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Weyl's Theorem, a-Weyl's Theorem and Single-Valued Extension Property

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1. Introduction and definitions

Let $T \in L(X)$ be a bounded operator on an infinite-dimensional complex Banach space X and denote by $\alpha(T)$ and $\beta(T)$ the dimension of the kernel $\ker T$ and the codimension of the range $T(X)$, respectively. Let us denote by

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\},$$

the class of all *upper semi-Fredholm* operators, and by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\},$$

the class of all *lower semi-Fredholm* operators. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, whilst the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. The *ascent* $p := p(T)$ of an operator T is the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* $q := q(T)$ of an operator T is the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [21, Proposition 38.3]. Two other important classes of operators in Fredholm theory are the class of all *upper semi-Browder operators*

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators*

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The two classes $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$ have been introduced in [19] and studied by several other authors, for instance [29]. The class of all *Browder operators* (known in the literature also as *Riesz-Schauder operators*) is defined by $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. Note that if $T \in \mathcal{B}_+(X)$ then the index, defined by $\text{ind}\, T := \alpha(T) - \beta(T)$ is less than or equal to 0, whilst if $T \in \mathcal{B}_-(X)$ then $\text{ind}\, T \geq 0$, see [21, Proposition 38.5]. The class of all *Weyl operators* $\mathcal{W}(X)$ is defined by

$$\mathcal{W}(X) := \{T \in \Phi(X) : \text{ind}\, T = 0\}.$$

Note that $\mathcal{B}(X) \subseteq \mathcal{W}(X)$, since every Fredholm operator with finite ascent and finite descent has necessarily index 0, see [21, Proposition 38.6].

The classes of operators defined above motivate the definition of several spectra. The *upper semi-Fredholm spectrum* is defined by

$$\sigma_{\text{uf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_+(X)\},$$

the *lower semi-Fredholm spectrum* is defined by

$$\sigma_{lf}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_-(X)\},$$

whilst the *semi-Fredholm spectrum* and the *Fredholm spectrum* are defined, respectively, by

$$\sigma_{\text{sf}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi_\pm(X)\}$$

and

$$\sigma_{\text{f}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(X)\}.$$

Analogously, the *upper semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

whilst the *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

Note that

$$\sigma_{\rm ub}(T) = \sigma_{\rm lb}(T^*), \quad \sigma_{\rm lb}(T) = \sigma_{\rm ub}(T^*).$$

The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{W}(X)\}.$$

Evidently,

$$\sigma_{\rm sf}(T) \subseteq \sigma_{\rm f}(T) \subseteq \sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T) = \sigma_{\rm w}(T) \cup \text{acc}\ \sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$

The single valued extension property was introduced by Dunford [13], [14] and has, successively, received a more systematic treatment in Dunford-Schwartz [15]. It also plays an important role in local spectral theory, see the monograph of Laursen and Neumann [24]. In this article we shall consider the following local version of this property, which has been studied in recent papers [6], [4], [5], [7] and previously by Finch [18].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Trivially, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, T has SVEP at every isolated point of $\sigma(T)$.

For an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, the *glocal spectral subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. The basic role of SVEP arises in local spectral theory since every decomposable operator enjoys this property. Recall that $T \in L(X)$ is said to have the *decomposition property* (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. The decomposability of $T \in L(X)$ may be defined in several way, for instance as the union of the property (β) and the property (δ), see [24, Theorem 2.5.19] for relevant definitions. Note that the property

(β) implies that T has SVEP, whilst the property (δ) implies SVEP for $T^\star$, see [24, Theorem 2.5.19].

Let us consider the *quasi-nilpotent part* of T, that is the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that ker $(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Moreover, $H_0(\lambda_0 I - T) = \mathcal{X}_T(\{\lambda_0\})$ for all $\lambda_0 \in \mathbb{C}$, see Vrbová [33] or Mbekhta [26].

The *analytic core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and $\delta > 0$ for which $x = u_0$, and $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$. It easily follows, from the definition, that $K(T)$ is a linear subspace of X and that $T(K(T)) = K(T)$. In general, $K(T) \subseteq T^\infty(X)$, where $T^\infty(X) := \bigcap_{n=1}^\infty T^n(X)$ is the *hyper-range* of T. Indeed, if $x \in K(T)$ and $(u_n) \subset X$ is a sequence for which $u_o = x$ and $Tu_{n+1} = u_n$ for every $n = 0, 1, \dots$ then $x = u_o = T^n u_n \in T^n(X)$ for every $n = 0, 1, \dots$.

