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Galois Coverings and the Problem of Axiomatization of the Representation Type of Algebras

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1. Introduction

Let K be an algebraically closed field. An important distinction in the class of finite dimensional associative K-algebras with identity is established by the representation types: such an algebra A is said to be of *finite type* if there are only finitely many indecomposable A-modules up to isomorphism; A is of *tame type* if in every dimension the indecomposable A-modules admit a parametrization by a finite number of one-parameter families; A is of *wild type* if there exists an A-$K\langle X,Y\rangle$-bimodule M free of finite rank over $K\langle X,Y\rangle$ such that the functor $M\otimes_{K\langle X,Y\rangle} - : \operatorname{mod} K\langle X,Y\rangle \to \operatorname{mod} A$ preserves indecomposability and sends non-isomorphic modules to non-isomorphic ones. A well-known result [10], [4] claims that every algebra is of one of these types.

Consider the class $\mathbf{Alg}(d)$ of all associative algebras with identity having dimension d over an algebraically closed field. This class is axiomatizable in a suitable first order language. In [18], Jensen an Lenzing proved that the subclass $\mathcal{F}(d)$ of $\mathbf{Alg}(d)$ formed by the algebras of finite representation type, as well as the class of algebras of infinite representation type, are finitely axiomatizable. Let $\mathbf{Alg}^p(d)$ denote the subclass of $\mathbf{Alg}(d)$ consisting of algebras over fields of characteristic p. In [19], the first author showed that the subclass $\mathcal{T}(d)$ of $\mathbf{Alg}(d)$ formed by the algebras of tame type is axiomatizable and an

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explicit system of axioms was given. Moreover, it was shown that finite axiomatizability of $\mathcal{T}^p(d) = \mathcal{T}(d) \cap \mathbf{Alg}^p(d)$ as a subclass of $\mathbf{Alg}^p(d)$ is equivalent to the fact that 'tame type is open', that is, the conjectured claim that the tame algebras form an open set in the variety of algebras $\mathbf{alg}_K(d)$ with the Zariski topology, for every algebraically closed field K (see [13]).

The Galois covering theory of algebras is one of the most important tools in representation theory. For a locally bounded K-category $\tilde{A}$ and a group G of K-automorphisms of $\tilde{A}$ acting freely on objects of $\tilde{A}$, the quotient $A = \tilde{A}/G$ of this action is again a K-category. In case A has finitely many objects (that is, the action of G has only finitely many orbits), the category A can be identified with a finite dimensional K-algebra. The natural projection functor

$$F : \tilde{A} \longrightarrow A$$

is called a Galois covering of A defined by the group G. The relations between the module categories $\operatorname{Mod} \tilde{A}$ and $\operatorname{Mod} A$ and other uses of Galois coverings may be seen in the extensive literature [2, 5, 6, 7, 9, 12, 16, 22, 26]. Of particular interest for us, are the results of Gabriel [12], completed in [23], claiming that $\tilde{A}$ is locally representation-finite (that is, for each object x of $\tilde{A}$ there are only finitely many indecomposable representations $M\colon \tilde{A} \to \operatorname{mod} K$, with $M(x) \neq 0$) if and only if A is of finite type; and the results [7, 9] showing that the tameness of A implies the tameness of $\tilde{A}$. Althought, it is known that tameness of $\tilde{A}$ does not imply the tameness of A [14], it has been conjectured (and a proof announced by Drozd and Ovsienko [11]) that for certain groups G the implication holds. See also [9] for an important partial result.

Conjecture GCPT (Galois Coverings Preserve Tameness). *Assume that G is a torsion-free group acting freely on objects of $\tilde{A}$. Then $\tilde{A}$ is tame if and only if A is tame.*

The aim of this work is to discuss some model-theoretic aspects of the theory of Galois coverings. The main idea is that preservation of representation type under Galois coverings is related with the axiomatizability of classes of algebras, and their coverings, defined by the representation type. For this purpose, the language of graded algebras is convenient. Indeed, every Galois covering of A with a group acting freely on objects can be obtained from a grading of A [16], [17].

