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On Convexity, Smoothness and Renormings in the Study of Faces of the Unit Ball of a Banach Space

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It is well known (see [6]) for a subset C of a bounded closed convex subset H of a T_2 locally convex topological vector space X that

(i) C is a face of H if it is closed convex and for every $x, y \in H$ and every $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)\, y \in C$, then $x, y \in C$;
(ii) C is an exposed face of H if there exists f in X^* such that $C = \{x \in H : f(x) = \sup(f(H))\}$;
(iii) C is a strongly exposed face of H if there exists f in X^* verifying that $C = \{x \in H : f(x) = \sup(f(H))\}$ and for every open subset U of H with $C \subseteq U$, there exists $\delta > 0$ such that $\operatorname{slc}(H, f, \delta) \subseteq U$ (where $\operatorname{slc}(H, f, \delta) = \{h \in H : f(h) \geq \sup(f(H)) - \delta\}$ is the slice of H determined by f and δ).

If c is an element of H, then

(i) c is an extreme point of H if $\{c\}$ is a face of H (see [1]);
(ii) c is an exposed point of H if $\{c\}$ is an exposed face of H (see [6]);
(iii) c is a strongly exposed point of H if $\{c\}$ is a strongly exposed face of H (see [6]).

It is well known for a point x of the unit sphere of a Banach space X that

(i) x is a rotund point of B_X if every y in S_X, such that $\|(x+y)/2\| = 1$, verifies that $x = y$ (see [8]);
(ii) x is a locally uniformly rotund point of B_X if every sequence $(y_n)_{n\in\mathbb{N}}$ in S_X, such that $(\|(x+y_n)/2\|)_{n\in\mathbb{N}}$ converges to 1, verifies that $(y_n)_{n\in\mathbb{N}}$ converges to x (see [4]).

It is said that a Banach space is (*locally uniformly*) *rotund* if every point of its unit sphere is a (locally uniformly) rotund point of its unit ball.

It is clear that every locally uniformly rotund point is a strongly exposed point (for the vector topology given by the norm). Nevertheless, there exist rotund points which are not strongly exposed points. A Banach space is said to be *strongly exposed* if every point of its unit sphere is a strongly exposed point of its unit ball.

It is well known (see [9, Chapter 5.3]) for a point x of the unit sphere of a Banach space X that

(i) x is a smooth point of B_X if every sequence $(f_n)_{n\in\mathbb{N}}$ in S_{X^*}, such that $(f_n(x))_{n\in\mathbb{N}}$ converges to 1, verifies that $(f_n)_{n\in\mathbb{N}}$ is ω^*-convergent;

(ii) x is a strongly smooth point of B_X if every sequence $(f_n)_{n\in\mathbb{N}}$ in S_{X^*}, such that $(f_n(x))_{n\in\mathbb{N}}$ converges to 1, verifies that $(f_n)_{n\in\mathbb{N}}$ is convergent.

It can be checked (see [7]) that

(i) x is a smooth point of B_X if and only if the norm of X is Gâteaux differentiable at x;

(ii) x is a strongly smooth point of B_X if and only if the norm of X is Fréchet differentiable at x.

It is said that a Banach space is (*strongly*) *smooth* if every point of its unit sphere is a (strongly) smooth point of its unit ball.

It is well known for a Banach space X that

(i) X has the *Efimov-Stechkin property* if for every sequence $(x_n)_{n\in\mathbb{N}}$ in S_X and for every f in S_{X^*} such that $(f(x_n))_{n\in\mathbb{N}}$ converges to 1, then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence (see [9, pp. 478–479] and [10]);

(ii) X is *almost-rotund* if all closed convex subsets of S_X are compact (see [3]).

We are very interested in (strongly exposed) faces, which allows us to characterize Efimov-Stechkin property and rotundity.

THEOREM 1. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the Efimov-Stechkin property.*

(ii) *X is reflexive, almost-rotund and every exposed face of B_X is a strongly exposed face of B_X.*

THEOREM 2. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X is rotund.*
(ii) *If C is a closed convex subset of S_X such that $\mathsf{B}_X \backslash C$ is convex, then C is a face of B_X.*

On the other hand, smoothness techniques can be used to characterize rotundity in a local way. Following this line, we extend a result of Bandyopadhyay and Lin (see [5]).