Recall that $T \in L(X)$ is said *bounded below* if T is injective and has closed range. Let $\sigma_a(T)$ denote the classical *approximate point spectrum* of T, i.e. the set

$$\sigma_a(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}.$$

THEOREM 1.2. *For a bounded operator $T \in L(X)$, X a Banach space, and any $\lambda_0 \in \mathbb{C}$ the following implications hold:*

(i) *$H_0(\lambda_0 I - T)$ closed $\Rightarrow H_0(\lambda_0 I - T) \cap K(\lambda_0 I - T) = \{0\} \Rightarrow T$ has SVEP at λ_0* [4].

(ii) *If $\sigma_a(T)$ does not cluster at λ_0 then T has SVEP at λ_0*, [7].

The following equivalences have been proved in [4] and [7]. These equivalences have been established in the more general situation of operators on Banach spaces which are of Kato-type. Recall that every semi-Fredholm operator is of Kato-type by the classical result of Kato [22].

THEOREM 1.3. *If $T \in \Phi_\pm(X)$ the following statements are equivalent:*

(i) *T has SVEP at λ_0;*
(ii) *$p(\lambda_0 I - T) < \infty$;*
(iii) *$\sigma_a(T)$ does not cluster at λ_0;*
(iv) *$H_0(\lambda_0 I - T)$ is finite-dimensional.*

2. Weyl's theorems

We first begin by establishing several equivalences for every bounded operators defined on Banach spaces. If $T \in L(X)$, we let

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\},$$

and, if we write $\operatorname{iso} K$ for the set of all isolated points of $K \subseteq \mathbb{C}$, then we let

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$$

denote the set of isolated eigenvalues of finite multiplicities. Clearly,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X).$$

Theorem 2.1. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $\pi_{00}(T) = p_{00}(T)$;
(ii) $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \emptyset$;
(iii) $\sigma_{\mathrm{sf}}(T) \cap \pi_{00}(T) = \emptyset$;
(iv) $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi_{00}(T)$;
(v) $H_0(\lambda I - T)$ *is finite-dimensional for all* $\lambda \in \pi_{00}(T)$;
(vi) $K(\lambda I - T)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;
(vii) $(\lambda I - T)^\infty(X)$ *is finite-codimensional for all* $\lambda \in \pi_{00}(T)$;
(viii) $\beta(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$;
(ix) $q(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$.

Proof. (i) ⇒ (ii) We have $p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T)$, so $\sigma_{\mathrm{b}}(T) \cap p_{00}(T) = \emptyset$ and this obviously implies $\sigma_{\mathrm{w}}(T) \cap \pi_{00}(T) = \emptyset$, since $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$.

(ii) ⇒ (iii) Obvious, since $\sigma_{\mathrm{sf}}(T) \subseteq \sigma_{\mathrm{w}}(T)$.

(iii) ⇒ (iv) If $\lambda \in \pi_{00}(T)$ then $\lambda I - T$ is semi-Fredholm, hence $(\lambda I - T)(X)$ is closed.

(iv) ⇒ (v) Let $\lambda \in \pi_{00}(T)$. If $(\lambda I - T)(X)$ is closed then $\lambda I - T \in \Phi_+(X)$. Since T has SVEP at every isolated point of $\sigma(T)$, by Theorem 1.3 then $H_0(\lambda I - T)$ is finite-dimensional.

(v) ⇒ (vi) If λ is an isolated point of $\sigma(T)$, then $H_0(\lambda I - T)$ is the range of the spectral projection P associated with the spectral set $\{\lambda\}$, see Proposition 49.1 of Heuser [21], whilst $K(\lambda I - T)$ is the kernel of P, see [30], so $X =$

$H_0(\lambda I - T) \oplus K(\lambda I - T)$. Hence, if $H_0(\lambda I - T)$ is finite-dimensional then $K(\lambda I - T)$ is finite-codimensional.

(vi) $\Rightarrow$ (vii) Immediate, since $K(\lambda I - T) \subseteq (\lambda I - T)^\infty(X)$ for every $\lambda \in \mathbb{C}$.

(vii) $\Rightarrow$ (viii) Clear, since $(\lambda I - T)^\infty(X) \subseteq (\lambda I - T)(X)$ for every $\lambda \in \mathbb{C}$.

(viii) $\Rightarrow$ (i) For every $\lambda \in \pi_{00}(T)$ we have $\alpha(\lambda I - T) < \infty$, and hence if $\beta(\lambda I - T) < \infty$ then $\lambda I - T \in \Phi(X)$. Since λ is an isolated point of $\sigma(T)$ SVEP of T and T^* at λ ensures that $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite, by Theorem 1.3 and Theorem 2.9 of [4]. Therefore $\pi_{00}(T) \subseteq p_{00}(T)$, and since the opposite inclusion is satisfied by every operator it then follows that $\pi_{00}(T) = p_{00}(T)$.

(i) $\Rightarrow$ (ix) Clear.