We show in Section 3 that the class $\mathcal{T}gr(d)$ of tame graded algebras is an axiomatizable subclass of $\mathbf{Gralg}(d)$ - the class of graded algebras of dimension d over an algebraically closed field, see Section 3 for the details. Let $\mathbf{Gralg}^p(d)$

be the subclass of $\mathbf{Gralg}(d)$ consisting of algebras over fields of characteristic p. The main result of the paper is the following.

THEOREM A. *If $\mathcal{T}gr^p(d) = \mathcal{T}gr(d) \cap \mathbf{Gralg}^p(d)$ is a finitely axiomatizable subclass of $\mathbf{Gralg}^p(d)$, then Conjecture GCPT holds for countable p'-residually finite groups G (in particular, for countable free groups).*

Recall that a group G is residually finite if for every element $g \neq 1$ there exists a normal subgroup H of G of finite index such that $g \notin H$. If p is a prime then we say that G is p'-residually finite if in addition the normal subgroup H can be chosen such that $[G : H]$ is not divisible by p. For convenience, we agree that p'-residually finite means just residually finite when $p = 0$. Note that every free group is p'-residually finite for every prime p, see [25].

Let $\mathbf{Tfgr}$ be the subclass of $\mathbf{Gralg}(d)$ consisting of tuples (K, A, G), where G is a torsion free group and A is basic G-graded K-algebra. We show the following partial converse of Theorem A.

THEOREM B. *Assume that the Conjecture GCPT holds and tame algebras induce open subsets of the varieties of algebras $\mathbf{alg}_K(d)$ for any algebraically closed field K of characteristic p. Then $\mathcal{T}gr^p(d) \cap \mathbf{Tfgr}$ is finitely axiomatizable as a subclass of $\mathbf{Tfgr}^p = \mathbf{Tfgr} \cap \mathbf{Gralg}^p(d)$.*

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2. GRADED ALGEBRAS AS MODELS FOR A LANGUAGE

Let us recall some basic notions of model theory. For more on this we refer to [3], [24], [18].

Fix a first order language $\mathbb{L}$. Let $\mathcal{C} \subseteq \mathcal{B}$ be classes of models for $\mathbb{L}$. A set Σ of sentences is a *set of axioms for $\mathcal{C}$ as a subclass of $\mathcal{B}$* if for every model M in $\mathcal{B}$, M belongs to $\mathcal{C}$ if and only if all sentences from Σ are satisfied in M. The class $\mathcal{C}$ is *finitely axiomatizable as a subclass of $\mathcal{B}$* provided there exists a finite set of axioms for $\mathcal{C}$ as a subclass of $\mathcal{B}$. This is equivalent to the existence of one axiom defining the subclass $\mathcal{C}$ of B.

By Łoś Ultraproduct Theorem (see [21], [18, Theorem 1.5], [3, Theorem 4.1.9]) if $\mathcal{C}$ is an axiomatizable class of models then it is closed under formation of ultraproducts. Let us recall this important construction.

A nonempty family $\mathcal{F}$ of subsets of a set I is called an *ultrafilter over* I if the following conditions are satisfied:

- $\emptyset \notin \mathcal{F}$;
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
- if $A \in \mathcal{F}$ and $B \subseteq I$ then $A \cup B \in \mathcal{F}$;
- if $A \notin \mathcal{F}$ then $I \setminus A \in \mathcal{F}$.

Given a family $(M_i)_{i\in I}$ of models the ultrafilter $\mathcal{F}$ over I induces an equivalence relation $\sim$ in the product $\prod_{i\in I} M_i$ defined by $(m_i)_{i\in I} \sim (m_i')_{i\in I}$ if and only if $m_i = m_i'$ for $\mathcal{F}$-*almost all* $i \in I$, that is, there exists $U \in \mathcal{F}$ such that $m_i = m_i'$ for all $i \in U$. We denote by $(m_i)^{\mathcal{F}}$ the equivalence class of an element (m_i) with respect to the relation "$\sim$", and by $\prod_{i\in I} M_i/\mathcal{F}$ the *ultraproduct* of $(M_i)_{i\in I}$ with respect to $\mathcal{F}$, that is, the set of all equivalence classes by "$\sim$". The ultraproduct is equipped with the canonical interpretation of symbols from $\mathbb{L}$.

We describe a first order language $\mathbb{GA}_d$ such that graded algebras of fixed dimension d over a field can be treated as models for $\mathbb{GA}_d$ forming an axiomatizable class.

