

Commutativity Criteria In Locally M -Convex Algebras

AIDA TOMA

*Mathematics Department, Academy of Economic Studies
Piata Romana no. 6, Bucharest, Romania
e-mail: aida_toma@yahoo.com*

(Presented by S. Dierolf)

AMS *Subject Class.* (2000): 46H05, 46J05

Received June 22, 2001

1. INTRODUCTION

In this paper we define the notions of semicommutativity and semicommutativity modulo a linear subspace. We prove some results regarding the semicommutativity or semicommutativity modulo a linear subspace of a sequentially complete m -convex algebra. We show how can be applied such results in order to obtain commutativity criteria for locally m -convex algebras.

Let A be a complex m -convex algebra, whose topology is defined by a separating family $(p_\alpha)_{\alpha \in I}$ of submultiplicative seminorms.

The unitization of A over \mathbb{C} , denoted by A_1 is the m -convex algebra consisting of the set $\mathbb{C} \times A$ with addition, scalar multiplication and product defined (for all $x, y \in A$ and $\alpha, \beta \in \mathbb{C}$) by

$$\begin{aligned}(\alpha, x) + (\beta, y) &= (\alpha + \beta, x + y) \\ \beta(\alpha, x) &= (\beta\alpha, \beta x) \\ (\alpha, x)(\beta, y) &= (\alpha\beta, xy + \alpha y + \beta x)\end{aligned}$$

and with the seminorms $(q_\alpha)_{\alpha \in I}$, defined by

$$q_\alpha((\lambda, x)) = |\lambda| + p_\alpha(x)$$

for all $\alpha \in I$, $\lambda \in \mathbb{C}$ and $x \in A$; A_1 is an m -convex algebra with unit element $(1, 0)$, $q_\alpha((1, 0)) = 1$ for all $\alpha \in I$, and the mapping $a \rightarrow (0, a)$ is an isomorphism of A onto a subalgebra of A_1 . It is a routine matter to verify that A_1

is sequentially complete when A is sequentially complete (i.e., every Cauchy sequence converges).

Recall that for an element x of an unital algebra A , the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \notin G(A)\}$$

is called the spectrum of x , and

$$\rho(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

is called the spectral radius of x , 1 being the unit element of A and $G(A)$ the set of all invertible elements of A .

If A_1 is a sequentially complete m -convex algebra, then (see [7])

$$\rho(y) = \sup_{\alpha \in I} \lim_{n \rightarrow \infty} q_\alpha(y^n)^{\frac{1}{n}}$$

for any $y \in A_1$. In particular, we have

$$\begin{aligned} \rho((0, x)) &= \sup_{\alpha \in I} \lim_{n \rightarrow \infty} q_\alpha((0, x)^n)^{\frac{1}{n}} \\ &= \sup_{\alpha \in I} \lim_{n \rightarrow \infty} q_\alpha((0, x^n))^{\frac{1}{n}} = \sup_{\alpha \in I} \lim_{n \rightarrow \infty} p_\alpha(x^n)^{\frac{1}{n}}. \end{aligned} \quad (1)$$

In the remainder of this paper we assume that A is a complex sequentially complete m -convex algebra with topology defined by a separating family $(p_\alpha)_{\alpha \in I}$ of submultiplicative seminorms.

DEFINITION 1. A is said to be commutative iff $xy = yx$ for any $x, y \in A$, and A is said to be semicommutative iff $xyz = zxy$ for any $x, y, z \in A$. Given a linear subspace E of A , A is said to be commutative modulo E iff $xy - yx \in E$ for any $x, y \in A$, and A is said to be semicommutative modulo E iff $xyz - zxy \in E$ for any $x, y, z \in A$.

For example, in the set $\mathcal{M}_4(\mathbb{C})$ of all square matrices with four columns and complex elements, we consider the subset

$$A = \{X \in \mathcal{M}_4(\mathbb{C}) : X = \alpha M + \beta N + \gamma MN, \alpha, \beta, \gamma \in \mathbb{C}\}$$

where

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If we define on A the family consisting of the submultiplicative seminorm

$$p(X) = |\alpha| + |\beta| + |\gamma|,$$

A become an m -convex algebra. This m -convex algebra is semicommutative but not commutative: on the one hand, $NM = 0$ (the null matrix) and

$$MN = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$

on the other hand, using the fact that $MNM = NMN = 0$ it is a routine matter to verify that $XYZ = ZXY = 0$ for any $X, Y, Z \in A$, and so we obtain the semicommutativity of A .