(ix) $\Rightarrow$ (viii) If $q(\lambda I - T) < \infty$ by Proposition 38.5 of [21] we have $\beta(\lambda I - T) \leq \alpha(\lambda I - T) < \infty$. ∎

The *reduced minimum modulus* of a non-zero operator T is defined by

$$\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}.$$

It is well-known that $T(X)$ is closed if and only if $\gamma(T) > 0$.

Theorem 2.2. *The statements* (i)-(ix) *of* Theorem 2.1 *are equivalent to the following condition:*

(x) *The mapping* $\lambda \to \gamma(\lambda I - T)$ *is not continuous at each* $\lambda_0 \in \pi_{00}(T)$.

Proof. Observe first that if $\lambda_0 \in \pi_{00}(T)$ there exists a punctured disc $\mathbb{D}_0$ centered at λ_0 such that

$$\gamma(\lambda I - T) \leq |\lambda - \lambda_0| \quad \text{for all } \lambda \in \mathbb{D}_0. \tag{1}$$

In fact, if λ_0 is isolated in $\sigma(T)$ then $\lambda I - T$ is invertible, and hence has closed range, in a punctured disc $\mathbb{D}$ centered at λ_0. Take $0 \neq x \in \ker(\lambda_0 I - T)$. Then

$$\begin{aligned} \gamma(\lambda I - T) &\leq \frac{\|(\lambda I - T)x\|}{\text{dist }(x, \ker(\lambda I - T))} = \frac{\|(\lambda I - T)x\|}{\|x\|} \\ &= \frac{\|(\lambda I - T)x - (\lambda_0 I - T)x\|}{\|x\|} = |\lambda - \lambda_0|. \end{aligned}$$

Clearly, from the estimate (1) it follows $\gamma(\lambda I - T) \to 0$ as $\lambda \to \lambda_0$ so the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at a point $\lambda_0 \in \pi_{00}(T)$ if and only if $\gamma(\lambda_0 I - T) > 0$, or equivalently, $(\lambda_0 I - T)(X)$ is closed. Therefore the condition (iv) of Theorem 2.1 is equivalent to the condition (x). ∎

Following Coburn [10], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T).$$

A classical result of H. Weyl [34] prove that hermitian operators satisfy Weyl's theorem. This result has been extended to several classes of operators, for instance to hyponormal operators, to Toeplitz operators, and to other classes of operators including seminormal operators [8], see [9] and [10].

Theorem 2.3. *Suppose that $T \in L(X)$, or T^* has SVEP. Then Weyl's theorem holds for T if and only if one of the equivalent conditions* (i)-(ix) *of* Theorem 2.1, *or the condition* (x) *of* Theorem 2.2, *holds. If both T and T^* have SVEP, then Weyl's theorem holds for T if and only if $\sigma_f(T) \cap \pi_{00}(T) = \emptyset$.*

Proof. If T or T^* has SVEP then $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, see Corollary 2.8 of [3]. If Weyl's theorem holds for T then

$$\pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T),$$

so the condition (i) of Theorem 2.1 is satisfied. Conversely, suppose that $\pi_{00}(T) = p_{00}(T)$. Then $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$.

If both T and T^* have SVEP then $\sigma_f(T) = \sigma_{\mathrm{w}}(T)$, see Corollary 2.12 of [6]. ∎

Theorem 2.3 improves a recent result obtained by Curto and Han [12], S. V. Djordjević, Duggal and Han [17]. They showed the result of Theorem 2.3 only in the case T has SVEP. Note that in [12, Theorem 2.2] the equivalence of the conditions (i)-(ix) of Theorem 2.1, and of the condition (x) of Theorem 2.2 was proved under the assumption that T has SVEP. Our methods, which are simpler, also show that these equivalences are valid for *any* operator $T \in L(X)$.

The condition (v) of Theorem 2.1, is an useful tool in order to prove that Weyl's theorem holds for several important classes of operators. To see this we need to show an intermediate result. Recall first that a bounded operator $T \in L(X)$ is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Theorem 2.4. *Suppose that for $T \in L(X)$, X a Banach space, the following condition holds:*

$$H_0(\lambda I - T) = \ker(\lambda I - T) \quad \textit{for all } \lambda \in \mathbb{C}. \tag{2}$$

Then Weyl's theorem holds for $f(T)$, where f is any analytic function defined on an open neighborhood of $\sigma(T)$.

Proof. Clearly, T has SVEP by Theorem 1.2. Moreover, by definition of $\pi_{00}(T)$ the quasi-nilpotent part $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$. Hence by Theorem 2.3 Weyl's theorem holds for T. Now, by Corollary 2.6 of Curto and Han [12] the spectral mapping theorem holds for T, i.e., $\sigma_{\mathrm{w}}(f(T)) = f(\sigma_{\mathrm{w}}(T))$, see also [2].