Let $\mathbb{L} = (x_1, x_2, \ldots, +, \cdot, 0, 1)$ be the first order language of the theory of fields and $\mathbb{G} = (g_1, g_2, \ldots, \circ, (-)^{-1}, 1)$ - the language of the theory of groups. We define $\mathbb{GA}_d$ as the disjoint union of two copies $\mathbb{L}_1$, $\mathbb{L}_2$ of $\mathbb{L}$ and $\mathbb{G}$. We call a variable from $\mathbb{L}_1$ (resp. $\mathbb{L}_2$, $\mathbb{G}$) a variable of first (resp. second, third) sort. We include into $\mathbb{GA}_d$ another function symbol "$\cdot$" associating to a pair of variables of first and second sort a variable of the second sort. Finally we add d constants $a_1, \ldots, a_d$ of second sort and d constants $g_1, \ldots, g_d$ of third sort. A model for $\mathbb{GA}_d$ is a triple

$$(K, A, G, \cdot : K \times A \to A, \{a_1, \ldots, a_d\} \subseteq A, \{g_1, \ldots, g_d\} \subseteq G)$$

where K, A, G are models for $\mathbb{L}_1$, $\mathbb{L}_2$ and $\mathbb{G}$ respectively (sometimes we identify models with underlying sets) and $\cdot : K \times A \longrightarrow A$ is an interpretation of the "new" function symbol "$\cdot$". In case it is not misleading we write (K, A, G) to denote the above model. Using the same symbols for constants and their interpretations does not lead to a confusion.

It is easy to write first order sentences in the language $\mathbb{GA}_d$ expressing the properties that K is an algebraically closed field, A is a d-dimensional associative K-algebra with a unit element, $a_1, \ldots, a_d$ is a K-basis of A, and G is a group (see [18]).

Consider the following sentence (easily expressible in the language $\mathbb{GA}_d$): *If* $\gamma = (\gamma_{ijk})_{i,j,k=1,\dots,d}$ *is a system of structure constants of* A *with respect to the basis* $a_1, \dots, a_d$*, that is,* $a_i a_j = \sum_{k=1}^{d} \gamma_{ijk} a_k$ *for* $i, j = 1, \dots, d$*, then* $g_i g_j = g_k$ *provided* $\gamma_{ijk} \neq 0$ *for every* $i, j, k = 1, \dots, d$*.*

If this sentence is satisfied in a model $(K, A, G, a_1, \dots, a_d, g_1, \dots, g_d)$ then A admits a G-grading

$$\bigoplus_{g \in G} A_g ,$$

where A_g is spanned by all a_i such that $g_i = g$ and $A_g = 0$ when $g \notin \{g_1, \dots, g_d\}$. Then $a_1, \dots, a_d$ is a homogeneous basis for A and g_i is the degree of a_i for $i = 1, \dots, d$.

Conversely, given a G-graded K-algebra A and its homogeneous basis we can identify it with a model for $\mathbb{GA}_d$. Note that nonisomorphic models can represent isomorphic graded algebras - fixing the model requires a choice of homogeneous basis of an algebra.

We see that d-dimensional algebras over algebraically closed fields which are graded by a group form an axiomatizable class of models for $\mathbb{GA}_d$, we denote this class by **Gralg** = **Gralg**(d). If p is 0 or a prime we denote by **Gralg**^{p} the subclass of **Gralg** consisting of all models (K, A, G) such that the characteristic of the field K is p. Note that **Gralg**^{0} is an axiomatizable subclass of **Gralg** and **Gralg**^{p} is finitely axiomatizable as a subclass of **Gralg**.

Given a model $(K, A, G, a_1, \dots, a_d, g_1, \dots, g_d)$ corresponding to a G-grading $\bigoplus_{g \in G} A_g$ of A and a normal subgroup H of G, we denote by $(K, A, G/H) = (K, A, G/H, a_1, \dots, a_d, g_1 H, \dots, g_d H)$ the model corresponding to the G/H-grading $A \cong \bigoplus_{\overline{g} \in G/H} \overline{A}_{\overline{g}}$, where $\overline{A}_{\overline{g}} = \bigoplus_{g \in G, gH = \overline{g}} A_g$.

