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On Normal Stratified Pseudomanifolds †

G. Padilla

Universidad Central de Venezuela, Escuela de Matemáticas, Facultad de Ciencias, Ciudad Universitaria, Caracas 1010, *Venezuela*

e-mail: gabrielp@euler.ciens.ucv.ve

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Devoted to the victims of the natural tragedy in Vargas, Dec. 15/1999, *who died under the rage of Waraira Repano.*

Foreword

A stratified pseudomanifold is normal if its links are connected. A normalization of a stratified pseudomanifold X is a normal stratified pseudomanifold X^N together with a finite-to-one projection $\mathfrak{n} : X^N \to X$ preserving the intersection homology. Recall that intersection homology is the suitable algebraic tool for the stratified point of view: it was first introduced by Goresky and MacPherson in the *pl*-category and later extended for any topological stratified pseudomanifold [2], [3]. Following Borel the map $\mathfrak{n}$ is usually required to satisfy the following property: For each $x \in X$ there is a distinguished neighborhood U such that the points of $\mathfrak{n}^{-1}(x)$ are in correspondence with the connected components of the regular part of U. A normalization satisfying the above condition always exists for any *pl*-stratified pseudomanifold [4], [6]. In this article we study the main properties of the map $\mathfrak{n}$. More precisely, we prove that $\mathfrak{n}$ can be required to satisfy a stronger condition: it is a locally trivial stratified morphism preserving the conical structure transverse to the strata. We make an explicit construction of such a normalization for any topological stratified pseudomanifold. Our construction is functorial, thus unique. We exhibit the relationship between the stratifications of X and X^N. Finally

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we prove that the normalization preserves the intersection homology with the family of perversities given in [7], see also [5], [9]. This family of perversities is larger than the usual one. With little adjust our procedure holds also in the C^∞ category.

1. Stratified pseudomanifolds

For a detailed treatment of the results contained in this section, see [8].

Manifolds considered in this paper will always be topological manifolds. A topological space is stratified if it can be written as a disjoint union of manifolds which are related by an incidence condition.

Definition 1.1. Let X be a Hausdorff, paracompact, second countable space. A *stratification* of X is a locally finite partition $\mathcal{S}$ of X in connected and locally closed subspaces called *strata*, satisfying

(1) Each stratum is a manifold with the induced topology.
(2) If a stratum S intersects the closure $\overline{S'}$ of another stratum S' then $S \subset \overline{S'}$ and we say S' *lies on* S.
(3) There exist open strata, all of them having the same dimension.

If this situation we say X is a *stratified space*. A stratum is *regular* if it is open and *singular* if not; the *dimension* of X is the dimension of the open strata; which we write $\dim(X)$. The *regular part* (resp. *singular part*) is the union of regular (resp. singular) strata. We write the singular part with the symbol Σ. For any paracompact subspace $Y \subset X$ the *induced partition* is the family $\mathcal{S}_Y$ whose elements are the connected component of $Y \cap S$, where S runs over the strata of X. If $\mathcal{S}_Y$ is a stratification of Y then we say that Y is a *stratified subspace*.

Examples 1.2. Here there are some examples of stratified spaces:

(1) For every manifold the canonical stratification is the family of its connected components. For every connected manifold M and every stratified space X the canonical stratification of the product $M \times X$ is the family of products $M \times S$ such that S is a stratum of X.

(2) The canonical stratification of the n-simplex $\Delta \subset \mathbb{R}^{n+1}$ is the family of connected components of $\Delta_j - \Delta_{j-1}$, where $0 \le j \le n$ and Δ_j is the j-skeleton of Δ. Any face of Δ is a stratified subpace.

(3) For every compact stratified space L the *cone of* L is the quotient space

$c(L) = L \times [0,\infty)/L \times \{0\}$. We write $[p,r]$ for the equivalence class of $(p,r) \in L \times [0,\infty)$, the symbol $\star$ denotes the equivalence class $L \times \{0\}$ called the *vertex*. By convention $c(\emptyset) = \{\star\}$. The radium is the function $\rho : c(L) \to [0,\infty)$ defined by $\rho[p,r] = r$. For every $\varepsilon > 0$ we write $c_\varepsilon(L) = \rho^{-1}[0,\varepsilon)$. The canonical stratification of the cone is the family

$$\mathcal{S}_{c(L)} = \{\star\} \sqcup \{S \times \mathbb{R}^+ : S \text{ is a stratum of } L\}.$$

Every $c_\varepsilon(L)$ is a stratified subspace of $c(L)$. A *basic model* is a product $M \times c(L)$ of a contractible manifold M with the cone.

