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a-Weyl's Theorem and the Single Valued Extension Property

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1. Introduction

Throughout this paper, $\mathcal{L}(X)$ denotes the algebra of all linear bounded operators on an infinite-dimensional complex Banach space X and $\mathcal{K}(X)$ its ideal of compact operators. For an operator $T \in \mathcal{L}(X)$, write T for its adjoint; $\mathrm{N}(T)$ for its kernel; $\mathrm{R}(T)$ for its range; $\sigma(T)$ for its spectrum; $\sigma_{\mathrm{ap}}(T)$ for its approximate point spectrum; $\sigma_{\mathrm{su}}(T)$ for its surjective spectrum and $\sigma_{\mathrm{p}}(T)$ for its point spectrum.

For an operator $T \in \mathcal{L}(X)$, the *ascent* $\mathrm{a}(T)$ and the *descent* $\mathrm{d}(T)$ are given by $\mathrm{a}(T) = \inf\{n \geq 0 : \mathrm{N}(T^n) = N(T^{n+1})\}$ and $\mathrm{d}(T) = \inf\{n \geq 0 : \mathrm{R}(T^n) = \mathrm{R}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be infinite. If the ascent and the descent of $T \in \mathcal{L}(X)$ are both finite, then $\mathrm{a}(T) = \mathrm{d}(T) = p$, $X = \mathrm{N}(T^p) \oplus \mathrm{R}(T^p)$ and $\mathrm{R}(T^p)$ is closed, [24].

Also, an operator $T \in \mathcal{L}(X)$ is called *semi-Fredholm* if $\mathrm{R}(T)$ is closed and either $\dim \mathrm{N}(T)$ or $\mathrm{codim}\mathrm{R}(T)$ is finite. For such an operator the *index* is defined by $\mathrm{ind}(T) = \dim \mathrm{N}(T) - \mathrm{codim}\mathrm{R}(T)$, and if the index is finite, T is said to be *Fredholm*. Let $T \in \mathcal{L}(X)$, the *essential spectrum* $\sigma_{\mathrm{e}}(T)$, the *semi-Fredholm spectrum* $\sigma_{\mathrm{SF}}(T)$, the *Weyl spectrum* $\sigma_{\mathrm{w}}(T)$, the *Browder spectrum* $\sigma_{\mathrm{b}}(T)$, the *essential approximate point spectrum* $\sigma_{\mathrm{ea}}(T)$ and the *Browder essential approximate point spectrum* $\sigma_{\mathrm{ab}}(T)$ are given respectively by

$$\begin{aligned}
\sigma_{\mathrm{e}}(T) &= \{\lambda \in \mathbb{C}\colon T-\lambda \text{ is not Fredholm}\},\\
\sigma_{\mathrm{SF}}(T) &= \{\lambda \in \mathbb{C}\colon T-\lambda \text{ is not semi-Fredholm}\},\\
\sigma_{\mathrm{w}}(T) &= \{\lambda \in \mathbb{C}\colon T-\lambda \text{ is not Fredholm of index } 0\},\\
\sigma_{\mathrm{b}}(T) &= \{\lambda \in \mathbb{C}\colon T-\lambda \text{ is not Fredholm of finite ascent and descent}\},\\
\sigma_{\mathrm{ea}}(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not semi-Fredholm of non-positive index}\},\\
\sigma_{\mathrm{ab}}(T) &= \{\lambda \in \mathbb{C} : T-\lambda \text{ is not semi-Fredholm of finite ascent}\}.
\end{aligned}$$

It is well known that

$$\sigma_{\mathrm{ea}}(T) \subseteq \sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$$

and

$$\sigma_{\mathrm{ea}}(T) \subseteq \sigma_{\mathrm{ab}}(T) \subseteq \sigma_{\mathrm{b}}(T).$$

For a subset K of $\mathbb{C}$, we shall write iso$K$ for its isolated points and acc$K$ for its accumulation points. A complex number λ is said to be *Riesz point* of $T \in \mathcal{L}(X)$ if $\lambda \in \mathrm{iso}\sigma(T)$ and the spectral projection corresponding to the set $\{\lambda\}$ has finite-dimensional range. The set of all Riesz points of T is denoted by $\Pi_{\mathrm{o}}(T)$.

