EXTRACTA MATHEMATICAE Vol. **17**, Núm. 2, 221 – 238 (2002)

# On Lopsided Systems †

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(Research paper presented by Jaume Llibre)

AMS *Subject Class.* (2000): 37G40 *Received June* 5, 2001

In this work, we study the Liapunov quantities, the problem of the center, and the local limit cycles of the lopsided systems $\dot{x} = y$, $\dot{y} = -x + p_k(x, y)$, when $k = 5, 7$ and when $k$ is odd. In general, the Liapunov quantities are derived from the focal values $\eta_{2k+2}$, but when $k = 5$, we show that they are derived from the focal values $\eta_{4k+2}$. Moreover, when $k = 5$, the origin is a center if and only if the system is time-reversible and if it is not, no more than five local limit cycles can bifurcate out of the origin. When $k = 7$, we show that the origin is a center if and only if the system is time-reversible and if it is not, no more than seven local limit cycles can bifurcate out of the origin under certain conditions. In general, when $k$ is odd, we conjecture that the origin is a center if and only if the system is time-reversible.

## 1. Introduction

The part of Hilbert's $16^{th}$-problem [8] that relates to the number of limit cycles of two-dimensional autonomous systems of the form

$$\dot{x} = P(x, y)\,, \qquad \dot{y} = Q(x, y)\,, \tag{1}$$

where $\cdot = \frac{d}{dt}$, $P$ and $Q$ are polynomials in $x$ and $y$, remains one of the outstanding unsolved problems in the theory of non-linear ordinary differential

†Keywords: Center-focus problem, nonlinear differential equations.

*Partially supported by a DGICYT Grant Number PB96-1153 and CONACIT Grant Number 199956 R 00349.

equations. The maximum possible number of *limit cycles* (isolated closed orbits) which can be bifurcate out of the origin of (1) is the question of interest in this part of Hilbert's $16^{th}$-problem. In the local study of these systems, we find the problem of a center is closely related to the problem of limit cycles. This problem consists in finding all necessary and sufficient conditions that bears on the coefficients of $P$ and $Q$, in order that all orbits in a neighborhood of the origin be periodic. Now when the origin is a center for the linearised systems of (1) we can choose co-ordinates in which (1) is of the form

$$\dot{x} = \lambda x + y + p(x,y)\,, \qquad \dot{y} = -x + \lambda y + q(x,y)\,. \tag{2}$$

We write $p(x,y) = p_2(x,y) + \cdots + p_n(x,y)$, $q(x,y) = q_2(x,y) + \cdots + q_n(x,y)$, where $p_k$ and $q_k$ are homogeneous polynomials of degree $k$. The linear part of (2) is in canonical form and the stability of the origin is determined by the sign of $\lambda$. If $\lambda = 0$ the origin is a centre for the linearised system and is said to be a *fine focus* (or a *weak focus*) of the non-linear system.

The Kukles system is the origin of the lopsided systems; in [9] Kukles has examined the conditions under which the origin is a centre for the differential system of the form

$$\dot{x} = y, \quad \dot{y} = -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3. \tag{3}$$

It was thought that the conditions given in [9] were necessary and sufficient conditions, but Xiaofan and Dongming [20] describe an example which was not covered by them and in which the computations suggest that the origin was a center, then Christopher and Lloyd [7] prove that the origin of the example suggested by Xiaofan and Dongming is indeed a centre. By transforming (3) to a system of Liénard type, Cherkas [6] also noted that the Kukles conditions were incomplete and he discussed some aspects of the problem. In [18] the author analyzes the center conditions given by Kukles and Cherkas. In [7] it was shown that for the class of systems (3) under the condition $a_7 = 0$, at most five limit cycles bifurcate from the origin. The Kukles conditions are complete under this restriction and a study of those centre conditions was developed in [16]. In [12], it was shown that, for the systems of type (3) under the condition $a_2 = 0$, at most six limit cycles bifurcate from the origin. Later, in [13], Lloyd and Pearson found another condition for a centre not covered by the preceding ones and they conjecture that there are no others conditions for a centre.