LEMMA 1.3. *If $\mathcal{S}$ is a stratification of X then*

(1) *The relationship $S \leq S' \Leftrightarrow S \subset \overline{S'}$ is a partial order on $\mathcal{S}$.*
(2) *For any stratum S the family of strata lying on S is finite.*
(3) *A stratum is maximal (resp. minimal) $\Leftrightarrow$ it is open (resp. closed).*
(4) *The regular part is a dense open subset.*
(5) *If X is compact then it has a finite number of strata.*

By the above lemma, the union of closed strata is called the *minimal part* of X, which we write Σ_{min}.

Since the stratification is locally finite, strict order chains

$$S_0 < S_1 < \cdots < S_p$$

are always finite. The *length* of X is the maximal p for which there exists such an order chain, we write it $\operatorname{len}(X)$. For instance, $\operatorname{len}(X) = 0$ if and only if X is a manifold with the canonical stratification.

A function $f : X \to X'$ between two stratified spaces is a *morphism* (resp. *isomorphism*) if

(1) f is a continuous function (resp. an homeomorphism).
(2) f preserves the regular part: $f(X - \Sigma) \subset (X' - \Sigma')$.
(3) f sends smoothly (resp. diffeomorphicaly) strata into strata.

For instance the change of radium $f_\varepsilon : c(L) \to c_\varepsilon(L)$ defined by $f_\varepsilon[p,r] = [p, 2\varepsilon \cdot \arctan(r)/\pi]$ is an isomorphism. A morphism f is an *immersion* when $f(X)$ is a stratified subspace of X' and f is an isomorphism from X onto $f(X)$. An *embedding* is an immersion whose image is open.

A stratified pseudomanifold is a stratified space having a conical behavior near the singular part, transversally to the singular strata.

Definition 1.4. Fix a stratified space X (and a stratification $\mathcal{S}$), take a stratum S and a point $x \in S$. A *chart* in x is the embedding of a basic model

$$\varphi : U \times c(L) \to X$$

such that $U \subset S$ is a contractible open neighborhood of x and $\varphi(u, \star) = u$ for each $u \in U$. The image $\mathrm{Im}(\varphi)$ is a *distinguished neighborhood* of x. The compact stratified space L is the *link* of the chart. Notice that $\mathrm{len}(L) < \mathrm{len}(X)$. An *atlas* of S is a family of charts with the same link $\mathcal{A}_S = \{\alpha : U_\alpha \times c(L) \to X\}_\alpha$ such that $\{U_\alpha\}_\alpha$ is a covering of S. We say that X is a *stratified pseudomanifold* if every stratum has an atlas whose link is a stratified pseudomanifold itself. In that case, an *atlas* of X is the choice of an atlas for each stratum.

Examples 1.5. These are the three examples of stratified pseudomanifolds the most frequently used in this work:

(1) Every manifold (with the canonical stratification) is a stratified pseudomanifold, the link of any stratum being the empty set. For every manifold M and every stratified pseudomanifold X the product $M \times X$ is a stratified pseudomanifold.

(2) If L is a compact stratified pseudomanifold then the open cone $c(L)$ is a stratified pseudomanifold. The link of the vertex is L.

(3) The canonical n-simplex $\Delta \subset \mathbb{R}^{n+1}$ is a compact stratified pseudomanifold, the links are faces of Δ.

Lemma 1.6. *If X is a stratified pseudomanifold then*

1. *Any two strata $S < S'$ satisfy* $\dim(S) < \dim(S')$.
2. *The family of distinguished neighborhoods is a base of the topology of X.*
3. *Every open subset of X is itself a stratified pseudomanifold.*

2. Normalizations and normalizers

From now on we fix a stratified pseudomanifold X and an atlas of X. In this section we will provide a detailed construction of the normalization of X. A stratified pseudomanifold is *normal* if its links are connected. For instance the canonical n-simplex $\Delta \subset \mathbb{R}^{n+1}$ is a normal stratified pseudomanifold. If X is a normal stratified pseudomanifold then any open subset $A \subset X$ is also normal, and any link of X is normal. The following result is straightforward.