We show now that T is isoloid. Since 0 is an isolated point in $\sigma(T)$, then 0 is a non-removable singularity of $(\lambda I - T)^{-1}$, and hence admits the Laurent expansion

$$(\lambda I - T)^{-1} = \sum_{n=1}^{\infty} \frac{P_n}{\lambda^n} + \sum_{n=0}^{\infty} \lambda^n Q_n$$

for every λ for which $0 < |\lambda| < \varepsilon$, with P_n, $Q_n \in L(X)$. Since $P_1 = P$ and $P_n = T^{n-1}P$, for all $n = 1, 2, \ldots$ (cf. [21, p. 209]), from $TP = 0$ it follows that $P_n = 0$ for all $n \geq 2$. Hence 0 is a simple pole of the resolvent $(\lambda I - T)^{-1}$. By Proposition 50.2 of [21] we conclude that $p(T) = q(T) = 1$ and 0 is an eigenvalue of T. Therefore T is isoloid. By [25, Lemma] it then follows that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_{\mathrm{w}}(T)),$$

and hence $\sigma_{\mathrm{w}}(f(T)) = \sigma(f(T)) \setminus \pi_{00}(f(T))$, so $f(T)$ satisfies Weyl's theorem. ∎

Remark 2.5. Theorem 2.4 applies to several operators and subsumes results proved for special classes of operators:

(a) A map $T : A \to A$ is said a *multiplier* if $(Tx)y = x(Ty)$ holds for all $x, y \in \mathcal{A}$. Every multiplier of a semi-simple Banach algebra A satisfies the condition (2). In fact, $\lambda I - T$ is a multiplier for every $\lambda \in \mathbb{C}$, so by Theorem 1.8 of [4] the condition (2) is satisfied. In particular, if T_μ is the convolution operator of the group algebra $L_1(G)$ of a locally compact Abelian group G then Weyl's theorem holds for T_μ.

(b) Recall that $T \in L(X)$ is said *paranormal* if $\|Tx\| \leq \|T^2x\|\|x\|$ for all $x \in X$. T is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$. Every totally paranormal operator satisfies the condition (2), see Laursen [23]. The fact that Weyl's theorem for $f(T)$ has been observed in [32, Theorem 2.7].

A bounded operator $T \in L(H)$ on a Hilbert space is said to be *hyponormal* if $\|T^*x\| \leq \|Tx\|$ for all $x \in X$. It is easily seen that every hyponormal operator is totally paranormal. The class of totally paranormal operators includes also subnormal operators and quasi-normal operators, since these operators are hyponormal, see [11]. Theorem 2.4 then implies Weyl's theorem for these classes of operators, see [8], [9] and [10].

(c) A bounded operator $T \in L(X)$ is said to be *transaloid* if the spectral radius $r(\lambda I - T)$ is equal to $\|\lambda I - T\|$ for every $\lambda \in \mathbb{C}$. If T is transaloid then T satisfies the condition (2), see Lemma 2.3 and Lemma 2.4 of [12]. Weyl's theorem for $f(T)$ has been proved in [12, Theorem 2.5]. It should be noted that the assumption of SVEP in Lemma 2.4 and Theorem 2.5 of [12] is redundant.

The condition (v) of Theorem 2.1 generally does not ensure that T has SVEP. This is the case, see [25], of operators for which the following *growth condition* G_m ($m \geq 1$) holds: there is a $K > 0$ for which

$$\|(\lambda I - T)^{-1}\| \leq \frac{K}{\text{dist }(\lambda, \sigma(T))^m} \quad \text{for all } \lambda \in \sigma(T)$$

Clearly, by Theorem 2.3 it follows that, if we add SVEP to the condition G_m, then Weyl's theorem holds for T, as it has been observed in [25, Corollary].

In the following theorem we shall consider operators $T \in L(X)$ for which the condition $K(T) = \{0\}$ holds. This condition may be viewed as an abstract shift condition since it is verified by every *weighted unilateral right shift* T on $\ell^p(\mathbb{N})$, with $1 \leq p < \infty$, defined by

$$Tx := \sum_{n=1}^{\infty} \omega_n x_n e_{n+1} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N}),$$

where the weight $\omega = (\omega_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive real numbers, and (e_n) stands for the canonical basis of $\ell^p(\mathbb{N})$. In fact for these operators it is easily seen that $T^\infty(X) = \{0\}$ and hence $K(T) = \{0\}$.

Observe that Theorem 2.3 works for unilateral weighted left shift T on $\ell^p(\mathbb{N})$. In fact, although these operators need not to have SVEP the dual T^* are right shifts and hence have SVEP, see Theorem 16 of [5]. Moreover, Theorem 2.3 applies also to bilateral weighted shifts on $\ell^p(\mathbb{Z})$, see [24] for relevant definitions, since, by Theorem 18 and Corollary 19 of [5], at least one of the operators T and T^* has SVEP.

