EXTRACTA MATHEMATICAE Vol. **20**, Núm. 3, 233 – 241 (2005)

Warfield Invariants in Abelian Group Rings

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(Presented by Avinoam Mann)

AMS *Subject Class.* (2000): 16U60, 20K21 *Received September* 8, 2005

1. Introduction

As a tradition, let the symbol RG denote the group ring of an abelian group G, with the torsion subgroup $G_t = \coprod_{\forall p} G_p$ and p-component G_p, over a commutative unitary ring R, let $V(RG)$ denote the group of normalized invertible elements (also called normed units) and let $V(RG)_p$ denote its p-primary component. All other unexplained notions and notation are standard and can be found in the existing literature on the subject (see, for nomenclature, [1] – [6]). Since for each $\alpha \geq 0$ we have that $(G^{p^\alpha})_p = (G_p)^{p^\alpha}$, we unambiguously write $G_p^{p^\alpha}$ to denote both of these expressions. Note that the same equality is true for the operator index "t" as well.

Important invariants in the abelian group theory and its generalizations are the Ulm-Kaplansky ones (often termed as *Ulm invariants*) defined for any abelian group A, for an arbitrary but fixed prime p and over each ordinal number α in the following manner (see, for example, [6]):

$$U_{\alpha,p}(A) = \mathrm{rank}(A^{p^\alpha}[p]/A^{p^{\alpha+1}}[p])\,.$$

For the modular and semi-simple aspect of group algebras these invariants were successfully calculated by some authors (see, for instance, the bibliography in [3] and [5]).

It is long-known and documented in [6] that the Ulm-Kaplansky functions determine up to isomorphism the simply presented torsion groups, whereas this is not the case for global simply presented (or, even more generally, Warfield) groups. Nevertheless, it was argued in [8] that the Ulm-Kaplansky invariants combined with new appropriate cardinal functions, named as

Warfield invariants, are absolutely enough for classifying those mixed groups isomorphically.

Thus, following Warfield [9], we define the Warfield p-invariants of an arbitrary abelian group A for any fixed prime number p and every ordinal α as follows:

$$W_{\alpha,p}(A) = \mathrm{rank}(A^{p^\alpha}/(A^{p^{\alpha+1}}A_t^{p^\alpha}))\,.$$

Since $A_t = A_p \times \coprod_{q\neq p} A_q$ and each q-component A_q for a prime q is p-divisible, that is $A_q^p = A_q$, we thereby easily observe that the so-defined functions of Warfield can equivalently be restated thus:

$$W_{\alpha,p}(A) = \mathrm{rank}(A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha}))\,.$$

Henceforth, since the factor-group $A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})$ is bounded at p, whence it is a vector space over the field of p-elements, we write

$$W_{\alpha,p}(A) = |A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})| \geq \aleph_0\,,$$

or

$$W_{\alpha,p}(A) = \log_p |A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha})| \quad \text{otherwise}\,.$$

Knowing this, it is a plain matter to see that $W_{\alpha,p}(A) = 0 \iff A^{p^\alpha} = A^{p^{\alpha+1}}A_p^{p^\alpha}$ for all $\alpha \geq 0$. Therefore, if A is a torsion group or A^{p^α} is p-divisible, then $A^{p^\alpha} = A^{p^{\alpha+1}}A_p^{p^\alpha}$ is always satisfied and thus $W_{\alpha,p}(A) = 0$.

It is the purpose of this manuscript to compute explicitly the foregoing listed functions due to Warfield for $V(FG)$ as expressions that depend only upon R and G under certain their minimal restrictions.

2. Preliminary technicalities

We start here with a series of technical conventions, necessary for proving the main result.

Lemma 1. ([3]) *For an abelian group G, an ordinal number α and a commutative ring R with 1 of prime* $\mathrm{char}(R) = p$, *the following equality holds:*

$$V^{p^\alpha}(RG) = V(R^{p^\alpha}G^{p^\alpha})\,.$$

Lemma 2. *Under the conditions on R and G alluded to above and for each ordinal number $\alpha \geq 0$, the following intersection dependence holds:*

$$G^{p^\alpha} \cap (V^{p^{\alpha+1}}(RG)V^{p^\alpha}(RG)_p) = G^{p^{\alpha+1}}G_p^{p^\alpha}\,.$$