LEMMA 2.1. *Each connected normal stratified pseudomanifold has only one regular stratum.*

The normalization of X is made up by cutting along the singular strata.

DEFINITION 2.2. A *normalization* of a stratified pseudomanifold X is a proper surjective morphism

$$\mathfrak{n} : X^N \to X$$

from a normal stratified pseudomanifold X^N to X, together with a family of normalizations of the links $\{\mathfrak{n}_L : L^N \to L\}_L$ (of some fixed atlas) satisfying

(1) The restriction $\mathfrak{n} : (X^N - \Sigma) \to (X - \Sigma)$ is an isomorphism.
(2) For each singular point z of X^N there is a commutative diagram

$$\begin{array}{ccc} U \times c(L)^N & \stackrel{\varphi^N}{\to} & X^N \\ \downarrow^{\mathfrak{n}_0} & & \downarrow^{\mathfrak{n}} \\ U \times c(L) & \stackrel{\varphi}{\to} & X \end{array} \tag{1}$$

satisfying

(a) φ is a chart of $\mathfrak{n}(z)$.
(b) $c(L)^N = \sqcup_j c(K_j)$ where $K_1, \ldots, K_m$ are the connected components of L^N.
(c) φ^N is an embedding and $\text{Im}(\varphi^N) = \mathfrak{n}^{-1}(\text{Im}(\varphi))$.
(d) $\mathfrak{n}_0(u, [p, r]_j) = (u, [\mathfrak{n}_L(p), r])$ where $[p, r]_j \in c(K_j)$.

In the above situation we will say X^N is a *normalizer* of X.

EXAMPLES 2.3. These are three easy examples of normalizations:
(1) For any normal stratified pseudomanifold Z the identity 1_Z is a normalization.
(2) Left vertical arrow $\mathfrak{n}_0$ of diagram (1) is a normalization.
(3) Fix a normalization $\mathfrak{n} : X^N \to X$. Then for every open subset A of X the restriction $\mathfrak{n} : \mathfrak{n}^{-1}(A) \to A$ is a normalization of A. If X' is another stratified pseudomanifold and $f : X \to X'$ is an isomorphism then the composition $f\mathfrak{n}$ is a normalization of X'. Finally, for every manifold M the map

$$1_M \times \mathfrak{n} : M \times X^N \to M \times X$$

is a normalization of $M \times X$.

The stratification of a normalizer can be written in terms of the starting stratified pseudomanifold.

Proposition 2.4. *If* $\mathfrak{n} : X^N \to X$ *is a normalization then*

(1) *For each stratum* S *the restriction* $\mathfrak{n} : \mathfrak{n}^{-1}(S) \to S$ *is a locally trivial finite covering.*
(2) *Every stratum of* X^N *is a connected component of* $\mathfrak{n}^{-1}(S)$ *for some stratum* S *of* X*.*
(3) $\mathfrak{n}^{-1}(\Sigma_{min}) = \Sigma_{min}$ *and* $\operatorname{len}(X^N) = \operatorname{len}(X)$*.*

Proof. (1) It follows directly from § 2.2. (2) Fix a stratum S of X. Since any two strata $R, R' \subset \mathfrak{n}^{-1}(S)$ have the same dimension of S, they cannot be compared. Consequently $\mathfrak{n}^{-1}(S) \cap \overline{R} = R$; so R is a closed, connected and codimensional submanifold of $\mathfrak{n}^{-1}(S)$. Hence it is a connected component. (3) In order to prove the first equality we first notice that

(a) *If* S *is a stratum and* $R \subset \mathfrak{n}^{-1}(S)$ *is a stratum then* $\mathfrak{n}(R) = S$: By step (1) of this proof and the fact that $\mathfrak{n}$ is proper; it follows that $\mathfrak{n}(R)$ is a closed, connected and codimensional submanifold of S.