Let $\mathcal{C}$ be a subclass of **Gralg**.

Definition 2.1. The class $\mathcal{C} \subseteq$ **Gralg** is called *essential* provided any model

$$(K, A, G, a_1, \dots, a_d, g_1, \dots, g_d) \in \mathbf{Gralg}$$

belongs to $\mathcal{C}$ if and only if $(K, A, G', a_1, \dots, a_d, g_1, \dots, g_d) \in \mathcal{C}$, where G' is the subgroup of G generated by $g_1, \dots, g_d$.

Let p be zero or a prime.

Lemma 2.2. *Let* G *be* p'*-residually finite, countable group. There exists a descending sequence* G_n*,* $n \in \mathbb{N}$*, of normal subgroups of* G *of finite index not divisible by* p*, such that* $\bigcap_{n \in \mathbb{N}} G_n = \{1\}$*.*

Proof. Given an element $g \neq 1$ of G let

$$\phi_g : G \longrightarrow H_g$$

be a group homomorphism with H_g of finite order not divisible by p and $\phi_g(g) \neq 1$. Enumerate the elements of $G = \{g_1, \dots, g_n, \dots\}$. Let

$$\psi_n : G \longrightarrow \prod_{m=1}^{n} H_{g_m}$$

be the homomorphism defined by $\psi_n(g) = (\phi_{g_m}(g))_{m \leq n}$ for $n = 1, 2, \dots$. We set $G_n = \operatorname{Ker} \psi_n$. Observe that in case $p \neq 0$ the index of G_n is not divisible by p as a divisor of $\prod_{m \leq n} |H_{g_m}|$. ∎

Definition 2.3. The class $\mathcal{C}$ is *closed under finite covers of p'-order* if $(K, A, G) \in \mathcal{C}$ provided there is a finite normal subgroup H of G with $|H|$ not divisible by p such that $(K, A, G/H) \in \mathcal{C}$. If $p = 0$, we say that $\mathcal{C}$ is just *closed under finite covers.*

Definition 2.4. The class $\mathcal{C}$ is *preserved under algebraically closed base field extensions* if for any model $\mathbf{A} = (K, A, G, a_1, \dots, a_d, g_1, \dots, g_d)$ and an algebraically closed field extension $K \subset L$, the model $\mathbf{A}$ belongs to $\mathcal{C}$ if and only if

$$\mathbf{A} \otimes L = (L, A \otimes_K L, G, a_1 \otimes 1, \dots, a_d \otimes 1, g_1, \dots, g_d) \in \mathcal{C}.$$

Throughout we omit the phrase "algebraically closed" in the above name because no other fields will be considered.

Proposition 2.5. *Let p be 0 or a prime. Assume that $\mathcal{C}$ is an essential class in $\mathbf{Gralg}(d)$ preserved under base field extensions and finite covers of p'-order. Suppose that $\mathcal{C}$ is axiomatizable. Let $(K, A, G) \in \mathbf{Gralg}$, where G is countable and p'-residually finite. Let H be a normal subgroup in G.*

Then $(K, A, G) \in \mathcal{C}$ provided $(K, A, G/H) \in \mathcal{C}$.

Proof. Let G_n, $n \in \mathbb{N}$, be a descending sequence of normal subgroups of G of finite index not divisible by p such that $\bigcap_{n \in \mathbb{N}} G_n = \{1\}$. Set $H_n = G_n \cap H$. Let $\mathcal{F}$ be an ultrafilter over $\mathbb{N}$ containing no finite sets.

Assume that $(K, A, G/H)$ belongs to $\mathcal{C}$. Since $\mathcal{C}$ is preserved by finite covers of p'-order it follows that $(K, A, G/H_n) \in \mathcal{C}$ for all $n \in \mathbb{N}$. Since $\mathcal{C}$ is axiomatizable, the ultraproduct

$$\Big(\prod_{n\in\mathbb{N}}(K, A, G/H_n)\Big)\Big/\mathcal{F},$$

which is isomorphic to the model

$$\Big(L, A\otimes_K L, \Big(\prod_{n\in\mathbb{N}}(G/H_n)\Big)\Big/\mathcal{F}, a_1\otimes 1, \ldots, a_d\otimes 1, (g_1H_n)^{\mathcal{F}}_{n\in\mathbb{N}}, \ldots, (g_dH_n)^{\mathcal{F}}_{n\in\mathbb{N}}\Big)$$

where L is the ultrapower $K^{\mathbb{N}}/\mathcal{F}$, belongs to $\mathcal{C}$ as well.

