Representation of Operators with Martingales and the Radon-Nikodým Property

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The Radon-Nikodým property was introduced to describe those Banach spaces X for which all operators acting between L_1 and X have a representation function. These spaces can be characterized in terms of martingales, as those spaces in which every uniformly bounded martingale converges. In the present work we study some classes of operators defined upon their behaviour with respect to the convergence of such martingales. We prove that an operator preserves the non-convergence of uniformly bounded martingales if and only if all of its compact perturbations have Asplund cokernel.

In the following, X, Y and Z will be Banach spaces, and $\mathcal{L}(X,Y)$ will be the class of all bounded linear operators acting between X and Y. Σ will stand for the Lebesgue σ-field on the unit interval $[0,1]$, and μ will be the Lebesgue measure on Σ.

DEFINITION 1. [3] Let Σ_0 be a subset of Σ. It is said that Σ_0 is a sub-σ-field of Σ if it is a σ-field in its own right. Given a μ-measurable function $f \in L_1(X)$, its conditional expectation with respect to Σ_0 is a $\mu|_{\Sigma_0}$-measurable function f_0 such that

$$\int_E f \, d\mu = \int_E f_0 \, d\mu$$

for every $E \in \Sigma_0$.

Even though the existence of such a conditional expectation is not clear from the definition, it can be proven [3] that it exists irrespectively of the function and the sub-σ-field chosen. We denote it by

$$f_0 = E_{\Sigma_0}(f).$$

For a fixed sub-σ-field Σ_0, the mapping E_{Σ_0} is a bounded linear operator from $L_1(X)$ to itself.

DEFINITION 2. Let $(\Sigma_n)_{n\in\mathbb{N}}$ be an increasing sequence of sub-σ-fields of Σ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions where each f_n is $\mu|_{\Sigma_n}$-measurable (and hence μ-measurable). The sequence $(f_n, \Sigma_n)_{n\in\mathbb{N}}$ is called a martingale if

$$f_n = E_{\Sigma_n}(f_m)$$

whenever $m > n$.

If there is no possible confusion about the family of sub-σ-fields involved, we will just say that $(f_n)_{n\in\mathbb{N}}$ is a martingale.

The easiest way to construct a martingale is to take any increasing sequence $(\Sigma_n)_{n\in\mathbb{N}}$ of sub-σ-fields of Σ, pick a μ-measurable function $f \in L_1(X)$ and then define $f_n = E_{\Sigma_n}(f)$. It is immediate to see that the sequence obtained this way is indeed a martingale. Martingales that can be built this way are called convergent martingales.

Martingales are known to be a useful tool in the study of operators from L_1 into X. In fact, they can be seen as an extension of the notion of representation function. Let us recall that an operator $T \in \mathcal{L}(L_1, X)$ is said to be representable if there exists a function $g \in L_\infty(X)$ such that

$$T(f) = \int fg\, d\mu$$

for every $f \in L_1$, in which case g is called the representation function of T. Unless X has the Radon-Nikodým property, not all operators in $\mathcal{L}(L_1, X)$ are representable, and this is the reason why martingales can aid in their study, since they can play the role of a representation function without imposing any special property on X to exist. The following results back up this statement.

PROPOSITION 3. *Let $(f_n, \Sigma_n)_{n\in\mathbb{N}}$ be a uniformly bounded martingale in X such that $\bigcup_{n\in\mathbb{N}} \Sigma_n$ generates Σ as a σ-field. Then there exists an operator $A \in \mathcal{L}(L_1, X)$ such that*

$$A(\chi_E) = \lim_n \int_E f_n\, d\mu$$

for every $E \in \Sigma$, whose norm is bounded by $\sup_{n \in \mathbb{N}} \|f\|_\infty$.

Whenever this situation arises, that is, whenever an operator $A \in \mathcal{L}(L_1, X)$ admits a uniformly bounded martingale $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ such that

$$A(\chi_E) = \lim_n \int_E f_n \, d\mu$$

for every $E \in \Sigma$, we will say that $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ is a *representation martingale* for A. Note that the representation martingale must be unique for a given family $(\Sigma_n)_{n \in \mathbb{N}}$.

Going the other way, which means getting a martingale from an operator, is also possible, albeit with some restrictions on the σ-fields involved.