DEFINITION 2. If A has unit element, the radical of A , denoted by $\text{Rad } A$ is the set

$$\text{Rad } A = \{x \in A : 1 - xy \in G(A) \text{ for any } y \in A\}.$$

If A hasn't unit element then we define the radical of A as

$$\text{Rad } A = \{x \in A : (0, x) \in \text{Rad } A_1\}.$$

2. SOME COMMUTATIVITY CRITERIONS

THEOREM 1. *If A is semicommutative then A is commutative modulo $\text{Rad } A$.*

Proof. We have to prove that $(0, xy - yx) \in \text{Rad } A_1$ for any $x, y \in A$. We prove first that

$$(xy)^n = x^n y^n \tag{2}$$

for any $n \in \mathbb{N}^*$ and $x, y \in A$. Indeed, for $n = 2$, from semicommutativity we have

$$(xy)^2 = (xy)(xy) = ((xy)x)y = x(yx)y = xy^2x = x^2y^2$$

for any $x, y \in A$, and by induction we obtain the equality (2).

Let now $x, y \in A$. For $\alpha \in I$ and $n \in \mathbb{N}^*$, we have

$$p_\alpha((xy)^n)^{\frac{1}{n}} = p_\alpha(x^n y^n)^{\frac{1}{n}} \leq p_\alpha(x^n)^{\frac{1}{n}} p_\alpha(y^n)^{\frac{1}{n}},$$

and by (1) we obtain

$$\rho((0, x)(0, y)) \leq \rho((0, x))\rho((0, y)). \quad (3)$$

Using again the semicommutativity of A , we find that $(xy - yx)^2 = 0$ for any $x, y \in A$. This implies that

$$\rho((0, xy - yx)) = \sup_{\alpha \in I} \lim_{n \rightarrow \infty} p_\alpha((xy - yx)^n)^{\frac{1}{n}} = 0 \quad (4)$$

for any $x, y \in A$. From (3) and (4) it follows that

$$\rho((0, xy - yx)(0, z)) = 0 \quad (5)$$

for any $x, y, z \in A$.

We prove now that for $u \in A$ with the properties

$$\rho(0, u) = 0 \quad \text{and} \quad \rho((0, u)(0, z)) = 0 \quad \text{for any } z \in A,$$

the equality

$$\rho((0, u)(\lambda, v)) = 0 \quad (6)$$

hold for any $(\lambda, v) \in A_1$. Indeed, using the semicommutativity of A we obtain that $(0, \lambda u)$ and $(0, uv)$ are permutable elements of A_1 . It is a well known fact that for permutable elements the spectral radius is submultiplicative so we have

$$\rho((0, u)(\lambda, v)) = \rho((0, \lambda u + uv)) \leq |\lambda|\rho((0, u)) + \rho((0, u)(0, v)) = 0$$

for any $(\lambda, v) \in A_1$.

From (4), (5) and (6) it follows that $\rho((0, xy - yx)t) = 0$ for any $x, y \in A$ and $t \in A_1$. We deduce that $1 \notin \sigma((0, xy - yx)t)$ for any $x, y \in A$ and $t \in A_1$. So $(1, 0) - (0, xy - yx)t \in G(A_1)$ for any $x, y \in A$ and $t \in A_1$, and consequently

$$(0, xy - yx) \in \text{Rad } A_1$$

for any $x, y \in A$, and this completes the proof. ■

DEFINITION 3. If $(q_\alpha)_{\alpha \in I}$ is a family of seminorms on A , the kernel of family $(q_\alpha)_{\alpha \in I}$, denoted by $\text{Ker}((q_\alpha)_{\alpha \in I})$, is the set

$$\text{Ker}((q_\alpha)_{\alpha \in I}) = \bigcap_{\alpha \in I} \text{Ker}(q_\alpha).$$

DEFINITION 4. If p, q are seminorms on A , q is said to be p -continuous if there exists $k > 0$ such that

$$q(x) \leq kp(x) \quad \text{for any } x \in A.$$

THEOREM 2. If $(q_\alpha)_{\alpha \in I}$ is a family of submultiplicative seminorms on A such that, for any $\alpha \in I$, q_α is a p_α -continuous seminorm and there exists $k_\alpha > 0$ such that

$$q_\alpha(xy) \leq k_\alpha p_\alpha(yx)$$

for any $x, y \in A$, then A is semicommutative modulo $\text{Ker}((q_\alpha)_{\alpha \in I})$.