The set of isolated points λ in the spectrum (resp. approximate point spectrum) for which $\ker(T-\lambda)$ is non-zero and finite-dimensional is denoted by $\Pi_{\mathrm{oo}}(T)$ (resp. $\Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$).

Definition. Let T be a bounded operator on X, we will say that

(i) Weyl's theorem holds for T if $\sigma_{\mathrm{w}}(T) = \sigma(T) \setminus \Pi_{\mathrm{oo}}(T)$.

(ii) a-Weyl's theorem holds for T if $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$.

(iii) Browder's theorem holds for T if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$.

(iv) a-Browder's theorem holds for T if $\sigma_{\mathrm{ea}}(T)\sigma_{\mathrm{ab}}(T)$.

The investigation of operators obeying Weyl's theorem was initiated by Hermann Weyl, who proved that for every hermitian operator on a complex Hilbert space H we have $\sigma_{\mathrm{w}}(T) = \sigma(T)\setminus\Pi_{\mathrm{o}}(T)$, [25]. This remarkable description of the largest subset of the spectrum remaining invariant under arbitrary compact perturbation, [23], was extended to several classes of operators including p-hyponormal [3], M-hyponormal and log-hyponormal operators, see [6] and [17]. Analogously, to conduct a similar study where the spectrum is replaced by the approximate point spectrum, the concept of a-Weyl's and a-Browder's theorem were introduced by V. Rakočević in [19]. Now it is well known that the following implications hold ([1], [19]):

$$\text{a-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem};$$

$$\text{a-Weyl's theorem} \Rightarrow \text{a-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

Also, it was shown by Y.M. Han and S.V. Djordjević [7] that if T^* is p-hyponormal, M-Hyponormal or log-hyponormal, then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$, where $\mathcal{H}(\sigma(T))$ denotes the space of all analytic functions on an open neighbourhood of $\sigma(T)$.

Let us introduce one of the basic notions of local spectral theory. An operator $T \in \mathcal{L}(X)$ is said to enjoy the *single valued extension property*, SVEP for brevity, if for every non-empty open set $U \subseteq \mathbb{C}$, the only analytic solution of the equation $(T-\lambda)f(\lambda) = 0$ for $\lambda \in U$ is the zero function. It is well known that every p-hyponormal, M-hyponormal and log-hyponormal operator satisfies the SVEP, see for instance [17].

In the present paper, we study a-Weyl's and a-Browder's theorem for an operator T such that T or T^* satisfies the SVEP. We establish that if T^* has the SVEP, then T obeys a-Weyl's theorem if and only if it obeys Weyl's theorem. Further, if T or T^* has the SVEP, we show that the spectral mapping theorem holds for the essential approximative point spectrum, and that a-Browder's theorem is satisfied by $f(T)$ whenever $f \in \mathcal{H}(\sigma(T))$. We also provide several conditions that force an operator with the SVEP to obey a-Weyl's theorem.

The author woulde like to precise that this paper constitute a part of his thesis [16].

2. Main results

We shall say that an operator $T \in \mathcal{L}(X)$ is *semi-regular* if $\mathrm{R}(T)$ is closed and $\mathrm{N}(T) \subseteq \mathrm{R}(T^n)$ for every $n \in \mathbb{N}$. The *semi-regular resolvent set* is the open set given by $s\text{-}reg(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is semi-regular}\}$, [13].

Let T be a bounded operator on X. The *quasi-nilpotent part* of T is defined by

$$\mathrm{H_o}(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\},$$

and the *analytic core* of T by

$$\mathrm{K}(T) := \{x \in X : \text{there exists } \{x_n\}_{n\geq 0} \subseteq X \text{ and } c > 0 \text{ such that } x = x_0, \\ Tx_{n+1} = x_n \text{ and } \|x_n\| \leq c^n \|x\| \text{ for all } n \geq 0\}.$$

These subspaces are T-invariant, and generally not closed. However if $\mathrm{H_o}(T)$ is closed then $T_{|\mathrm{H_o}(T)}$ is quasi-nilpotent, also if Y is a T-invariant closed subspace of X such that $TY = Y$ then $Y \subseteq \mathrm{K}(T)$. It is straightforward to see that $T(\mathrm{K}(T)) = \mathrm{K}(T)$ and $\mathrm{N}(T^n) \subseteq \mathrm{H_o}(T)$ for all $n \in \mathbb{N}$. For more detail on these subspaces we refer the reader to [13], [14], and [12].