PROPOSITION 4. *Let $A \in \mathcal{L}(L_1, X)$ be an operator and let $(\Sigma_n)_{n \in \mathbb{N}}$ be an increasing family of sub-σ-fields of Σ where every Σ_n is finite and $\bigcup_{n \in \mathbb{N}} \Sigma_n$ generates Σ as a σ-field. Then there exist functions $(f_n)_{n \in \mathbb{N}}$ such that $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ is a representation martingale for A.*

So it can be seen that any operator lends itself to having a representation martingale, at least for the right choice of σ-fields. It should be noted that this is not overly restrictive, because the existence of just one such family of σ-fields is enough for our purposes; for instance, Σ_n could be the (finite) σ-field spawned by the 2^n n-order dyadic intervals in $[0, 1]$.

Now that we have defined what a representation martingale is, the natural question to ask is what relationship it holds with representation functions. For the first thing, it is clear that, given an operator A with representation function g, the constant martingale $(g, \Sigma)_{n \in \mathbb{N}}$ represents g. Furthermore, given any increasing family $(\Sigma_n)_{n \in \mathbb{N}}$ of sub-σ-fields of Σ, the convergent martingale $(E_{\Sigma_n}(g), \Sigma_n)_{n \in \mathbb{N}}$ also represents A. In fact, we have the following:

THEOREM 5. *Let $A \in \mathcal{L}(L_1, X)$ be an operator and let $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ be a representation martingale for A. Then A is representable if and only if $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ converges, in which case its limit is the representation function for A.*

Just as representation of operators is the key to the definition of Radon-Nikodým spaces, we can also define some operator classes based on representation martingales. Let us recall that a Banach space X is said to have the Radon-Nikodým property if every operator in $\mathcal{L}(L_1, X)$ is representable; X is said to be Asplund if X^* has the Radon-Nikodým property. Since any operator

must preserve martingales (if $(f_n, \Sigma_n)_{n \in \mathbb{N}}$ is a martingale, so is $(Tf_n, \Sigma_n)_{n \in \mathbb{N}}$), we will define a couple of operator classes based on their behaviour with convergent martingales. These classes are:

$$\mathrm{RN}(X, Y) = \{\, T \in \mathcal{L}(X, Y) : T \text{ takes any uniformly bounded martingale}$$
$$\text{into a convergent martingale}\}$$

$$\mathrm{RN}_+(X, Y) = \{\, T \in \mathcal{L}(X, Y) : T \text{ preserves non-convergence of uniformly}$$
$$\text{bounded martingales}\}$$

The class RN is an operator ideal, in the sense of Pietsch [6]; on the other hand, the structure of RN_+ is that of an operator semigroup, as described in [1]. An operator in RN_+ is used, for instance, to prove that the class of separable $\mathcal{L}^1$-spaces not containing a copy of L_1 has no universal element [2].

Our interest now will be in finding a perturbative characterization for the class $\mathrm{RN}_+(X, Y)$. By a perturbative characterization we mean the assertion that an operator belongs to this class if and only if every of its compact perturbations has some related property, in our case the Radon-Nikodým property. Typical examples of operators having a perturbative characterization include Fredholm and semi-Fredholm operators [5] and tauberian operators [4]. These examples also show that such a characterization is important because it summarizes the essential behaviour of the operators in the class. In our case, we have the following:

PROPOSITION 6. *Let $T \in \mathcal{L}(X, Y)$ be an operator. Then:*

(i) *If $T \in \mathrm{RN}_+(X, Y)$, then $N(T)$ has the Radon-Nikodým property.*

(ii) *If $T \in \mathrm{RN}_+(X, Y)$ and $A \in \mathrm{RN}(X, Y)$, then $T + A \in \mathrm{RN}_+(X, Y)$.*

(iii) *If T is compact, then $T \in \mathrm{RN}(X, Y)$.*

By combining these facts, it can be seen that for any operator $T \in \mathrm{RN}_+(X, Y)$ it holds that $N(T+K)$ has the Radon-Nikodým property for every compact perturbation K. The question now is whether the opposite holds, and we can give a partial affirmative answer, when the operator involved is a conjugate operator. The Radon-Nikodým property is more tractable in dual spaces, where it has been studied intensively [7], [8].

How do representation martingales help in this task? Although a representation martingale may have a rather general form, in practice, as said earlier, we will just restrict ourselves to a specific subset of them, namely those whose underlying σ-fields are spawned by the dyadic intervals.