(b) *Strata contained in* $\mathfrak{n}^{-1}(\Sigma_{min})$ *are not comparable*: It follows from step (2), and the fact that minimal strata in X are not comparable (they are disjoint closed subsets).

Next fix a stratum R of X^N. By § 1.3 it suffices to show that that R is closed $\Leftrightarrow S = \mathfrak{n}(R)$ is closed. Then $\Leftarrow$ follows from step (2) and the continuity of $\mathfrak{n}$. By the other hand, since $\mathfrak{n}$ is a proper map; the converse $\Rightarrow$ follows from step (a). This proves the first equality. The second equality is straightforward, it can be deduce from the first one by an inductive argument. ∎

In order to show the uniqueness of the normalization we will establish its functoriality.

Proposition 2.5. (The lifting property) *Let* $\mathfrak{n} : Y^N \to Y$ *be another normalization,* $f : X \to Y$ *a continuous map preserving the regular part. Then there is a unique continuous function* f^N *making commutative the following diagram*

$$\begin{array}{ccc} X^N & \stackrel{f^N}{\to} & Y^N \\ \downarrow^{\mathfrak{n}} & & \downarrow^{\mathfrak{n}} \\ X & \stackrel{f}{\to} & Y \end{array}$$

Proof. If there exists such a map f^N, then in $X^N - \Sigma$ it satisfies

$$f^N = \mathfrak{n}^{-1} f \mathfrak{n}$$

So f^N is unique because $X^N - \Sigma$ is an open dense. The above equation shows that we only need to define f^N in the singular part. Fix a singular point z of X^N; we will say that $f^N(z) = v$ if there exists a sequence $\{z_j\} \subset X^N - \Sigma$ converging to z, such that $\{f^N(z_j)\}_j$ converges to v. Such a v always exists because $X^N - \Sigma$ is an open dense and the normalizations are proper maps. Notice that, by definition

$$\mathfrak{n}(v) = \mathfrak{n}\left(\lim_j f^N(z_j)\right) = \lim_j \mathfrak{n} f^N(z_j) = \lim_j f\mathfrak{n}(z_j) = y \qquad (2)$$

Consequently, $v \in \mathfrak{n}^{-1}(y)$. Then:
• *The lifting is well defined*: Since this is a local matter, we can suppose that X^N is connected and little. By §2.1 X^N has a unique regular stratum R (and so does X). Take a chart $\psi : V \times c(L) \to Y$ of $y = f\mathfrak{n}(z)$ as in diagram (1). Assume that $f(X) \subset \mathrm{Im}(\psi)$, so we get a diagram

$$\begin{array}{ccccc} X^N & \overset{f^N}{\dashrightarrow} & Y^N & \overset{\psi^N}{\longleftarrow} & V \times c(L)^N \\ \downarrow^{\mathfrak{n}} & & \downarrow^{\mathfrak{n}} & & \downarrow^{\mathfrak{n}_0} \\ X & \overset{f}{\longrightarrow} & Y & \overset{\psi}{\longleftarrow} & V \times c(L) \end{array}$$

where the dashed arrow f^N is well defined in the regular part. So the composition $(\psi^N)^{-1} f^N$ sends R in a regular stratum $U \times R' \times \mathbb{R}^+ \subset U \times c(L)^N$. Notice that $\overline{R'} = K_j$ is a connected component of L^N. So, for any sequence $\{z_i\}_i \subset R$ converging to z the sequence $\{(\psi^N)^{-1} f^N(z_i)\}_i$ is contained in $U \times c(K_j)$. Hence

$$\lim_i (\psi^N)^{-1} f^N(z_i) = (f\mathfrak{n}(z), \star_j) \in U \times \{\star_j\}$$

This implies that $f^N(z) = \psi^N(f(x), \star_j)$ is well defined.
• *The lifting is continuous*: This is easily seen by taking limits. ∎

Theorem 2.6. (Functoriality) *In the same situation of § 2.5, if f is a morphism (resp. an embedding or an isomorphism) then so is f^N.*

Proof. We consider three cases:
(a) *f is a morphism*: Since f^N is continuous and $\mathfrak{n} f^N = f\mathfrak{n}$, by §2.4, it sends strata into strata. Hence f^N is a morphism.