Let $T \in \mathcal{L}(X)$, we denote by $\sigma_{\mathrm{p}}^{\mathrm{f}}(T)$ the set of all eigenvalues of T of finite multiplicity.

PROPOSITION 2.1. *Let T be a bounded operator on X. If $\mathrm{H_o}(T-\lambda)$ is closed for every $\lambda \in \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, then T satisfies a-Browder's theorem.*

The proof of this proposition requires the following elementary lemma:

LEMMA 2.2. *Let T be a semi-Fredholm operator, then*

$$T \text{ has finite ascent } \Leftrightarrow \mathrm{H_o}(T) \text{ is finite-dimensional}\,.$$

Moreover, 0 is an isolated point of $\sigma_{ap}(T)$ if and only if $\mathrm{H_o}(T)$ is a non-zero closed subspace.

Proof. First, since T is semi-Fredholm, then the Kato decomposition, [8, Theorem 4], provides two closed T-invariant subspaces X_1, X_2 such that $X = X_1 \oplus X_2$, X_1 is finite-dimensional, $T_1 := T_{|X_1}$ is nilpotent and $T_2 := T_{|X_2}$ is semi-regular. Therefore $X_1 \subseteq \mathrm{H_o}(T)$ and $\mathrm{H_o}(T) = X_1 \oplus \mathrm{H_o}(T) \cap X_2$.

For the first part, suppose that T has finite ascent $p = \mathrm{a}(T_2)$. Because T_2 is semi-regular, Lemma 1.1 of [14] ensures that $\overline{\mathrm{H_o}(T_2)} = \overline{\cup_n \mathrm{N}(T_2^n)}$. Therefore $\mathrm{H_o}(T_2) \subseteq \mathrm{N}(T_2^p)$ and consequently $\mathrm{H_o}(T_2) = \mathrm{N}(T_2^p)$ is closed. But T_2 is semi-regular, hence $\mathrm{H_o}(T) \cap X_2 = \mathrm{H_o}(T_2) = \{0\}$, [11]. Thus $\mathrm{H_o}(T) = X_1$ is finite-dimensional. The other implication is obvious.

For the second part suppose that $\mathrm{H_o}(T)$ is a non-zero closed subspace. It follows easily from the above argument that 0 is an isolated point of $\sigma_{\mathrm{ap}}(T)$. Reciprocally, if $0 \in \mathrm{iso}\sigma_{\mathrm{ap}}(T)$, and because $\mathrm{R}(T)$ is closed, we obtain that $\mathrm{N}(T)$, and consequently $\mathrm{H_o}(T)$, is non-zero. Let λ in a deleted connected neighborhood of 0 such that $T-\lambda$ is injective with closed range. Then $T_2-\lambda$ is injective with closed range and $\mathrm{H_o}(T_2-\lambda) = \{0\}$, which implies that $\mathrm{H_o}(T_2) = \{0\}$ by Lemma 1.3 of [14]. Finally $\mathrm{H_o}(T) = X_1$ is finite-diemnsional. ∎

It is interesting to note that, in the literature, the Browder essential approximate point spectrum is defined to be the complementary in $\mathbb{C}$ of the complex numbers λ for which $T-\lambda$ is semi-Fredholm, $\dim \mathrm{N}(T-\lambda)$ and $\mathrm{a}(T-\lambda)$ are finite. However, by the preceding lemma, the condition of finiteness of $\dim \mathrm{N}(T-\lambda)$ is redundant.

For an operator T, we denote by $\Pi_{\mathrm{o}}^{\mathrm{a}}(T)$ the set of all isolated points λ of $\sigma_{\mathrm{ap}}(T)$ for which $T-\lambda$ is semi-Fredholm. It is clear by Lemma 2.2 that $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) \subseteq \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$.

Remark. Let T be a bounded operator on X, as immediate consequences of Lemma 2.2, we derive the following assertions:

(i) $\sigma_{\mathrm{ab}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \Pi_{\mathrm{o}}^{\mathrm{a}}(T) \mathrm{acc} \sigma_{\mathrm{ap}}(T) \cup \sigma_{\mathrm{SF}}(T)$.