Proof. We consider only the case when $\alpha = 0$ since the general one follows via the substitutions $R \to R^{p^\alpha}$ and $G \to G^{p^\alpha}$ along with Lemma 1. By virtue of Lemma 1, we write $g(f_1g_1 + \cdots + f_sg_s) = r_1a_1 + \cdots + r_sa_s \in V(RG)_p$ where $g \in G; f_1, \ldots, f_s \in R^p$ and $g_1, \ldots, g_s \in G^p; r_1, \ldots, r_s \in R$ and $a_1, \ldots, a_s \in G; s \in \mathbb{N}$. The canonical records imply $gg_i = a_i$ for all $1 \leq i \leq s$. Since there is an index $j \in \{1, \ldots, s\}$ with $a_j \in G_p$, we infer at once that $g \in G^pG_p$, as required. ∎

COROLLARY 3. *Under the foregoing circumstances, the following inequality is fulfilled:*

$$W_{\alpha,p}(G) \leq W_{\alpha,p}(V(RG)) .$$

Proof. Invoking the preceding Lemma 2 and the classical second Noether isomorphism theorem, we derive that $V^{p^\alpha}(RG)/[V^{p^{\alpha+1}}(RG)V^{p^\alpha}(RG)_p] \supseteq G^{p^\alpha}V^{p^{\alpha+1}}(RG)V^{p^\alpha}(RG)_p/V^{p^{\alpha+1}}(RG)V^{p^\alpha}(RG)_p \cong G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})$, as needed. ∎

LEMMA 4. *For any infinite abelian group G and any commutative ring R with unity without zero divisors, the following equivalence is true:*

$$V(RG) = G \iff G_t = 1 .$$

Proof. Referring to a classical result of Higman [7], the group G being torsion-free implies that the normed units in RG are only trivial, hence $V(RG) = G$ does occur.

Let us now presume in a way of contradiction that $G_t \neq 1$ and suppose that $V(RG) = G$. Since the modular case is straightforward, we shall consider only the semi-simple one. Since $|G| \geq \aleph_0$, one can choose a finite subgroup $1 \neq C < G$ so that $e^2 = e \in RC \setminus \{0, 1\}$ and $g \in G \setminus C$, whence $eg \neq e$. Furthermore, we construct the element $x_g = 1 + e(g - 1)$. It is a plain technical exercise to verify that x_g^{-1} exists, precisely $x_g^{-1} = g^{-1} + (1 - e)(1 - g^{-1})$, and that $x_g \in V(RG) \setminus G$. We therefore infer that $V(RG) \neq G$. But this is against our assumption on G to have proper torsion elements, whence as a final conclusion $G_t = 1$, thus finishing the checking of the point. ∎

Remark. In the case when $\text{char}(R) = 0$, in [7] was proved that if G is a finite abelian group and R is an algebraic number field, then RG has only trivial torsion units, hence $V(RG)_t = G$.

Here we recall for future application the calculation of the cardinality of finite p-primary $V(RG)$. In fact, suppose G is finite and R is finite without zero divisor elements of prime characteristic p. It is well-known that $|V(RG)_p| = |R|^{|G|-|G|/|G_p|}$. In particular, when $G = G_p$, we derive that $|V(RG)| = |R|^{|G|-1}$ and we are done.

Proposition 5. *For a mixed or torsion-free abelian group G and a commutative ring R with 1 of* $\mathrm{char}(R) = p$ *with no zero divisor elements, the following equivalence is true:*

$$V(RG) = GV(RG)_p \iff G_t = G_p\,.$$

Proof. "$\Leftarrow$". It follows directly by [2].

"$\Rightarrow$". Replacing G by G/G_p in the given equality, and bearing in mind that $V(R(G/G_p))_p = 1$, we have $V(R(G/G_p)) = G/G_p$. Utilizing now the previous Lemma 4 and its proof, we deduce that $(G/G_p)_t = G_t/G_p = 1$, whence $G_t = G_p$, provided $|G/G_p| \geq \aleph_0$. In what remains G/G_p is a finite group hence $G/G_p = (G/G_p)_t = G_t/G_p$, i.e., $G = G_t$ sustained. So, in all cases, we conclude for the group G that $G_t = G_p$ or that $G = G_t$ where the second equality must be excluded by owing to our supposition. ∎

Remark. To justify the terminology, such groups G for which $G_t = G_p$, i.e., for which the only torsion is p-torsion, are called *p-mixed*.

The next affirmation is our crucial tool.

Proposition 6. *Assume that G is an abelian group and R is a commutative ring with identity of prime characteristic p. Then*

$$V(RG) = V(RG^p)V(RG)_p \iff G = G^pG_p\,.$$

Proof. "$\Rightarrow$". Choose $g \in G \subseteq V(RG)$, hence $g \in G \cap (V(RG^p)V(RG)_p)$. Consequently, owing to Lemmas 1 and 2, we obtain $g \in G^pG_p$, and so we are done.