THEOREM 2.6. *Suppose that for a bounded operator $T \in L(X)$ on a Banach space X we have $K(T) = \{0\}$. If T is not quasi-nilpotent then T obeys Weyl's theorem.*

Proof. The condition $K(T) = \{0\}$ entails that T has SVEP. In fact, since $\ker\,(\lambda I - T) \subseteq K(T)$ for all $\lambda \neq 0$, from our assumption we obtain that $\ker\,(\lambda I - T) \cap K(\lambda I - T) = \{0\}$ for every $\lambda \in \mathbb{C}$, and hence, by Corollary 3 of [5], T has SVEP at every $\lambda \in \mathbb{C}$. Moreover, $\sigma(T) = \sigma_{\rm b}(T) = \sigma_{\rm w}(T)$ and $\sigma(T)$ is connected, see [31]. From this it follows that $\sigma(T) \setminus \sigma_{\rm w}(T) = \emptyset$. Since $\ker\,(\lambda I - T) = \{0\}$ for all $\lambda \neq 0$ we then conclude that $\pi_{00}(T) \subseteq \{0\}$. Now, suppose that T does not satisfy Weyl's theorem. Then $\pi_{00}(T) = \{0\}$, so 0 is an isolated point of $\sigma(T)$. Since $\sigma(T)$ is connected it then follows that $\sigma(T) = \{0\}$, i.e., T is quasi-nilpotent. ∎

Corollary 2.7. *If $T \in L(X)$ is a non quasi-nilpotent weighted unilateral right shift T on $\ell^p(\mathbb{N})$, with $1 \leq p < \infty$, then T obeys Weyl's theorem.*

Note that, by Theorem 1.6.3 of [24], the condition of being non-quasi-nilpotent for weighted unilateral right shift T is equivalent to several other conditions, for instance to be not decomposable, or not to be a Riesz operator.

Remark 2.8. The previous theorem cannot, in general, be reversed. In fact, if V is the quasi-nilpotent Volterra operator on the Banach space $X := C[0,1]$, defined by

$$(Vf)(t) := \int_0^t f(s)\mathrm{d}s \text{ for all } f \in C[0,1] \text{ and } t \in [0,1],$$

then $K(V) = \{0\}$, since V is quasi-nilpotent, see [26]. Moreover, since V is injective we have $\pi_{00}(V) = \emptyset = \sigma(V) \setminus \sigma_{\rm w}(V)$. We note in passing that this argument also shows, in general, that if $K(T) = \{0\}$ and T is injective then T obeys Weyl's theorem. Consequently, every weighted right shift with none of the weights $\omega_n = 0$ obeys Weyl's theorem.

Define

$$\pi^{\rm a}_{00}(T) := \{\lambda \in \text{iso } \sigma_{\rm a}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Clearly, for every $T \in L(X)$, we have

$$p_{00}(T) \subseteq \pi_{00}(T) \subseteq \pi^{\rm a}_{00}(T)$$

Theorem 2.9. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) $\sigma_{\rm ub}(T) \cap \pi^{\rm a}_{00}(T) = \emptyset$;
(ii) $\sigma_{\rm uf}(T) \cap \pi^{\rm a}_{00}(T) = \emptyset$;

(iii) $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi^{\rm a}_{00}(T)$*;*
(iv) $H_0(\lambda I - T)$ *is finite-dimensional for all* $\lambda \in \pi_{00}(T)$ *and* $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi^{\rm a}_{00}(T) \setminus \pi_{00}(T)$*;*
(v) $q(\lambda I - T) < \infty$ *for all* $\lambda \in \pi_{00}(T)$ *and* $(\lambda I - T)(X)$ *is closed for all* $\lambda \in \pi^{\rm a}_{00}(T) \setminus \pi_{00}(T)$*;*
(vi) *The mapping* $\lambda \to \gamma(\lambda I - T)$ *is not continuous at each* $\lambda_0 \in \pi^{\rm a}_{00}(T)$*.*

Proof. (i) $\Rightarrow$ (ii) is clear, since $\sigma_{\rm uf}(T) \subseteq \sigma_{\rm ub}(T)$.

The implication (ii) $\Rightarrow$ (iii) follows since $\lambda I - T \in \Phi_+(X)$ for every $\lambda \in \pi^{\rm a}_{00}(T)$.

(iii) $\Rightarrow$ (iv) From $\pi_{00}(T) \subseteq \pi^{\rm a}_{00}(T)$ we see that $\lambda I - T \in \Phi_+(X)$ for each $\lambda \in \pi_{00}(T)$. The SVEP at every $\lambda \in \pi_{00}(T)$ is equivalent, by Theorem 1.3, to saying that $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$. The second assertion is obvious.