THEOREM 3. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following assertions are equivalent:*

(i) *x is a rotund point of B_X.*
(ii) *For every $y \in \mathsf{S}_X \backslash \{x\}$,*

$$\lim_{t \to 0^+} \left(\frac{\|x + ty\| - \|x\|}{t} \right) < 1\,.$$

THEOREM 4. *Let X be a Banach space and let $x \in \mathsf{S}_X$. If x is a strongly exposed point of B_X and a strongly smooth point of B_X, then it is a locally uniformly rotund point of B_X.*

COROLLARY 5. *Let X be a Banach space. Then, X is locally uniformly rotund if it is strongly exposed and its norm is Fréchet differentiable in S_X.*

Finally, exposed faces can be characterized using some renorming techniques. In this way, we can prove the following theorems.

THEOREM 6. *Let X be a Banach space. Let C be a nonempty subset of S_X. The following statements are equivalent:*

(i) *C is an exposed face of B_X.*
(ii) *There exists an equivalent norm $\|\ \|_0$ on X such that $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = C \cup -C$, and C is a maximal face of B_{X_0}, where X_0 denotes the space X with the norm $\|\ \|_0$.*

COROLLARY 7. *Let X be a Banach space and let $x \in \mathsf{S}_X$. The following statements are equivalent:*

(i) *x is an exposed point of B_X.*

(ii) *There exists an equivalent norm* $\| \|_0$ *on* X *such that* $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = \{x, -x\}$, *and* x *is a rotund point of* B_{X_0}, *where* X_0 *denotes the space* X *with the norm* $\| \|_0$.

Theorem 8. *Let* X *be a Banach space and let* $x \in \mathsf{S}_X$. *The following statements are equivalent:*

(i) x *is a strongly exposed point of* B_X.

(ii) *There exists an equivalent norm* $\| \|_0$ *on* X *such that* $\mathsf{B}_X \subseteq \mathsf{B}_{X_0} \subseteq \sqrt{2}\mathsf{B}_X$, $\mathsf{S}_{X_0} \cap \mathsf{S}_X = \{x, -x\}$, *and* x *is a locally uniformly rotund point of* B_{X_0}, *where* X_0 *denotes the space* X *with the norm* $\| \|_0$.

Part of these results will appear in [2].

References

[1] Aizpuru, A., On operators which attain their norm at extreme points, *Arch. Math.* **69** (1997), 1 – 5.

[2] Aizpuru, A., García-Pacheco, F.J., Some questions about rotundity and renormings in Banach spaces, *J. Austral. Math. Soc.* (to appear).

[3] Banas, J., On drop property and nearly uniformly smooth Banach spaces, *Nonlinear Analysis, Theory, Meth. & Appl.* **14** (11) (1990), 927 – 933.

[4] Bandyopadhyay, P., Huang, D., Lin, B.-L., Troyanski, S.L., Some generalizations of locally uniform rotundity, *J. Math. Anal. Appl.* **252** (2000), 906 – 916.

[5] Bandyopadhyay, P., Lin, B.-L., Some properties related to nested sequence of balls in Banach spaces, *Taiwanese J. Math.* **5** (2001), 19 – 34.

[6] Bourgin, R.D., "Geometric Aspects of Convex Sets with the Radon-Nikodym Property", Lecture Notes in Math. 993, Springer-Verlag, Berlin, 1983.

[7] Deville, R., Godefroy, G., Zizler, V., "Smoothness and Renormings in Banach Spaces", Pitman Monographs and Surveys in Pure and Appl. Math. 64, Longman, Brunt Mill, 1993.

[8] Giles, J.R., Strong differentiability of the norm and rotundity of the dual, *J. Austral. Math. Soc.* **26** (1978), 302 – 308.

[9] Megginson, R.E., "An Introduction to Banach Space Theory", Graduate Texts in Math. 183, Springer-Verlag, New York, 1998.

[10] Singer, I., Some remarks on approximative compactness, *Romanian J. of Pure and Appl. Math.* **9** (1964), 167 – 177.