"$\Leftarrow$". Given $x \in V(RG)$, hence $x = r_1g_1 + \cdots + r_sg_s$ with $r_i \in R$, $\sum_{1\leq i\leq s} r_i = 1$ and $g_i \in G$, $1 \leq i \leq s \in \mathbb{N}$. Write $g_i = a_i^pc_i$ where $a_i \in G$ and $c_i \in G_p$. Furthermore, $x = r_1a_1^pc_1 + \cdots + r_sa_s^pc_s = 1 + r_1(a_1^pc_1 - 1) + \cdots + r_s(a_s^pc_s - 1) = 1 + r_1(a_1^p(c_1 - 1) + a_1^p - 1) + \cdots + r_s(a_s^p(c_s - 1) + a_s^p - 1) = 1 + r_1a_1^p(c_1 - 1) + r_1(a_1^p - 1) + \cdots + r_sa_s^p(c_s - 1) + r_s(a_s^p - 1) = 1 + r_1(a_1^p - 1) + \cdots + r_s(a_s^p - 1) + r_1a_1^p(c_1 - 1) + \cdots + r_sa_s^p(c_s - 1)$. Since there is $k \in \mathbb{N}$ such that $c_i^{p^k} = 1$ for all i

$(1 \le i \le s)$, we deduce that $x^{p^k} = [1+r_1(a_1^p-1)+\cdots+r_s(a_s^p-1)]^{p^k}$. By setting $u = 1 + r_1(a_1^p - 1) + \cdots + r_s(a_s^p - 1)$, we plainly conclude that $u \in V(RG^p)$ since $x \in V(RG)$. Finally, we decompose $x = u(1 + r_1 a_1^p u^{-1}(c_1 - 1) + \cdots + r_s a_s^p u^{-1}(c_s - 1)) = uv$, where it is an easy matter to see that $v \in V(RG)_p$. Therefore the wanted decomposition holds, as expected. ∎

As an immediate consequence, we yield the following.

COROLLARY 7. *Under the foregoing given restrictions, the following relation is valid:*

$$W_{\alpha,p}(V(RG)) = 0 \iff W_{\alpha,p}(G) = 0\,.$$

DEFINITION 8. The subgroup C of an abelian group A is said to be *p-isotype* in A for some prime number p if $C \cap A^{p^\delta} = C^{p^\delta}$, for each ordinal δ. If this equality holds over every prime p, the subgroup C is named as *isotype* in A.

We mention that if C is (p-) isotype in A, then C^{p^α} is so in A^{p^α} for all α. The following assertion can be interpreted as classical and well-known folklore. However, for completeness of the exposition, we provide a proof.

LEMMA 9. *If C is a p-isotype subgroup of an abelian group A such that A/C is p-torsion, then $W_{\alpha,p}(A) = W_{\alpha,p}(C)$, for all ordinals α.*

In particular, if C is isotype in A so that A/C is torsion, then the above equality holds over all primes as well.

Proof. We shall verify only the first part-half. The second one is its direct consequence via the usage of the same idea.

And so, since $A/C \supseteq A^{p^\alpha}C/C \cong A^{p^\alpha}/(A^{p^\alpha} \cap C) = A^{p^\alpha}/C^{p^\alpha}$ is a p-group, we conclude because of the p-purity of C^{p^α} in A^{p^α} that $A^{p^\alpha}/C^{p^\alpha} = (A^{p^\alpha}/C^{p^\alpha})_p = A_p^{p^\alpha}C^{p^\alpha}/C^{p^\alpha}$, hence $A^{p^\alpha} = A_p^{p^\alpha}C^{p^\alpha}$. Consequently, the claim follows immediately via the isomorphism relationships $A^{p^\alpha}/(A^{p^{\alpha+1}}A_p^{p^\alpha}) = A_p^{p^\alpha}C^{p^\alpha}/A_p^{p^\alpha}C^{p^{\alpha+1}} \cong C^{p^\alpha}/[C^{p^\alpha} \cap (C^{p^{\alpha+1}}A_p^{p^\alpha})] = C^{p^\alpha}/(C^{p^{\alpha+1}}C_p^{p^\alpha})$, where the last equality is materialized by consulting with the modular law from [6]. ∎

Remark. Other interesting properties of the Warfield invariants are that $W_{\alpha,p}(A) = W_{0,p}(A^{p^\alpha})$, that $W_{\alpha,p}(A) = W_{\alpha,p}(A/A_p)$ for all α $(0 \le \alpha < \omega)$, and even that $W_{\alpha,p}(A) = W_{\alpha,p}(A/A_p)$ for all $\alpha \ge 0$ provided A_p is a nice subgroup of A.