(b) f *is an isomorphism*: The inverse morphism f^{-1} lifts to a unique morphism $g : Y^N \to X^N$ such that $\mathfrak{n}g = f^{-1}\mathfrak{n}$. Notice that

$$\mathfrak{n}gf^N = f^{-1}\mathfrak{n}f^N = f^{-1}f\mathfrak{n} = 1_Y\mathfrak{n}$$

so gf^N is a lifting of the identity 1_Y. Since the lifting is unique $gf^N = 1_Y^N$ is the identity of Y^N. With the same argument $f^Ng = 1_X^N$ is the identity of X^N, thus g is the inverse of f^N.

(c) f *is an embedding*: Notice that the restriction $\mathfrak{n} : \mathfrak{n}^{-1}(f(X)) \to f(X)$ is a normalization, because $f(X)$ is open. Now apply step (b) to the isomorphism $f : X \to f(X)$. ∎

Corollary 2.7. *Normalizations and normalizers are unique up to isomorphisms.*

Now we will prove the existence of a normalization of X.

Theorem 2.8. *Each stratified pseudomanifold has a normalization.*

Proof. Fix a stratified pseudomanifold X and an atlas.

• *Reduction to a local matter*: Proceed by induction on $p = \mathrm{len}(X)$. For $p = 0$ it is trivial, assume the inductive hypothesis. Then each link of X has a normalization and for each chart $\varphi : U \times c(L) \to X$ the composition $\varphi\mathfrak{n}_0$ is a normalization of $\mathrm{Im}(\varphi)$, where $\mathfrak{n}_0$ is the map given in § 2.2. The family

$$\mathcal{U} = \{A \subset X : A \text{ is open and has a normalizer}\}$$

is a basis of the topology of X (see § 1.6), and it is closed by finite intersections (see § 2.3) and by arbitrary disjoint unions. By a Mayer-Vietoris argument known as the *Bredon's Trick* [1], it suffices to consider the case when $X = A \cup B$ for two open subsets A, B having normalizations

$$\mathfrak{n}_A : A^N \to A \qquad \mathfrak{n}_B : B^N \to B$$

By Theorem 2.6 there is an isomorphism ϕ such that the following diagram commutes

$$\begin{array}{ccc} \mathfrak{n}_A^{-1}(A \cap B) & \xrightarrow{\phi} & \mathfrak{n}_B^{-1}(A \cap B) \\ \downarrow^{\mathfrak{n}_A} & & \downarrow^{\mathfrak{n}_B} \\ A \cap B & \xrightarrow{1} & A \cap B \end{array} \qquad \mathfrak{n}_B\phi = \mathfrak{n}_A \tag{3}$$

• *Construction of the normalizer*: Define X^N as the sum of A^N and B^N amalgamated by the isomorphism ϕ, i.e.;

$$X^N = A^N \sqcup B^N / \sim \qquad z \sim \phi(z)\ \forall z \in \mathfrak{n}_A^{-1}(A \cap B) \tag{4}$$

We write$[z]$ for the equivalence class of $z \in A^N \sqcup B^N$. Endow X^N with the quotient topology induced by the canonical projection $q : z \mapsto [z]$. Since ϕ is an isomorphism it's easy to see that X^N is a stratified space and q is a surjective morphism.

• *Embedding of A^N and B^N in X^N*: The arrow $q : A^N \to X^N$ is continuous and injective by definition. For every subset $R \subset A^N$ we have

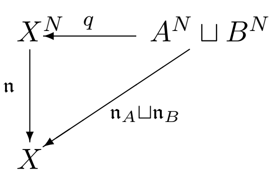
$$q^{-1}q(R) = R \sqcup \phi(R \cap \mathfrak{n}_A^{-1}(B))$$

If R is open then so is $q(R)$, which implies $q\mid_{A^N}$ is open. By the other hand, if R is a stratum then q sends it into a stratum of X^N and $q : R \to q(R)$ is a homeomorphism. Hence, $q\mid_{A^N}$ is an embedding; the same holds for $q\mid_{B^N}$.