Note that the map $G \to (\prod_{n\in\mathbb{N}}(G/H_n))/\mathcal{F}$ defined by $g \mapsto (gH_n)^{\mathcal{F}}_{n\in\mathbb{N}}$ is an injection, since $\bigcap_{n\in\mathbb{N}} H_n = \{1\}$.

Now $(K, A, G, a_1, \ldots, a_d, g_1, \ldots, g_d)$ belongs to $\mathcal{C}$ because $\mathcal{C}$ is essential and closed under base field extensions. ∎

3. The class of tame graded algebras

Let $\mathcal{T}gr(d)$ denote the subclass of $\mathbf{Gralg} = \mathbf{Gralg}(d)$ formed by all models (K, A, G) such that A is representation tame as graded algebra. By $\mathcal{W}gr(d)$ we denote the complement $\mathbf{Gralg} \setminus \mathcal{T}gr(d)$ consisting of all wild graded algebras.

Let $(K, A, G) \in \mathbf{Gralg}$. Let $\text{gr-mod}^G(A)$ denote the category of all finitely generated G-graded A-modules.

Given a finite subset S of G let $A(S)$ be the K-algebra defined as

$$A(S) = \bigoplus_{s_1,s_2\in S} B_{s_1,s_2}$$

where $B_{s_1,s_2} = A_{s_1^{-1}s_2}$. We equip $A(S)$ with a multiplication: the product of $b_{s_1,s_2} \in B_{s_1,s_2}$ and $b_{t_1,t_2} \in B_{t_1,t_2}$ equals the product $b_{s_1,s_2}b_{t_1,t_2}$ in A if $s_2 = t_1$, $s_1^{-1}t_2 \in S$ and it is zero otherwise. Observe that $b_{s_1,s_2}b_{t_1,t_2} \in B_{s_1,t_2}$.

Let $\text{gr-mod}^G_S(A)$ be the full subcategory of $\text{gr-mod}^G(A)$ formed by all graded modules $M = \bigoplus_{g\in G} M_g$ such that $g \in S$ whenever $M_g \neq 0$.

Lemma 3.1. (1) *For every finite set* $S \subseteq G$ *the categories* $\text{gr-mod}^G_S(A)$ *and* $\text{mod}(A(S))$ *are equivalent.*

(2) *The graded algebra* A *is tame (that is, the category* $\text{gr-mod}^G(A)$ *is tame) if and only if for every finite set* $S \subseteq G$ *the algebra* $A(S)$ *is tame.*

Proof. For the assertion (1) see [16, Theorem 2.5], whereas (2) follows from (1) by standard arguments (see also [8]). ∎

Observe that a system of structure constants of $A(S)$ can be derived from a system of structure constants of A with respect to a homogeneous basis by an elementary formula. It follows that given a finite set S every first order property of $A(S)$ can be expressed by a first order property of (K, A, G).

Fix G, S and a natural number l. Given a sentence ϕ in the language $\mathbb{A}$ of the theory of algebras over fields, let $\phi^{l,S}$ be a sentence in the language $\mathbb{GA}_d$ such that $(K, A, G) \models \phi^{l,S}$ if and only if $(K, A(S)) \models \phi$ and $\dim_K A(S) = l$, for every model $(K, A, G) \in \mathbf{Gralg}(d)$. Note that $\dim_K A(S) \leq |S| \dim_K A$.

Let Σ_l be a set of axioms for the class of tame algebras in $\mathbf{Alg}(l)$, the class of l-dimensional algebras over algebraically closed fields [19]. The following theorem is an immediate consequence of the above remarks and Lemma 3.1 (2).

THEOREM 3.2. *Let Ξ be the set of all sentences of the form*

$$\phi_1^{1,S} \vee \cdots \vee \phi_{d|S|}^{d|S|,S},$$

where $\phi_l \in \Sigma_l$, $l = 1, \ldots, d|S|$ and S runs through all finite subsets of G. Then Ξ is a set of axioms for the class $\mathcal{T}gr(d)$ as a subclass of $\mathbf{Gralg}(d)$.