Proof. Let $x, y \in A$, $z \in A_1$, and $f : \mathbb{C} \rightarrow A$ given by

$$f(\lambda) = \exp(\lambda z)xy \exp(-\lambda z)$$

for any $\lambda \in \mathbb{C}$, where

$$\exp(\lambda z) = (1, 0) + \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^n z^n.$$

The function f is well defined, because A is an ideal of A_1 . We consider the linear space $A/\text{Ker}((q_\alpha)_{\alpha \in I})$ and the family of seminorms $(p'_\alpha)_{\alpha \in I}$ given by

$$p'_\alpha(\widehat{x}) = q_\alpha(x)$$

for any $\widehat{x} \in A/\text{Ker}((q_\alpha)_{\alpha \in I})$ and $\alpha \in I$. We immediately obtain that the seminorms p'_α are well defined and that $(p'_\alpha)_{\alpha \in I}$ is a separating family. So the linear space $A/\text{Ker}((q_\alpha)_{\alpha \in I})$ endowed with the family $(p'_\alpha)_{\alpha \in I}$ is a locally convex space. We will denote this space by \widetilde{A} .

Let now $\widetilde{f} : \mathbb{C} \rightarrow \widetilde{A}$ defined by $\widetilde{f}(\lambda) = \widehat{f(\lambda)}$ for any $\lambda \in \mathbb{C}$. The function \widetilde{f} is differentiable on \mathbb{C} . Indeed, for any $\alpha \in I$, from the fact that q_α is a p_α -continuous seminorm, we get the existence of a constant β_α such that

$$\begin{aligned} p'_\alpha \left(\frac{\widetilde{f}(\lambda) - \widetilde{f}(\lambda_0)}{\lambda - \lambda_0} - \widehat{f'(\lambda_0)} \right) &= p'_\alpha \left(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right) \\ &= q_\alpha \left(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right) \\ &\leq \beta_\alpha p_\alpha \left(\frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} - f'(\lambda_0) \right) \end{aligned}$$

because f is a differentiable function as a product of differentiable functions. It follows that \tilde{f} is differentiable on \mathbb{C} and

$$\left(\tilde{f}\right)'(\lambda) = \widehat{f'(\lambda)} \quad \text{for any } \lambda \in \mathbb{C}.$$

Let $\alpha \in I$ and $\lambda \in \mathbb{C}$. We have

$$\begin{aligned} p'_\alpha\left(\tilde{f}(\lambda)\right) &= p'_\alpha\left(\widehat{f(\lambda)}\right) = q_\alpha(f(\lambda)) \\ &= q_\alpha(\exp(\lambda z)xy \exp(-\lambda z)) \leq k_\alpha p_\alpha(yz). \end{aligned}$$

So \tilde{f} is differentiable and bounded on \mathbb{C} and using Liouville Theorem we get that \tilde{f} is a constant function. This implies that $\left(\tilde{f}\right)'(\lambda) = \widehat{0}$ for any $\lambda \in \mathbb{C}$. We have

$$\left(\tilde{f}\right)'(\lambda) = \widehat{f'(\lambda)} = z \exp(\lambda z)xy \exp(-\lambda z) - \exp(\lambda z)xy \exp(-\lambda z)z$$

for any $\lambda \in \mathbb{C}$. For $\lambda = 0$ we obtain $\widehat{0} = zxy - xyz$. So $zxy - xyz \in \text{Ker}((q_\alpha)_{\alpha \in I})$ for any $x, y, z \in A$ and this completes the proof. ■

COROLLARY 1. *If there exists a separating family $(q_\alpha)_{\alpha \in I}$ of submultiplicative seminorms on A with the properties that for any $\alpha \in I$, q_α is a p_α -continuous seminorm and there exists $k_\alpha > 0$ such that*

$$q_\alpha(xy) \leq k_\alpha p_\alpha(yx) \quad \text{for any } x, y \in A,$$

then A is semicommutative.

Proof. From the fact that $(q_\alpha)_{\alpha \in I}$ is a separating family, we have

$$\text{Ker}((q_\alpha)_{\alpha \in I}) = \bigcap_{\alpha \in I} \text{Ker}(q_\alpha) = \{0\}$$

and now we use Theorem 2. ■

COROLLARY 2. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(xy) \leq k_\alpha p_\alpha(yx) \quad \text{for any } x, y \in A,$$

then A is commutative.

Remark. If the conditions of Corollary 1 or Corollary 2 are satisfied then A is commutative modulo $\text{Rad } A$. In addition, if A has unit element, then A is commutative.

Now we consider that A has unit element. We denote by $S = S(A)$ the set of all states on A , i.e., the set of all continuous functionals s on A with the properties that $s(1) = 1$ and there exists $\alpha \in I$ such that

$$|s(x)| \leq p_\alpha(x) \quad \text{for any } x \in A.$$

Recall that, for an element $x \in A$, the set

$$V(x) = \{s(x) : s \in S\}$$

is called the numerical range of x , and

$$v(x) = \sup\{|s(x)| : s \in S\}$$

is called the numerical radius of x . We recall the generalization of Bohnenblust and Karlin theorem for m -convex algebras (see [3], [4]). Let A be an unital m -convex algebra and $x \in A$. Then

$$\frac{1}{e} \sup_{\alpha \in I} p_\alpha(x) \leq v(x) \leq \sup_{\alpha \in I} p_\alpha(x).$$

COROLLARY 3. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$v(xy) \leq k_\alpha p_\alpha(yx) \quad \text{for any } x, y \in A,$$

then A is commutative.