• *Definition of the normalization*: Define $\mathfrak{n} : X^N \to X$ by the rule

$$\mathfrak{n}[z] = \begin{cases} \mathfrak{n}_A(z) & \text{if } z \in A^N \\ \mathfrak{n}_B(z) & \text{if } z \in B^N \end{cases} \tag{5}$$

Then $\mathfrak{n}$ is well defined by diagram (3); besides it's continuous and surjective. We obtain a commutative diagram



where q and $\mathfrak{n}_A \sqcup \mathfrak{n}_B$ are morphisms, then so is $\mathfrak{n}$. It's easy to see that restricted to the regular part $\mathfrak{n}$ is a diffeomorfism, so we verify the other properties of § 2.2.

• *Local triviality*: For every point $z \in X^N$ its image $x = \mathfrak{n}(z)$ lies in A or in B. Suppose $x \in A$ and let $\varphi : U \times c(L) \to A$ a chart of x. We get a commutative diagram

$$\begin{array}{ccccc} U \times c(L)^N & \stackrel{\varphi^N}{\to} & A^N & \stackrel{q}{\to} & X^N \\ \downarrow^{1 \times \mathfrak{n}_L} & & \downarrow^{\mathfrak{n}_A} & & \mathfrak{n} \\ U \times c(L) & \stackrel{\varphi}{\to} & A^N & \hookrightarrow & X \end{array}$$

By restricting $q\varphi^N$ to the connected component of $U \times c(L)^N$ which contains $(\varphi^N)^{-1}(z)$ we get a chart of z.

From the above discussion it is straightforward that $\mathfrak{n}$ is a proper map and X^N is a normal stratified pseudomanifold. ∎

Remark 2.9. If we ask $\mathfrak{n}$ to be smooth on each stratum and a diffeomorphism in the regular part; then Theorems 1 and 2 still hold in the C^∞ category.

Example 2.10. *Normalizing a tubular neighborhood*: A tubular neighborhood is a locally trivial fiber bundle $\xi = (T, \tau, S, c(L))$ where

(1) T is a stratified pseudomanifold, $S \subset T$ is the unique minimal stratum. The fiber $c(L)$ is the cone of a compact stratified pseudomanifold.

(2) $\tau : T \to S$ is a morphism, the restriction $\tau\mid_S$ is the identity of S.

(3) The structure group of the fiber bundle is $G = Iso(L)$. In other words, there is a trivializing atlas

$$\mathcal{U} = \{\alpha : U_\alpha \times c(L) \to \tau^{-1}(U_\alpha)\}_\alpha$$

such that, if $U_\alpha \cap U_\beta \neq \emptyset$, then the change of charts satisfies

$$\beta\alpha^{-1} : U_\alpha \cap U_\beta \times c(L) \to U_\alpha \cap U_\beta \times c(L) \qquad (x, [p, r]) \mapsto (x, [g_{\alpha\beta}(x, p), r])$$

Notice that $g_{\alpha\beta}(x, -)$ is an isomorphism and the global extension of the radium

$$\rho : T \to [0, \infty) \qquad \rho\alpha(x, [p, r]) = r$$

is a stratified morphism which makes sense.

Fix two normalizations $\mathfrak{n} : T^N \to T$ and $\mathfrak{n}_L : L^N \to L$; take $\mathfrak{n}_0 : c(L)^N \to c(L)$ as in § 2.2. We will show that there is a fiber bundle

$$\xi^N = (T^N, \tau^N, \mathfrak{n}^{-1}(S), c(L)^N)$$

where each object is induced by the process of normalization and the structure group is $G = Iso(L^N)$. But this in immediate since, by Theorem 2.6, there is a unique lifting $\tau^N : T^N \to \mathfrak{n}^{-1}(S)$ such that the following diagram commutes

$$\begin{array}{ccc} T^N & \xrightarrow{\tau^N} & \mathfrak{n}^{-1}(S) \\ \downarrow{\scriptstyle \mathfrak{n}} & & \downarrow{\scriptstyle \mathfrak{n}} \\ T & \xrightarrow{\tau} & S \end{array}$$

Each trivialization α in the atlas $\mathcal{U}$ lifts to a trivialization

$$\alpha^N : U_\alpha \times c(L)^N \to (\tau^N)^{-1}(\mathfrak{n}^{-1}(U))$$

By the existence and uniqueness of liftings, the family of cocycles $\{g_{\alpha\beta}\}$ lifts to a family $\{g^N_{\alpha\beta}\}$ satisfying $g^N_{\alpha\beta}g^N_{\beta\delta} = g^N_{\alpha\delta}$ for all α, β, δ; thus it is again a family of cocycles. It is straightforward that the radium ρ lifts in a consistent way to a radium ρ^N.