We shall need a geometric criterion for tameness of graded algebras. Let us present it without details since it is a realisation of a well known idea. Given a graded algebra (K, A, G) and a function $m : G \to \mathbb{N}$ such that $m(g) = 0$ for all but a finite number of $g \in G$, let $\mathbf{grmod}_A(m)$ denote the variety of graded modules $M = \bigoplus_{g \in G} M_g$ such that $\dim_K M_g = m(g)$ for $g \in G$. The group $Gl(m) = \prod_{g \in G} Gl(m(g), K)$ acts on $\mathbf{grmod}_A(m)$ in such a way that the orbits of this action correspond to the isomorphism classes of graded A-modules. Let $\mathrm{par}(A, G, m)$ be the "number of parameters" of this action, that is, the maximum of the values $\dim X - \dim Gl(m)x$, where $x \in X$, taken over all irreducible components X of $\mathbf{grmod}_A(m)$. The following fact is essentially contained in [27] and [13].

LEMMA 3.3. *Let $(K, A, G) \in \mathbf{Gralg}$.*

(1) *The graded algebra A is tame if and only if* $\mathrm{par}(A, G, m) \leq \sum_{g \in G} m(g)$ *for every function $m : G \to \mathbb{N}$ such that $m(g) = 0$ for almost all $g \in G$.*

(2) *The graded algebra A is wild if and only if there exists a function $m : G \to \mathbb{N}$ such that $m(g) = 0$ for almost all $g \in G$ and a positive real number c satisfying* $\mathrm{par}(A, G, lm) \geq cl^2$ *for every $l \in \mathbb{N}$.*

Let us recall in the language of graded algebras the concept of the push-down functor. Let $(K, A, G) \in$ **Gralg** and assume that H is a normal subgroup of G. We consider the G/H-grading of A induced by the previous G-grading. There is a pair of functors

$$\text{gr-mod}^G(A) \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \text{gr-mod}^{G/H}(A)$$

such that if $M = \bigoplus_{g \in G} M_g$ is a G-graded A-module then

$$F_\lambda(M) = \bigoplus_{\overline{g} \in G/H} F_\lambda(M)_{\overline{g}},$$

where $F_\lambda(M)_{\overline{g}} = \bigoplus_{g \in G, gH = \overline{g}} M_g$.

The functor $F_\bullet$ sends a G/H-graded A-module $N = \bigoplus_{\overline{g} \in G/H} N_{\overline{g}}$ to the G-graded A-module $\bigoplus_{g \in G} F_\bullet(N)_g$, where $F_\bullet(N)_g = N_{gH}$.

The above functors are defined on homomorphisms in the obvious way and it is not hard to prove that F_λ is the left adjoint to $F_\bullet$. Given a G-graded module $M = \bigoplus_{g \in G} M_g$, we denote by

$$\nabla_M : F_\lambda(F_\bullet(M)) \longrightarrow M$$

the homomorphism adjoint to the identity map on $F_\bullet(M)$. Observe that $F_\lambda(F_\bullet(M)) = \bigoplus_{g \in G} M'_g$, where $M'_g = \bigoplus_{g' \in gH} M_{g'}$ and if $(m_{g'})_{g' \in gH} \in M'_g$ then $\nabla_M((m_{g'})_{g'}) = \sum_{g'} m_{g'}$.

If H is a finite group then there is also a map

$$\Delta_M : M \longrightarrow F_\lambda(F_\bullet(M))$$

sending $m \in M_g$ to $(m_{g'})_{g' \in gH}$, where $m_{g'} = m$ for all g'.

Note that when H is finite the functor F_λ coincides with the right adjoint to $F_\bullet$ and Δ_M is the homomorphism adjoint to the identity map on $F_\bullet(M)$.