(iv) $\Rightarrow$ (v) Since, as observed in the proof of Theorem 2.1, for every isolated point $\lambda \in \sigma(T)$ we have $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$, then $K(\lambda I - T)$ has finite codimension for every $\lambda \in \pi_{00}(T)$. For every $\lambda \in \mathbb{C}$ the inclusion $K(\lambda I - T) \subseteq (\lambda I - T)(X)$ holds, which obviously implies that $\beta(\lambda I - T) < \infty$. Therefore, $\lambda I - T \in \Phi(X)$ for all $\lambda \in \pi_{00}(T)$. Finally, the SVEP of T^* at λ implies, by Theorem 2.9 of [4], that $q(\lambda I - T) < \infty$ for all $\lambda \in \pi_{00}(T)$.

(v) $\Rightarrow$ (i) Suppose that $\lambda \in \pi^{\rm a}_{00}(T) \setminus \pi_{00}(T)$. Then $(\lambda I - T)(X)$ is closed and from the inclusion $\alpha(\lambda I - T) < \infty$ we see that $\lambda I - T \in \Phi_+(X)$. By assumption $\sigma_{\rm a}(T)$ does not cluster at λ, hence by Theorem 1.3 $p(\lambda I - T) < \infty$. Therefore $\lambda \notin \sigma_{\rm ub}(T)$.

Suppose now the other case, $\lambda \in \pi_{00}(T)$. By [21, Proposition 38.5] the condition $q(\lambda I - T) < \infty$ implies that $\beta(\lambda I - T) \le \alpha(\lambda I - T) < \infty$, and hence $\lambda I - T \in \Phi(X)$. Finally, by Theorem 1.3 the SVEP at the isolated point λ of $\sigma(T)$ implies that $p(\lambda I - T) < \infty$, and hence $\lambda \notin \sigma_{\rm ub}(T)$.

(vi) $\Leftrightarrow$ (iv) The proof is analogous to that of Theorem 2.2. ∎

The *Weyl (or essential) approximate point spectrum* $\sigma_{\rm wa}(T)$ of a bounded operator $T \in L(X)$ is the complement of those $\lambda \in \mathbf{C}$ for which $\lambda I - T \in \Phi_+(X)$ and ind $(\lambda I - T) \le 0$. Note that $\sigma_{\rm wa}(T)$ is the intersection of all approximate point spectra $\sigma_{\rm a}(T + K)$ of compact perturbations K of T, see [27]. The *Weyl surjectivity spectrum* $\sigma_{\rm ws}(T)$ is the complement of those $\lambda \in \mathbf{C}$ for which $\lambda I - T \in \Phi_-(X)$ and ind $(\lambda I - T) \ge 0$. Note that $\sigma_{\rm ws}(T)$ is the intersection of all surjectivity spectra $\sigma_s(T + K)$ of compact perturbations K of T, see [27]. Clearly,

$$\sigma_{\rm wa}(T) \subseteq \sigma_{\rm ub}(T) \quad \text{and } \sigma_{\rm ws}(T) \subseteq \sigma_{\rm lb}(T).$$

Moreover, the two spectra $\sigma_{\rm wa}(T)$ and $\sigma_{\rm ws}(T)$ are dual each other, i.e.,

$$\sigma_{\rm wa}(T) = \sigma_{\rm ws}(T^\star) \quad \text{and} \quad \sigma_{\rm ws}(T) = \sigma_{\rm wa}(T^\star).$$

Furthermore, $\sigma_{\rm w}(T) = \sigma_{\rm wa}(T) \cup \sigma_{\rm ws}(T)$.

Theorem 2.10. *If T or T^* has SVEP then*

$$\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T) \quad \textit{and} \quad \sigma_{\rm ws}(T) = \sigma_{\rm lb}(T).$$

Proof. Suppose first that T has SVEP. By Corollary 2.4 of [28] we know that $\sigma_{\rm ub}(T) = \sigma_{\rm wa}(T) \cup \text{acc}\ \sigma_a(T)$, so, to show that $\sigma_{\rm ub}(T) = \sigma_{\rm wa}(T)$, it suffices to prove that acc $\sigma_a(T) \subseteq \sigma_{\rm wa}(T)$.

Suppose that $\lambda \notin \sigma_{\rm wa}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and the SVEP at λ ensures that $\sigma_a(T)$ does not cluster at λ, by Theorem 1.3. Hence $\lambda \notin$ acc $\sigma_{ap}(T)$.

To show the equality $\sigma_{\rm lb}(T) = \sigma_{\rm ws}(T)$ suppose that $\lambda \notin \sigma_{\rm lb}(T)$. Then $\lambda I - T \in \Phi_-(X)$ and $q(\lambda I - T) < \infty$. Since $\lambda I - T \in \Phi_-(X)$ the SVEP at λ implies by Theorem 1.3 that $p(\lambda I - T) < \infty$. Hence $p(\lambda I - T) = q(\lambda I - T) < \infty$, and consequently $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$, see Proposition 38.6 of Heuser [21]. Therefore $\lambda \notin \sigma_{\rm ws}(T)$ and this shows that $\sigma_{\rm ws}(T) \subseteq \sigma_{\rm lb}(T)$.