3. Intersection homology of the normalizer

Intersection homology was first introduced by Goresky and MacPherson in the category of *pl*-stratified pseudomanifolds, and later extended for any stratified pseudomanifold [2], [3]. In order to show that the normalization preserves the intersection homology, we extend Borel's procedure to the family of perversities given in [7]; see also [4], [5], [9]. Those preversities are more general than the usual ones.

Definition 3.1. Fix a stratified pseudomanifold X. A *perversity* in X is a function $\overline{p} : \mathcal{S} \to \mathbb{Z}$ from the family of strata $\mathcal{S}$ to the integers. A singular simplex $\sigma : \Delta \to X$ is *$\overline{p}$-admissible* if it satisfies the following properties:

(1) σ sends the interior of Δ in $X - \Sigma$.

(2) $\sigma^{-1}(S) \subset (\dim(\Delta) - \operatorname{codim}(S) + \overline{p}(S))$-skeleton of Δ, for each singular stratum S of X.

A singular chain $\xi = \Sigma_{j=1}^m r_j \sigma_j$ is *$\overline{p}$-admissible* if every σ_j is $\overline{p}$-admissible. We will say that ξ is a *$\overline{p}$-chain* if ξ and its boundary $\partial\xi$ are both $\overline{p}$-admissible. We write $SC^{\overline{p}}_*(X)$ for the complex of $\overline{p}$-chains. The *$\overline{p}$-intersection homology* of X is the homology $H^{\overline{p}}_*(X)$ of the complex $SC^{\overline{p}}_*(X)$.

Given a normalization $\mathfrak{n} : X^N \to X$, each perversity $\overline{p}$ in X induces trivially a perversity in X^N which we write again $\overline{p}$ by abuse of language.

Theorem 3.2. *$H^{\overline{p}}_*(X^N) = H^{\overline{p}}_*(X)$ for any perversity $\overline{p}$ in X.*

Proof. We claim that the usual chain morphism $\mathfrak{n}_* : SC^{\overline{p}}_*(X^N) \to SC^{\overline{p}}_*(X)$ is an isomorphism. It suffice to show it in the $\overline{p}$-simplexes.

• *The arrow is well defined*: We will proof that if σ is a simplex $\overline{p}$-admissible in X^N then $\mathfrak{n}\sigma$ is $\overline{p}$-admissible in X. Condition (1) is trivial, so we verify (2):

Fix a singular stratum S of X. Let S' be a stratum contained in $\mathfrak{n}^{-1}(S)$. Then $\sigma^{-1}(S') \subset \dim(\Delta) - \operatorname{codim}(S') + \overline{p}(S')$-skeleton of Δ. Since $\dim(S) = \dim(S')$ then we conclude that $(\mathfrak{n}\sigma)^{-1}(S) \subset (\dim(\Delta) - \operatorname{codim}(S) + \overline{p}(S))$-skeleton of Δ.

• *The arrow is injective*: Take two singular simplexes $\sigma, \sigma' : \Delta \to X^N$ and suppose $\mathfrak{n}\sigma = \mathfrak{n}\sigma'$; then σ and σ' coincide in the interior of Δ which implies that $\sigma = \sigma'$.

• *The arrow is surjective:* Recall that Δ is a normal stratified pseudomanifold, thus the identity $1_\Delta : \Delta \to \Delta$ is a normalization. Take a $\overline{p}$-admissible singular simplex $\sigma : \Delta \to X$. By §2.5; lifts to a unique singular simplex $\sigma^N : \Delta \to X^N$ satisfying $\mathfrak{n}\sigma^N = \sigma$. Conditions (1) and (2) of § 3.1 are easily verified for σ^N; so it is $\overline{p}$-admissible. ∎

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