LEMMA 3.4. *Assume that H is a finite normal subgroup of G and M is a G-graded A-module. We have:*

(a) $\nabla_M \Delta_M = |H|\, id_M$;

(b) *if* char K *does not divide* $|H|$, *then the module M is a direct summand of* $F_\lambda(F_\bullet(M))$;

(c) *if* char K *does not divide* $|H|$ *and* $(K, A, G/H)$ *is wild, then* (K, A, G) *is wild.*

Proof. The assertions (a) and (b) are immediate. Let us remark that they are essentially contained in [12, 3.4]. The point (c) follows from (b) thanks to Lemma 3.3: let $m : G/H \to \mathbb{N}$ be a function such that

$$\mathrm{par}(A, G/H, lm) \geq cl^2$$

for some $c > 0$ and all natural l; one can prove using (b) that

$$\mathrm{par}(A, G, |H| l\widetilde{m}) \geq cl^2$$

for all l, where $\widetilde{m}$ is the composition of the natural projection $G \to G/H$ with m. ∎

LEMMA 3.5. *Let p be 0 or a prime. The classes $\mathcal{W}gr(d)$ and $\mathcal{W}gr^p(d) := \mathcal{W}gr(d) \cap \mathbf{Gralg}^p$ are essential and closed under base field extensions. The class $\mathcal{W}gr^p(d)$ is closed under finite covers with p'-orders.*

Proof. It is obvious that $\mathcal{W}gr(d)$ and $\mathcal{W}gr^p(d)$ are essential. The assertion about base field extensions follows from [19] (applied to the algebras $A(S)$, $S \subseteq G$). Finally, $\mathcal{W}gr^p(d)$ is closed under finite covers with p'-orders by Lemma 3.4 (c). ∎

THEOREM 3.6. *Let p be 0 or a prime. Assume that the class $\mathcal{T}gr^p(d)$ is finitely axiomatizable as a subclass of $\mathbf{Gralg}^p$. Let $(K, A, G) \in \mathbf{Gralg}^p$, where G is a countable p'-residually finite group and let H be a normal subgroup of G such that $(K, A, G/H)$ is a wild graded algebra. Then (K, A, G) is also a wild graded algebra.*

Proof. Follows from 3.5 and 2.5. ∎

Theorem A is a corollary of 3.6. In order to see this, recall from [16] the relationship between gradings and Galois coverings of algebras. Every Galois covering $\widetilde{A} \to A$ with covering group G acting freely on objects of $\widetilde{A}$ induces a grading $A = \bigoplus_{g \in G} A_g$. The first assertion in the following theorem follows by [16] whereas the second one is routine.

PROPOSITION 3.7. (1) *The category* $\text{gr-mod}^G(A)$ *of G-graded finite dimensional A-modules is equivalent to* $\mathrm{mod}(\widetilde{A})$.

(2) *The representation types of* $\text{gr-mod}^G(A)$ *and* $\mathrm{mod}(\widetilde{A})$ *coincide.*

Proof of Theorem A. In view of Proposition 3.7, Theorem A follows from Theorem 3.6 applied to $H = G$. ∎

4. Final remarks and comments

4.1. The formulation of the Conjecture GCPT as well as known results on representation finite algebras [12], [2] suggest that the algebras graded by torsion-free groups form a relevant class.

Let **Tfgr** be the subclass of $\mathbf{Gralg}(d)$ consisting of models (K, A, G) such that A is basic and G is torsion-free. Note that this class is axiomatizable. Indeed, the property "A is basic" is equivalent to the invertibility of $1 - c(ab - ba)$ for any $a, b, c \in A$. Set $\mathbf{Tfgr}^p = \mathbf{Tfgr} \cap \mathbf{Gralg}^p(d)$.

Let $(K, A, G) \in \mathbf{Tfgr}$. Then the Jacobson radical of A is a homogeneous ideal and the simple A-modules are gradable, therefore the grading is induced by a Galois covering $\widetilde{A} \to A$, as described at the end of Section 3 (see [16, Theorem 2.4]). The group G acts freely on objects of $\widetilde{A}$ thus, since it is torsion-free, it acts freely on finite dimensional indecomposable $\widetilde{A}$-modules.

Denote by $\mathcal{LRF}$ the subclass of **Tfgr** consisting of all locally representation-finite graded algebras, that is, such graded algebras (K, A, G) that given an element $g \in G$ there exists only finitely many isomorphism classes of graded A-modules M with $M_g \neq 0$. We denote by $\mathcal{LRI}$ the complementary subclass of **Tfgr** consisting of all locally representation-infinite graded algebras.