Proof. From the generalization of Bohnenblust and Karlin theorem we have

$$\frac{1}{e} \sup_{\alpha \in I} p_\alpha(xy) \leq v(xy) \quad \text{for any } x, y \in A.$$

Now using Corollary 2 it follows that A is a commutative algebra. \blacksquare

THEOREM 3. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(x)^2 \leq k_\alpha p_\alpha(x^2) \quad \text{for any } x \in A,$$

then A is commutative.

Proof. Let $\alpha \in I$ and $x \in A$. An induction argument lead us to

$$p_\alpha(x) \leq k_\alpha^{1-\frac{1}{2^n}} (p_\alpha(x^{2^n}))^{\frac{1}{2^n}}.$$

We denote

$$\rho_\alpha(x) = \lim_{n \rightarrow \infty} (p_\alpha(x^{2^n}))^{\frac{1}{2^n}}.$$

Letting $n \rightarrow \infty$, we obtain $p_\alpha(x) \leq k_\alpha \rho_\alpha(x)$. As it is known, $\rho_\alpha(xy) = \rho_\alpha(yx)$ for any $x, y \in A$ (see [7]). So

$$p_\alpha(xy) \leq k_\alpha \rho_\alpha(xy) = k_\alpha \rho_\alpha(yx) \leq k_\alpha p_\alpha(yx)$$

for any $x, y \in A$, and using Corollary 2 we obtain that A is commutative. ■

COROLLARY 4. (See [2]) *Let A be a complex Banach algebra with unit such that, for some $k > 0$,*

$$\|xy\| \leq k\|yx\| \quad \text{for any } x, y \in A.$$

Then A is commutative.

THEOREM 4. *If A has unit element and for any $\alpha \in I$ there exists $k_\alpha > 0$ such that*

$$p_\alpha(x) \leq k_\alpha \rho_\alpha(x) \quad \text{for any } x \in A,$$

then A is commutative.

Proof. Let $x, y \in A$ and $f : \mathbb{C} \rightarrow A$ given by

$$f(\lambda) = \exp(\lambda x)y \exp(-\lambda x)$$

for any $\lambda \in \mathbb{C}$. For any $\alpha \in I$ and $\lambda \in \mathbb{C}$, we have

$$p_\alpha(f(\lambda)) = p_\alpha(\exp(\lambda x)y \exp(-\lambda x)) \leq k_\alpha \rho_\alpha(\exp(\lambda x)y \exp(-\lambda x)) = k_\alpha \rho_\alpha(y)$$

because the spectral radius has the property $\rho(xy) = \rho(yx)$ for any $x, y \in A$ (see [7]). So f is a bounded and differentiable function on \mathbb{C} and using Liouville theorem we obtain that f is a constant function. This implies that $f'(\lambda) = 0$ for any $\lambda \in \mathbb{C}$ and consequently

$$x \exp(\lambda x)y \exp(-\lambda x) - \exp(\lambda x)yx \exp(-\lambda x) = 0 \quad \text{for any } \lambda \in \mathbb{C}.$$

For $\lambda = 0$ we have $xy - yx = 0$, and this completes the proof. ■

COROLLARY 5. (See [1] and [2]) *Let A be a complex Banach algebra with unit such that, for some $k > 0$,*

$$\|x\| \leq k\rho(x) \quad \text{for any } x \in A.$$

Then A is commutative.

REFERENCES

- [1] AUPETIT, B., "Propriétés Spectrales des Algèbres de Banach", Lecture Notes Math. 735, Springer, Berlin, 1979.
- [2] BONSALL, F.F., DUNCAN, J., "Complete Normed Algebras", New York, Springer-Verlag, 1973.
- [3] BONSALL, F.F., DUNCAN, J., "Numerical Ranges of Operators and of Elements of Normed Algebras", LMS Lecture Note Series 2, Cambridge University Press, London, 1971.
- [4] GILES, J.R., KOEHLER, D.O., On numerical ranges of elements of locally m -convex algebras, *Pacific J. Math.* **49** (1973), 79–91.
- [5] MALLIOS, A., "Topological Algebras, Selected Topics", North-Holland Math. Studies 124, Amsterdam, 1986.
- [6] MOCANU, GH., On numerical radius of an element of a normed algebra, *Glasgow Math. J.* **15** (1974), 90–92.
- [7] ŻELAZKO, W., "Selected Topics in Topological Algebras", Lecture Note Series No. 31, Aarhus Universitet, Aarhus, 1971.