(ii) if T satisfies a-Browder's theorem, then a-Weyl's theorem holds for T if and only if $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$.

(iii) if a-Weyl's theorem holds for T then so does a-Browder's theorem. Indeed, if we assume that T satisfies a-Weyl's theorem, we have $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \cap \sigma_{\mathrm{SF}}(T) \subseteq \Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \cap \sigma_{\mathrm{ea}}(T) = \emptyset$, and so $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T) \subseteq \Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \mathrm{iso} \sigma_{\mathrm{ap}}(T) \cap \rho_{\mathrm{SF}}(T)$. Thus, $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$ and $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{ab}}(T)$.

Proof of Proposition 2.1. Let us show that $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{ab}}(T)$. Suppose $\lambda \notin \sigma_{\mathrm{ea}}(T)$. If $T-\lambda$ is injective then it has a finite ascent, and hence $\lambda \notin \sigma_{\mathrm{ab}}(T)$. Suppose that $\mathrm{N}(T-\lambda)$ is a non-zero subspace. Since $T-\lambda$ is semi-Fredholm with non-positive index, $\mathrm{N}(T-\lambda)$ is of finite dimension. Consequently $\lambda \in \sigma_{\mathrm{p}}^{\mathrm{f}}(T)$, and so $\mathrm{H}_{\mathrm{o}}(T-\lambda)$ is closed, by hypothesis. Therefore Lemma 2.2 implies that $T-\lambda$ has finite ascent and $\lambda \notin \sigma_{\mathrm{ab}}(T)$. The other inclusion is clear. ∎

Proposition 2.3. *Let T be a bounded operator on X.*

(i) *If T^* has the SVEP, then T satisfies a-Weyl's theorem if and only if it satisfies Weyl's theorem.*

(ii) *If T has the SVEP, then T^* satisfies a-Weyl's theorem if and only if it satisfies Weyl's theorem.*

Proof. (i) Suppose that T^* has the SVEP, then Proposition 1.3.2 of [9] implies that $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, and consequently $\Pi_{\mathrm{oo}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$. Therefore it suffices to show that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{ea}}(T)$. Let $\lambda \notin \sigma_{ea}(T)$, then $T-\lambda$ is semi-Fredholm and $\mathrm{ind}(T-\lambda) \leq 0$, hence, by Proposition 2.2 of [17], we get that $\mathrm{ind}(T-\lambda) = 0$. Thus $\lambda \notin \sigma_w(T)$. The other inclusion is clear and the equivalence between Weyl's theorem and a-Weyl's theorem is proved for T.
(ii) Outlines the proof of the first statement. ∎

Theorem 2.4. *Let T be a bounded operator on X. If T, or its adjoint T^*, satisfies the SVEP, then a-Browder's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Let us show first that a-Browder's theorem holds for T. Suppose that T^* has the SVEP, then by [17, Theorem 2.7] it follows that Browder's theorem holds for T, i.e. $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Also from the proof of the previous Proposition we have $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{w}}(T)$. Therefore, to show that a-Browder's theorem holds for T, it suffices to establish that $\sigma_{\mathrm{ab}}(T) = \sigma_{\mathrm{b}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ab}}(T)$ then $\mathrm{ind}(T-\lambda) \leq 0$. But the SVEP for T^* implies that $\mathrm{ind}(T-\lambda) \geq 0$, [17, Proposition 2.2], therefore $\mathrm{ind}(T-\lambda) = 0$ and so $\lambda \notin \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. The other inclusion is obvious.

Now assume that T satisfies the SVEP and let $\lambda \in \sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ea}}(T)$. Then $T-\lambda$ is semi-Fredholm and consequently, by the Kato decomposition, there exists a $\delta > 0$ for which $\{\mu \in \mathbb{C} \colon 0 < |\mu - \lambda| < \delta\} \subseteq s\text{-reg}(T)$. On the other hand $s\text{-reg}(T) = \rho_{ap}(T)$ because T has the SVEP, [17, Lemma 2.1], and consequently $\lambda \in \mathrm{iso}\sigma_{\mathrm{ap}}(T) \cap \rho_{\mathrm{SF}}(T) = \Pi_{\mathrm{o}}^{\mathrm{a}}(T)$; which proves that $\sigma_{\mathrm{ap}}(T) \setminus \sigma_{\mathrm{ea}}(T) \subseteq \Pi_{\mathrm{o}}^{\mathrm{a}}(T)$. The other inclusion is clear, hence $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{ap}}(T) \setminus \Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \sigma_{\mathrm{b}}(T)$ and a-Browder's theorem holds for T.