It follows by finite axiomatizability of the class of representation finite algebras [18] and Gabriel's result [12] that the classes $\mathcal{LRF}$ and $\mathcal{LRI}$ are finitely axiomatizable subclasses of **Tfgr**. Theorem B follows by analogous reasoning:

Proof of Theorem B. If Conjecture GCPT holds then any set of axioms for the class $\mathcal{T}(d)$ of tame d-dimensional algebras is at the same time a set of axioms for $\mathcal{T}gr \cap \mathbf{Tfgr}$. Therefore it is enough to recall that if the tame algebras induce open sets in the varieties of algebras $\mathbf{alg}_K(d)$, for any algebraically closed field K of characteristic p, then the class $\mathcal{T}^p(d)$ is a finitely axiomatizable subclass of $\mathbf{Alg}^p(d)$ by [19]. ∎

4.2. It is natural to ask how restrictive is the assumption of residually finiteness of the covering group in the context of our considerations.

Following [15] call a grading $A = \bigoplus_{g \in G} A_g$ of the algebra A *critical* if there is no surjective group homomorphism $f : H \to G$ which is not injective and a grading $A = \bigoplus_{h \in H} A'_h$ such that:

(i) H is generated by elements h such that $A'_h \neq 0$ (that is, the grading is full in the terminology of [16]), and

(ii) $A_g = \bigoplus_{h \in f^{-1}(g)} A'_h$.

We do not know if there exists a tame critically G-graded algebra A such that G is not residually finite.

In view of the correspondence between gradings and Galois coverings, this problem is closely related to the problem of the existence of a tame algebra $A = KQ/I$, for a quiver Q and an admissible ideal I, such that the fundamental group $\Pi_1(Q,I)$ of the bound quiver (Q,I), see [22], is not residually finite.

A case-by-case inspection shows that there is no such example among local algebras: if KQ/I is a tame local algebra, then $\Pi_1(Q,I)$ is either abelian or free with two generators or it is a group of the form $D_{n,m} = \langle x, y \mid x^n = y^m \rangle$ for some $m, n > 0$ (case of quotients of the algebra $K\langle x,y\rangle/(xy, yx, x^n - y^m)$). All those groups are residually finite, for the case of $D_{n,m}$ we give the following argument.

Let H be the subgroup of $D_{n,m}$ generated by the element x^n. Then H is normal. Indeed, since $y^{-1} = x^{-n}y^{m-1}$ we have $yx^ny^{-1} = y^m = x^n$. The quotient $D_{n,m}/H$ is isomorphic to the free product $\mathbb{Z}_n * \mathbb{Z}_m$ of cyclic groups, hence it is residually finite by the results of [1].

Now let $1 \neq g \in D_{n,m}$. If $g \notin H$, then by the above remark there is a normal subgroup N of $D_{n,m}$ such that $g \notin N$. Otherwise let $g = x^{kn}$. Let p be a natural number not dividing km. There is a group homomorphism

$$f : D_{n,m} \longrightarrow \mathbb{Z}_p$$

defined by $f(x) = \overline{m}$, $f(y) = \overline{n}$, where $\overline{t}$ denotes the coset of t in $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ and $f(g) \neq \overline{0}$. Then $\operatorname{Ker} f$ is a required normal subgroup of $D_{n,m}$. This proves that $D_{n,m}$ is residually finite.

One of the best known examples of a not residually finite group is $G = \langle b, t \mid b^2t = tb^3 \rangle$. This group does not have the *Hopf property* [25]: there exists a surjective group endomorphism $\phi : G \to G$ which is not injective. Indeed, one can define ϕ by setting $t \mapsto t$, $b \mapsto b^2$. Then it is easy to observe that ϕ is surjective but the commutator $[b, t^{-1}bt]$, which is not the identity in G by Britton's Lemma [20, Chap. IV.2], belongs to the kernel of ϕ. Therefore G is not residually finite by [20, Theorem 4.10]; see also [25]. Note also that G is torsion-free by the Torsion Theorem for HNN Extensions [20, Theorem 2.4]. Let us finish the paper with the following:

Problem. Let A be an algebra graded critically by $G = \langle b, t \mid b^2t = tb^3 \rangle$. Is A necessarily wild? Does the existence of the endomorphism ϕ reflects on the structure of the category of (graded) A-modules?

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