other inequality, that is

$$\Lambda'_{\mu-}(q) \leq -T'_{\mu,\nu-}(0),$$

it is sufficient to show that

$$\Lambda_\mu(p+q) \geq \Lambda_\mu(q) - T_{\mu,\nu}(p) \tag{2}$$

for all $p < 0$.

Let $\rho > 0$ and $\varepsilon > 0$. It results from the definitions of $T_{\mu,\nu}$ and $\overline{P}_{\mu}^{p+q,\Lambda_\mu(q)-T_{\mu,\nu}(p)-\rho}$ that for $r > 0$ small enough,

$$1 \leq \int_{S_\mu} \mu\big(B(x,r)\big)^p r^{-T_{\mu,\nu}(p)-\rho} \mathrm{d}\nu(x),$$

and

$$\overline{P}_{\mu,\frac{r}{2}}^{p+q,\Lambda_\mu(q)-T_{\mu,\nu}(p)-\rho}(S_\mu) < \overline{P}_{\mu}^{p+q,\Lambda_\mu(q)-T_{\mu,\nu}(p)-\rho}(S_\mu) + \varepsilon.$$

By applying covering Besicovich lemma [4], we have

$$(\overline{K}\xi)^{-1} \leq \overline{P}_{\mu}^{p+q,\Lambda_\mu(q)-T_{\mu,\nu}(p)-\rho}(S_\mu) + \varepsilon.$$

So

$$\overline{P}_{\mu}^{p+q,\Lambda_\mu(q)-T_{\mu,\nu}(p)-\rho}(S_\mu) > 0,$$

in other words,

$$\Lambda_\mu(p+q) \geq \Lambda_\mu(q) - T_{\mu,\nu}(p) - \rho.$$

The arbitrary in ρ implies the inequality (2), which achieves the proof of the proposition. ∎

Proof of Theorem 3.1. We have, $\dim \overline{X}_{\widetilde{\mu}}(\alpha) \geq \inf\big(m, \dim \overline{X}_\mu(\alpha)\big)$ for $\gamma_{n,m}$-almost every $V \in G_{n,m}$, since $p_V\big(\overline{X}_\mu(\alpha)\big) \subset \overline{X}_{\widetilde{\mu}}(\alpha)$. In particular, condition (H_4) implies

$$\dim \overline{X}_{\widetilde{\mu}}\big(T'_{\mu,\nu}(0)\big) \geq \dim \overline{X}_\mu\big(T'_{\mu,\nu}(0)\big) \quad \text{for } \gamma_{n,m} \text{-almost all } V \in G_{n,m} \tag{3}$$

Hence, the proposition and the assumptions (H_2) and (H_3) give that $\Lambda'_\mu(q) = -T'_{\mu,\nu}(0)$. As a consequence, it follows from Theorem 2.1 that

$$\Lambda'_\mu(q) = -T'_{\widetilde{\mu},\widetilde{\nu}}(0) \text{ for } \gamma_{n,m} \text{-almost every } V \in G_{n,m}. \tag{4}$$

Thus, the result is a consequence from (3), (4) and the inequality

$$\dim \overline{X}_\mu\big(-\Lambda'_\mu(q)\big) \geq \Lambda^*_\mu\big(-\Lambda'_\mu(q)\big),$$

established in [1]. ∎

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