Finally, if $f \in \mathcal{H}(\sigma(T))$, then by Theorem 3.3.6 of [9], $f(T)$, or $f(T)^*$, satisfies the SVEP, and the above argument implies that a-Browder's theorem holds for $f(T)$. ∎

For an operator satisfying the SVEP, the conclusion of the preceding Theorem was recently established by R. Curto and Y. Han in [4]. However, the arguments used here are different from the ones given in [4].

As immediate consequence of Theorem 2.4, we have:

Corollary 2.5. *Let T be a bounded operator on X. If T or T^* has the SVEP, then a-Weyl's theorem holds for T if and only if $\Pi_{\mathrm{o}}^{\mathrm{a}}(T) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$.*

From [20], we recall that for $T \in \mathcal{L}(X)$, the spectral mapping theorem holds for $\sigma_{\mathrm{ab}}(T)$, but may fail to hold for $\sigma_{\mathrm{ea}}(T)$.

Theorem 2.6. *If $T \in \mathcal{L}(X)$, or its adjoint T^*, satisfies the SVEP, then $f(\sigma_{\mathrm{ea}}(T)) = \sigma_{\mathrm{ea}}(f(T))$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. Since by the preceding Theorem, a-Browder's theorem holds for both T and $f(T)$, we have

$$f(\sigma_{\mathrm{ea}}(T)) = f(\sigma_{\mathrm{ab}}(T)) = \sigma_{\mathrm{ab}}(f(T)) = \sigma_{\mathrm{ea}}(f(T)).$$

This completes the proof. ∎

In [12], the class of the operators $T \in \mathcal{L}(X)$ for which $\mathrm{K}(T) = \{0\}$ was studied. It was shown that for such operators, the spectrum is connected and the SVEP holds.

Proposition 2.7. *Let* $T \in \mathcal{L}(X)$*, if there exists a complex number* λ *for which* $\mathrm{K}(T-\lambda) = \{0\}$*, then* $f(T)$ *satisfies a-Browder's theorem for every* $f \in \mathcal{H}(\sigma(T))$*. Moreover, if in addition,* $\mathrm{N}(T-\lambda) = \{0\}$*, then a-Weyl's theorem holds for* $f(T)$ *for every* $f \in \mathcal{H}(\sigma(T))$*.*

Proof. Let $f \in \mathcal{H}(\sigma(T))$, without loss of generality we can suppose that f is a non-constant analytic function on an open neighbourhood Ω of $\sigma(T)$. Since T has the SVEP, then so does $f(T)$, and hence, by Theorem 2.4, a-Browder's theorem holds for $f(T)$.

Now suppose that $N(T-\lambda) = \{0\}$, we claim that $\sigma_{\mathrm{p}}(f(T)) = \emptyset$. Let $\alpha \in \sigma(f(T))$ and write $f(z) - \alpha = p(z)g(z)$, where g is analytic on Ω and without zeros in $\sigma(T)$, while p is a polynomial of the form $p(z) = \Pi_{i=1}^{n}(z-\lambda_i)^{d_i}$ with distinct roots $\lambda_1, \lambda_2, \dots, \lambda_n \in \sigma(T)$. Because $g(T)$ is invertible, we have

$$\mathrm{N}(f(T)-\alpha) = \mathrm{N}(p(T)) \oplus_{i=1}^{n} \mathrm{N}(T-\lambda_i)^{d_i}.$$

On the other hand, from the fact that $\ker(T-\lambda) = \{0\}$ and $\ker(T-\mu) \subseteq \mathrm{K}(T-\lambda)$ for all complex number $\mu \neq \lambda$, we obtain that $\sigma_{\mathrm{p}}(T) = \emptyset$. Consequently $\mathrm{N}(f(T)-\alpha) = \{0\}$; which proves that $\sigma_{\mathrm{p}}(f(T)) = \emptyset$. Thus $\Pi_{\mathrm{o}}^{\mathrm{a}}(f(T)) = \Pi_{\mathrm{oo}}^{\mathrm{a}}(f(T))\emptyset$ and a-Weyl's theorem holds for $f(T)$. ∎

If $T \in \mathcal{L}(X)$ is a *semi-shift*, i.e. T is an isometry such that $\cap_{n=1}^{\infty}\mathrm{R}(T)^n = \{0\}$, then by the preceding proposition, a-Weyl's theorem holds for T.