On the other hand, if $\lambda \notin \sigma_{\rm ws}(T)$ then $\lambda I - T \in \Phi_-(X)$ with $\beta(\lambda I - T) \leq \alpha(\lambda I - T)$. Again, the SVEP at λ gives by Theorem 1.3 that $p(\lambda I - T) < \infty$, hence, always by Proposition 38.5 of [21], $\alpha(\lambda I - T) = \beta(\lambda I - T)$. At this point, by Proposition 38.6 of Heuser [21] the finiteness of $p(\lambda I - T)$ implies that also $q(\lambda I - T)$ is finite, and hence $\lambda \notin \sigma_{\rm lb}(T)$. Therefore, $\sigma_{\rm lb}(T) \subseteq \sigma_{\rm ws}(T)$ and the proof of the second equality is complete in the case that T has SVEP.

Suppose now that T^* has SVEP. Then by the first part $\sigma_{\rm ub}(T^*) = \sigma_{\rm wa}(T^*)$ and $\sigma_{\rm lb}(T^*) = \sigma_{\rm ws}(T^*)$. By duality it then follows that $\sigma_{\rm lb}(T) = \sigma_{\rm ws}(T)$ and $\sigma_{\rm ub}(T) = \sigma_{\rm wa}(T)$. ∎

We say that a-Weyl's theorem holds for $T \in L(X)$ if

$$\pi^{\rm a}_{00}(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm wa}(T).$$

It is known [27] that

$$a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem}.$$

In [27] it is shown that cohyponormal operators obey to the a-Weyl's theorem. In [16] it is shown that if T^* is quasihyponormal then a-Weyl's theorem holds for T.

Theorem 2.11. *If T or T^* has SVEP, then a-Weyl's theorem holds for T if and only if one of the equivalent conditions* (i)-(vi) *of* Theorem 2.9 *holds.*

Proof. Observe first that both $\pi_{00}^{\mathrm{a}}(T)$ and $\sigma_{\mathrm{ub}}(T)$ are subsets of $\sigma_{\mathrm{a}}(T)$ and the condition $\sigma_{\mathrm{ub}}(T) \cap \pi_{00}^{\mathrm{a}}(T) = \emptyset$ obviously implies that

$$\pi_{00}^{\mathrm{a}}(T) \subseteq \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) \subseteq \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T).$$

To establish the a-Weyl's theorem for T we need to show the reverse inclusion $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) \subseteq \pi_{00}^{\mathrm{a}}(T)$.

Suppose that T has SVEP and $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$. Then $\lambda \in \sigma_{\mathrm{a}}(T)$ and $\lambda I - T \in \Phi_+(X)$. Clearly, since $(\lambda I - T)(X)$ is closed, $\lambda I - T$ is not injective and hence $0 < \alpha(\lambda I - T) < \infty$. On the other hand, since $\lambda I - T \in \Phi_+(X)$, by Theorem 1.3 the SVEP at λ is equivalent to saying that $\sigma_{\mathrm{a}}(T)$ does not cluster at λ, and consequently $\lambda \in \pi_{00}^{\mathrm{a}}(T)$.

Finally, suppose that T^* has SVEP and $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$. The SVEP of T^* yields, by Corollary 2.9 of [3] and Theorem 2.10, that

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T).$$

Moreover, $\sigma_{\mathrm{a}}(T) = \sigma(T)$ by [24, Proposition 1.3.2], and hence

$$\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = p_{00}(T) \subseteq \pi_{00}^{\mathrm{a}}(T),$$

so the proof is complete. ∎

The equivalences established in Theorem 2.11 was proved by Curto and Han [12, Corollary 3.3] under the condition that T has SVEP. Our methods are, which are considerably simpler, show that the equivalences established in Theorem 2.9 are valid also without assuming SVEP.

Theorem 2.12. *If T^* has SVEP then the following statements are equivalent:*

(i) *Weyl's theorem holds for T;*
(ii) *a-Weyl's theorem holds for T.*

Proof. We have only to show the implication (i) ⇒ (ii). If T^* has SVEP then $\sigma_{\mathrm{a}}(T) = \sigma(T)$, see [24, Proposition 1.3.2], so $\pi_{00}^{\mathrm{a}}(T) = \pi_{00}(T)$. Again, as in the proof of Theorem 2.11, since T^* has SVEP then