To this end, let us define $I_i^n = \left[(i-1)/2^n, i/2^n\right)$ and $\chi_i^n = \chi_{I_i^n}$ for every $n \in \mathbb{N}$ and $1 \leq i \leq 2^n$, and let Σ_n be the (finite) σ-field spawned by $(I_i^n)_{i=1}^{2^n}$. Then any $\mu|_{\Sigma_n}$-measurable function f will take the form $f = \sum_{i=1}^{2^n} x_i \chi_i^n$ for some $x_1, \ldots, x_{2^n} \in X$. Moreover, the conditional expectation of any $\mu|_{\Sigma_{n+1}}$-measurable function $f = \sum_{i=1}^{2^{n+1}} x_i \chi_i^{n+1}$ with respect to Σ_n will be

$$E_{\Sigma_n}(f) = \sum_{i=1}^{2^n} \frac{x_{2i-1} + x_{2i}}{2} \chi_i^n,$$

so a martingale on $(\Sigma_n)_{n \in \mathbb{N}}$ can be identified with a family $(x_i^n)_{n,i}$ in X, where $n \in \mathbb{N}$ and $1 \leq i \leq 2^n$, which satisfies that every

$$x_i^n = \frac{x_{2i-1}^{n+1} + x_{2i}^n}{2}.$$

Such a family is called a tree in X.

PROPOSITION 7. *Let $T \in \mathcal{L}(X, Y)$ be an operator such that $T^* \notin$ $\mathrm{RN}_+(Y^*, X^*)$ and let $\lambda > 1$. Then there exist $\alpha > 0$ and a bounded tree $(z_i^n)_{n,i} \subseteq Y^*$ such that the difference sequence*

$$d_i^n = z_{2i}^{n+1} - z_{2i-1}^{n+1}$$

is a basic sequence with basis constant less than λ and whose coordinate functionals $(b_i^n)_{n,i} \subseteq Y$ satisfy, for every n, i,

$$\alpha/2 < \|d_i^n\| < 3;$$
$$\|T^* d_i^n\| < \alpha/4^n;$$
$$1/3 < \|b_i^n\| < 12/\alpha.$$

It must be noted that the subspace spawned by $(d_i^n)_{n,i}$ lacks the Radon-Nikodým property, since it contains the shifted tree $(z_i^n - z_1^0)_{n,i}$, which is bounded and $\alpha/4$-separated [3].

Now the path is clear to our main result:

THEOREM 8. *Let $T \in \mathcal{L}(X, Y)$ be an operator such that $T^* \notin$ $\mathrm{RN}_+(Y^*, X^*)$. Then there exists a nuclear operator $K \in \mathcal{L}(X, Y)$ such that $N(T^* - K^*)$ does not have the Radon-Nikodým property.*

The proof is as follows: Using the sequences $(b_i^n)_{n,i}$ and $(d_i^n)_{n,i}$ provided by Proposition 7, the operator K can be defined as

$$K(x) = \sum_{n,i} \langle T^* d_i^n, x \rangle b_i^n.$$

It is immediate to prove that K is nuclear and that $K^*(d_i^n) = T^*(d_i^n)$ for every n and i, so $N(T^* - K^*)$ contains the subspace spawned by $(d_i^n)_{n,i}$ and, therefore, cannot have the Radon-Nikodým property.

As a corollary, we obtain the following:

COROLLARY 9. *Let $T \in \mathcal{L}(X, Y)$ be an operator. The following are equivalent:*

(i) $T^* \in \mathrm{RN}_+$;

(ii) $A \in \mathrm{RN}(Z, Y^*)$ *whenever* $T^*A \in \mathrm{RN}(Z, X^*)$;

(iii) $N(T^* + K)$ *has the Radon-Nikodým property for every compact operator* $K \in \mathcal{L}(Y^*, X^*)$;

(iv) $Y / \overline{R(T + K)}$ *is Asplund for every compact operator* $K \in \mathcal{L}(X, Y)$.

It is clear from the definition and the remarks after Proposition 6 that (i) implies (ii) and also that (ii) implies (iii). The fact that (iii) implies (iv) comes from $N(T^* + K^*)$ being (isomorphic to) the dual of $Y / \overline{R(T + K)}$. Lastly, (iv) implies (i) because of the previous theorem.

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