3. The main result

The following attainment is our central result in the present article. It was previously announced as Theorem 13 in [1]. Well, we already have at our disposal all the information necessary and so we are ready to attack what follows.

Theorem 10. *Let R be a perfect commutative ring with identity of characteristic $p \neq 0$ which has no zero divisors and G an abelian group. Then, for every ordinal α, the following formulae are realized:*

$$W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G) \neq 0\,,$$

$$G^{p^\alpha} \neq G^{p^{\alpha+1}} G_p^{p^\alpha}\,, \quad G_t^{p^\alpha} = G_p^{p^\alpha}\,;$$

or

$$W_{\alpha,p}(V(RG)) = \max\{|R|, W_{\alpha,p}(G)\}\,,$$

$$G^{p^\alpha} \neq G^{p^{\alpha+1}} G_p^{p^\alpha}\,, \quad G_t^{p^\alpha} \neq G_p^{p^\alpha}\,, \quad |R|.W_{\alpha,p}(G) \geq \aleph_0\,;$$

or

$$W_{\alpha,p}(V(RG)) = (p^{W_{\alpha,p}(G)} - 1)\log_p |R|\,,$$

$$G^{p^\alpha} \neq G^{p^{\alpha+1}} G_p^{p^\alpha}\,, \quad G_t^{p^\alpha} \neq G_p^{p^\alpha}\,, \quad |R| < \aleph_0\,, \quad W_{\alpha,p}(G) < \aleph_0\,;$$

or

$$W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G) = 0\,,$$

$$G^{p^\alpha} = G^{p^{\alpha+1}} G_p^{p^\alpha}\,.$$

Proof. Foremost, taking into account the quoted above first property of the Warfield functions, namely that $W_{\alpha,p}(A) = W_{0,p}(A^{p^\alpha})$, and besides that Lemma 1 holds, we may restrict our attention to $\alpha = 0$.

First of all, we consider the case when $G_t = G_p$, i.e., when G is p-mixed, but $G \neq G^p G_p$ (we deal with both the finite and infinite powers). According to [4], we extract that G must be isotype in $V(RG)$. Moreover, Proposition 5 assures that the quotient $V(RG)/G = GV(RG)_p/G \cong V(RG)_p/G_p$ is p-primary. That is why, Lemma 9 substantiates our claim that the Warfield invariants of $V(RG)$ and G are equal.

After this, let we concentrate our attention on the possibility when G_t with G_p as well as G with $G^p G_p$ are para-wise different. To verify that the stated above formulas sustained, as we already have seen, one needs to compute the cardinality $|V(RG)/[V^p(RG)V(RG)_p]|$, i.e., by Lemma 1, the cardinality $|V(RG)/[V(RG^p)V(RG)_p]|$, of the factor-group $V(RG)/[V^p(RG)V(RG)_p]$. In order to do this, we will show below that

$$(*)\qquad \begin{aligned} &(r_1g_1+\cdots+r_sg_s)V(RG^p)V(RG)_p=(f_1h_1+\cdots+f_sh_s)V(RG^p)V(RG)_p\\ &\iff g_iG^pG_p=h_jG^pG_p \text{ and } r_i \text{ is a linear combination of}\\ &\quad\text{the } f_j\text{'s for all } i,j\ (1\le i,j\le s\in\mathbb{N}),\end{aligned}$$

where $r_1g_1+\cdots+r_sg_s\in V(RG)\setminus[V(RG^p)V(RG)_p]$ and $f_1h_1+\cdots+f_sh_s\in V(RG)\setminus[V(RG^p)V(RG)_p]$; thus Proposition 6 insures that there are at least two indices $1\le i\le s$ and $1\le j\le s: g_i\in G\setminus(G^pG_p)$ and $h_j\in G\setminus(G^pG_p)$. Note also that since $G\ne G^pG_p$ we have that G is not torsion ($G=G_t$ forces that $G=G_p\times\coprod_{q\ne p}G_q\subseteq G_pG^p$ because the latter direct component is p-divisible) and so, by Proposition 5, the elements $r_1g_1+\cdots+r_sg_s$ and $f_1h_1+\cdots+f_sh_s$ may be selected so that $r_1g_1+\cdots+r_sg_s\notin GV(RG)_p$ and $f_1h_1+\cdots+f_sh_s\notin GV(RG)_p$.