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T).$$

If Weyl's theorem holds for T then

$$\pi^{\mathrm{a}}_{00}(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T),$$

and hence a-Weyl's theorem holds for T. ∎

Let us consider for a weighted unilateral right shift T on $\ell^p(\mathbb{N})$, with weight $\omega = (\omega_n)_{n\in\mathbb{N}}$, the following quantity:

$$c(T) := \lim_{n\to\infty} \inf(\omega_1 \cdots \omega_n)^{1/n}.$$

Corollary 2.13. *If $T \in L(X)$ is a non quasi-nilpotent weighted unilateral right shift T on $\ell^p(\mathbb{N})$, with $1 \le p < \infty$. If $c(T) = 0$, then T obeys a-Weyl's theorem.*

Proof. If $c(T) = 0$ then T^* has SVEP, see Theorem 16 of [5]. Moreover, by Corollary 2.7 Weyl's theorem holds for T, so Theorem 2.12 applies to T. ∎

A bounded operator $T \in L(X)$ is said to be *reguloid* if for every isolated point λ of $\sigma(T)$, $\lambda I - T$ is relatively regular, i.e., there exists $S_\lambda \in L(X)$ such that $(\lambda I - T)S_\lambda(\lambda I - T) = \lambda I - T$. It is well known that $T \in L(X)$ is relatively regular operator if and only $\ker T$ and $T(X)$ are complemented. It is easily seen that every reguloid operator is isoloid [20, Theorem 14].

Examples of reguloid operators are multipliers of semi-simple Banach algebras, because every isolated point λ_0 of $\sigma(T)$ is a simple pole of the resolvent $R(\lambda, T) := (\lambda I - T)^{-1}$, see [1], so $\ker\ (\lambda_0 I - T)$ is the range of the spectral projection P_0 associated with λ_0, whilst $(\lambda_0 I - T)(X)$ is the kernel of P_0, see [21, Proposition 50.2]. Clearly, by Theorem 2.3 every reguloid operator T for which T or T^* has SVEP obeys to Weyl's theorem, since the condition (iv) of Theorem 2.1 is satisfied.

Theorem 2.14. *Suppose that $T \in L(X)$ is reguloid and let f be any analytic function defined on an open neighborhood $\mathcal{U}$ of $\sigma(T)$. The following assertions hold:*

(i) *If T or T^* has SVEP then Weyl's theorem holds for $f(T)$.*
(ii) *If T^* has SVEP, then a-Weyl's theorem holds for $f(T)$.*

Proof. Every reguloid operator is isoloid. Therefore, by [25, Lemma]

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)).$$

If T or T^* has SVEP then $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T)$, by Corollary 2.8 of [3], and by Corollary 2.6 of [12] we also have $f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T))$. Since T or T^* has SVEP then Weyl's theorem holds for T, so $\sigma(T) \setminus \pi_{00}(T) = \sigma_{\mathrm{w}}(T)$. Therefore $\sigma(f(T)) \setminus \pi_{00}(f(T)) = \sigma_{\mathrm{w}}(f(T))$, i.e. Weyl's theorem holds for $f(T)$.

If T^* has SVEP then by Theorem 3.3.6 [24] $f(T^*) = f(T)^*$ has SVEP, so, by part (i) and by Theorem 2.12, a-Weyl's theorem holds for $f(T)$. ∎

Part (i) of Theorem 2.14 improves a similar result of [12]. In fact, in Corollary 3.4 of [12] the assumption of the Dunford property (C) for T or T^* is stronger than the SVEP for T or T^*. An example of operator T having SVEP but not property (C) may be found in [4]. Other examples of operators which have SVEP but without property (C) may be found among multipliers of the group algebra $L^1(G)$, see Chapter 4 of [24]. It should be noted that Theorem 2.14 also extends the result of Theorem 2.8 of [17] where Weyl's theorem was established only for reguloid operators having SVEP.

For a commutative semi-simple Banach algebra A, let $M(A)$ denote the commutative Banach algebra of all multipliers. The following result is immediate, since property (δ) for any operator T entails SVEP for T^*.

Corollary 2.15. *Suppose that $T \in M(A)$, A a commutative semi-simple Banach algebra, has property (δ). Then a-Weyl's theorem holds for $f(T)$.*

The previous result leads to the following interesting application. Let $M(G)$ be the canonical measure algebra of a locally compact Abelian group G and denote by $T_\mu : L^1(G) \to L^1(G)$ the convolution operator on the group algebra $L_1(G)$. Denote by $\hat{\mu}$ the *Fourier-Steltjes transform* of μ defined on the dual group $\widehat{G}$, and let $M_0(G)$ be the ideal in $M(G)$ of all measures μ for which $\hat{\mu}$ vanish at infinity on $\widehat{G}$.

Corollary 2.16. *Suppose that G is a compact Abelian group. If $\mu \in M_0(G)$ has denumerable spectrum, then a-Weyl's theorem holds for $f(T_\mu)$.*

Proof. By Theorem 4.11.8 of [24] T_μ has property (δ). ∎

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