For an operator $T \in \mathcal{L}(X)$, the *reduced minimum modulus* is defined by

$$\gamma(T) = \inf\{\|Tx\| : x \in X \text{ and } d(x, \mathrm{N}(T)) = 1\};$$

obviously $\gamma(T) > 0$ if and only if $\mathrm{R}(T)$ is closed, and $\gamma(T) = \|T^{-1}\|^{-1}$ if T is invertible, see [8].

The next result was established in [4], we provide here a short proof for it.

Theorem 2.8. *let* T *be a bounded operator on* X *satisfying the SVEP, the following assertions are equivalent:*

(i) T *obeys a-Weyl's theorem,*
(ii) $\mathrm{R}(T-\lambda)$ *is closed for every* $\lambda \in \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$*,*
(iii) γ *is discontinuous at every point of* $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$*.*

Proof. (i) $\Leftrightarrow$ (ii). It is straightforward to see that $\Pi_{\mathrm{oo}}^{\mathrm{a}}(T) = \Pi_{\mathrm{o}}^{\mathrm{a}}(T)$ if and only if $\mathrm{R}(T-\lambda)$ is closed for every $\lambda \in \Pi_{\mathrm{oo}}^{\mathrm{a}}(T)$. Hence the equivalence between (i) and (ii) follows immediately from Corollary 2.5.

(ii) $\Rightarrow$ (iii). Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$ be such that $\mathrm{R}(T-\lambda)$ is closed. Since T has the SVEP, $\sigma_{\mathrm{ap}} = \mathbb{C} \setminus \mathrm{s}-\mathrm{reg}(T)$ and consequently $T-\lambda$ is not semi-regular. Therefore, by Theorem 4.1 of [13], γ is discontinuous at λ.

(iii) $\Rightarrow$ (ii). Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$ and choose a non-zero element x in $\mathrm{N}(T-\lambda)$. For μ in a small deleted neighbourhood of λ, we have

$$\gamma(T-\mu)\|x\| \leq \|(T-\mu)x\| = |\lambda-\mu|\|x\|,$$

and so $\gamma(T-\mu) \leq |\lambda-\mu|$. Therefore, $\lim_{\mu\to\lambda}\gamma(T-\mu) = 0$, and since γ is discontinuous at λ, we get that $\gamma(T-\lambda) > 0$, that is, $\mathrm{R}(T-\lambda)$ is closed. ∎

Proposition 2.9. *let T be a bounded operator on X satisfying the SVEP. If $T-\lambda$ has finite descent at every $\lambda \in \Pi^{a}_{00}(T)$, then T obeys a-Weyl's theorem.*

Proof. Let $\lambda \in \Pi^{\mathrm{a}}_{\mathrm{oo}}(T)$. Since $d = \mathrm{d}(T-\lambda)$ is finite, it follows that $X = \mathrm{N}(T-\lambda)^d + \mathrm{R}(T-\lambda)$. Moreover, $\mathrm{N}(T-\lambda)$ is finite-dimensional, then by an inductive argument we get that also $\mathrm{N}(T-\lambda)^d$ is finite-dimensional. Therefore $\mathrm{R}(T-\lambda)$ is finite-codimensional and hence is closed. Now to conclude that a-Weyl's theorem holds for T, we use part (ii) of Theorem 2.8. ∎

Now let us consider the class $\mathcal{P}(X)$ defined as those operators $T \in \mathcal{L}(X)$ such that for every complex number λ there exists a positive integer d_λ for which $\mathrm{H_o}(T-\lambda) = \mathrm{N}(T-\lambda)^{d_\lambda}$. This class has been introduced and studied in [17], it was shown that it contains every M-hyponormal, log-hyponormal, p-hyponormal and totally paranormal operator. Also, it was established that the SVEP is shared by all the operators of $\mathcal{P}(X)$ and that Weyl's theorem holds for $f(T)$ whenever $T \in \mathcal{P}(X)$ and $f \in \mathcal{H}(\sigma(T))$.