In fact, we observe that the equality $(r_1g_1+\cdots+r_sg_s)V(RG^p)V(RG)_p=(f_1h_1+\cdots+f_sh_s)V(RG^p)V(RG)_p$ can be written in an equivalent form as $r_1g_1+\cdots+r_sg_s=(f_1h_1+\cdots+f_sh_s)(\alpha_1b_1^p+\cdots+\alpha_sb_s^p)(\beta_1a_1+\cdots+\beta_sa_s)$, where $\alpha_1b_1^p+\cdots+\alpha_sb_s^p\in V(RG^p)$ and $\beta_1a_1+\cdots+\beta_sa_s\in V(RG)_p$. Thereby, there exists $k\in\mathbb{N}$ such that $r_1^{p^k}g_1^{p^k}+\cdots+r_s^{p^k}g_s^{p^k}=(f_1^{p^k}h_1^{p^k}+\cdots+f_s^{p^k}h_s^{p^k})(\alpha_1^{p^k}b_1^{p^{k+1}}+\cdots+\alpha_s^{p^k}b_s^{p^{k+1}})$. Therefore $r_1^{p^k}g_1^{p^k}+\cdots+r_s^{p^k}g_s^{p^k}=\sum_l\sum_m f_l^{p^k}\alpha_m^{p^k}h_l^{p^k}b_m^{p^{k+1}}=\gamma_1^{p^k}h_{l_1}^{p^k}b_{m_1}^{p^{k+1}}+\cdots+\gamma_s^{p^k}h_{l_s}^{p^k}b_{m_s}^{p^{k+1}}$, where the last sum is in canonical record and the coefficients $\gamma_1,\cdots,\gamma_s$ are linear combinations of the f_i's. Furthermore, because of the nonexistence of nilpotent elements, $r_i=\gamma_i$ and besides $g_i^{p^k}=h_{l_i}^{p^k}b_{m_i}^{p^{k+1}}$ for all i $(1\le i\le s)$. Thus $g_i\in h_{l_i}G^pG_p$ and, by the useful observation that there are no zero divisors whence all group products which do not belong to the support must be in relations one with other, we establish the desired claim in (*).

So, we are in position to treat the infinite case, that is, either $|R|\ge\aleph_0$ or $W_{0,p}(G)\ge\aleph_0$. By what we have just shown above, it obviously follows that $|V(RG)/[V(RG^p)V(RG)_p]|=|R||G/[G^pG_p]|=|R|.W_{0,p}(G)=\max\{|R|,W_{0,p}(G)\}$, as asserted.

Next, we become to the finite case, that is, $|R|<\aleph_0$ and $W_{0,p}(G)<\aleph_0$. Thus, by what we have already argued above, it follows at once that $|V(RG)/[V(RG^p)V(RG)_p]|=|R|^{|G/[G^pG_p]|-1}=|R|^{p^{W_{0,p}(G)}-1}$, as promised.

Finally, Corollary 7 exhausts the trivial situation when $G=G^pG_p$. ∎

The following immediate consequence is essential. It ensures that the Warfield p-invariants of the group basis of the group algebra are structural invariants for the algebra and thus they can be determined from it. This is a major fact and it was used intensively many times when examining the isomorphism problem for commutative modular group algebras (see e.g. [4]).

Corollary 11. *Suppose G is an abelian group and R is a perfect commutative unitary ring of characteristic $p > 0$ which possesses no zero divisor elements such that $G_t = G_p$ or such that $\aleph_0 \leq |R| \leq W_{\alpha,p}(G)$. Then, for each ordinal number α,*

$$W_{\alpha,p}(V(RG)) = W_{\alpha,p}(G)\,.$$

In particular, in all of these cases, $W_{\alpha,p}(G)$ can be recaptured from RG. Moreover, if $|R| < \aleph_0$ and $W_{\alpha,p}(G) < \aleph_0$ then $W_{\alpha,p}(G)$ is recovered by RG as well.

Proof. Because $V(RG)$ can be retrieved by RG, we directly use Theorem 10 to get the claim. ∎

Problem. Calculate $W_{\alpha,p}(V(RG))$ when R is not necessarily perfect.

4. Concluding discussion

In closing, we emphasize that the computation of the Warfield p-invariants of $V(RG)/G$, under the validity of the foregoing given restrictions on R and G, will definitely be of importance. Of interest, and to be the achievement complete, is also the calculating of the Warfield q-invariants for $V(RG)$ and $V(RG)/G$ whenever $q \neq p$ is a prime.

However, these two matters will be a theme of some other appropriate research exploration.

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