Theorem 2.10. *Let T be a bounded operator on X. If there exists a function $h \in \mathcal{H}(\sigma(T))$ non-constant in any connected component of its domain, and such that $h(T^*) \in \mathcal{P}(X^*)$, then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. By Theorem 3.4 of [17] it follows that $T^* \in \mathcal{P}(X^*)$. Let us show first that a-Weyl's theorem holds for T. Since T^* has the SVEP, then by Proposition 2.3 it suffices to prove that Weyl's theorem holds for T, that is, by [17, Corollary 2.10], $\Pi_{\mathrm{oo}}(T) = \Pi_{\mathrm{o}}(T)$. To this aim, suppose $\lambda \in \Pi_{\mathrm{oo}}(T)$, then λ is an isolated point of $\sigma(T^*)$, and hence by Theorem 1.6 of [11], we have $X^* = \mathrm{H_o}(T^*-\lambda) \oplus \mathrm{K}(T^*-\lambda)$ where the direct sum is topological. On the other hand, $T^* \in \mathcal{P}(X^*)$ implies that $\mathrm{H_o}(T^*-\lambda) = \mathrm{N}(T^*-\lambda)^d$ for some integer d, therefore $X^* = \mathrm{N}(T^* - \lambda)^d \oplus \mathrm{K}(T^* - \lambda)$, and $\mathrm{R}(T^* - \lambda)^d = (T^* - \lambda)^d \mathrm{K}(T^* - \lambda) = \mathrm{K}(T^* - \lambda)$ is closed. Moreover, since $\dim \mathrm{N}(T - \lambda)$ is finite, we get that $\mathrm{N}(T - \lambda)^d$ is also finite-dimensional, and so $\mathrm{R}(T^* - \lambda)^d$ is finite-codimensional. Consequently $(T^* - \lambda)^d$ is Fredholm and hence so is $T - \lambda$. Thus $\lambda \in \mathrm{iso}\sigma(T) \cap \rho_{\mathrm{e}}(T) = \Pi_{\mathrm{o}}(T)$. The other inclusion is clear and Weyl's theorem holds for T.

Now if $f \in \mathcal{H}(\sigma(T))$, [17, Theorem 3.4] ensures that $f(T)^* \in \mathcal{P}(X^*)$, and from the above argument we conclude that a-Weyl's theorem holds for $f(T)$. ∎

Corollary 2.11. *If $T^* \in \mathcal{P}(X^*)$, then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proposition 2.12. *Let $T \in \mathcal{P}(X)$ be such that $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. By the spectral mapping theorem for the spectrum and the approximate point spectrum, and the fact that $f(T) \in \mathcal{P}(X)$, it suffices to establish a-Weyl's theorem for T. Also, because Weyl's theorem holds for T and $\sigma(T) = \sigma_{\mathrm{ap}}(T)$, we have only to prove that $\sigma_{\mathrm{ea}}(T) = \sigma_{\mathrm{w}}(T)$. Let $\lambda \notin \sigma_{\mathrm{ea}}(T)$, it follows that $\mathrm{H}_{\mathrm{o}}(T - \lambda) = \mathrm{N}(T - \lambda)^d$ is finite-dimensional, where d is a positive integer. If $T - \lambda$ is invertible then $\lambda \notin \sigma_{\mathrm{w}}(T)$. Therefore we may suppose that $\lambda \in \sigma(T) = \sigma_{\mathrm{ap}}(T)$. Since $T - \lambda$ is semi-Fredholm, $\mathrm{H}_{\mathrm{o}}(T - \lambda)$ is non-zero, and hence Lemma 2.2 implies that $0 \in \mathrm{iso}\sigma_{\mathrm{ap}}(T) = \mathrm{iso}\sigma(T)$. Consequently, [11, Theorem 1.6],

$$
\begin{aligned}
X &= \mathrm{N}(T - \lambda)^d \oplus \mathrm{K}(T - \lambda) \\
&= \mathrm{N}(T - \lambda)^d \oplus \mathrm{R}(T - \lambda)^d.
\end{aligned}
$$

This shows that $(T - \lambda)^d$, and so $T - \lambda$, is Fredholm of indice 0. Thus $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{ea}}(T)$. The other inclusion is trivial, then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{ea}}(T)$ and a-Weyl's theorem holds for T. ∎

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