This work is a continuation of Kukles System and the systems of type (2), where $p(x,y) = p_n(x,y)$ and $q(x,y) = q_n(x,y)$. The systems of type (2), where $p(x,y) = p_n(x,y)$ and $q(x,y) = q_n(x,y)$ has been thoroughly studied by many researchers. In particular, we should highlight the works of Bautin [2] when $n = 2$, and the works of Lunkevich and Sibirskii [14] when $n = 3$, for the fact that they characterize all the centers. Some conditions of a center are given in [4] and [5] when $n = 4$ and $n = 5$, respectively.

For the systems of type (2), Poincaré introduced an important technique which is developed by Liapunov [10] in order to determine whether the origin is a center. It consists in looking for a formal power series of $x$ and $y$ of the form $V(x,y) = \sum_{k=2}^{\infty} V_k(x,y)$, where $V_2(x,y) = \frac{1}{2}(x^2+y^2)$, so that

$$\dot{V} = \sum_{k=1}^{\infty} \eta_{2k}(x^2+y^2)^k \,,$$

where the coefficients $\eta_{2k}$ are the *focal values* and they are polynomials in $\lambda$ and the coefficients in p and q. It is known that the origin is stable or unstable according to whether the first non-zero focal value is negative or positive, and that the origin is a centre if all the focal values are zero. What we really need are the so-called *Liapunov quantities* $L(0), L(1), \ldots$; these are the non-zero expressions obtained by calculating each $\eta_{2k}$ under the condition $\eta_2 = \eta_4 = \cdots = \eta_{2k-2} = 0$. Then the origin is a center if all the Liapunov quantities are zero. The origin of (2) is said to be a fine focus of *order* $k$ if $\eta_2 = \eta_4 = \cdots = \eta_{2k} = 0$, but $\eta_{2k+2} \neq 0$. In general $L(k)$ is derived from $\eta_{2k+2}$, but it may happen that a reduced focal value is necessarily zero, in which case it does not contribute a Liapunov quantity, as we shall show for a lopsided quintic systems in the next section.

*Remark.* The origin of the system (2) is a fine focus of order $k$ if $L(0) = L(1) = \cdots = L(k-1) = 0$, but $L(k) \neq 0$.

A *reversible system* [17] is a planar differential system $\dot{X} = f(X)$, $X \in \mathbb{R}^2$, for which there exists a diffeomorphism $R : \mathbb{R}^2 \to \mathbb{R}^2$ such that $R^2$ is the identity and $f(R(X)) = -R(f(X))$. We say that system $\dot{X} = f(X)$ is *time-reversible* if after a rotation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

the system becomes invariant by the transformation of the form $(X,t) \mapsto (R(X), -t)$. The time-reversible systems are characterized for the existence of at least a straight line through the origin, which is a symmetry axis of the phase portrait. This line has the slope $\tan(\frac{\alpha}{2})$, then after a rotation of the angle $\frac{\alpha}{2}$ the system is reversible with respect to the diffeomorphism $R(x,y) = (x,-y)$. Note that a vector field $(p(x,y), q(x,y))$ is reversible with respect to the map $R$ if and only if $p(x,-y) = -p(x,y)$ and $q(x,-y) = q(x,y)$; if the system of type (2) is reversible then the origin is a centre (the symmetry principle, see [15, p. 135]). A general study on reversible vector fields can be found in [18], [19] and [21].

In this work, we study lopsided systems of the form

$$\dot{x} = \lambda x + y\,, \qquad \dot{y} = -x + \lambda y + q_k(x,y)\,, \tag{4}$$

where $q_k(x,y)$ is homogeneous polynomial of degree $k$. For $k=5$ and $q_5(x,y) = a_1x^5 + a_2x^4y + a_3x^3y^2 + a_4x^2y^3 + a_5xy^4 + a_6y^5$, we refer to the system (4) as a lopsided quintic for which we have the following results

Theorem 1. *For a lopsided quintic system, the Liapunov quantities $L(k)$ is derived from the focal values $\eta_{4k+2}$ in each cases.*

Theorem 2. *For a lopsided quintic system, we have:* (i) *The origin is a centre if and only if the system is time-reversible.* (ii) *If the system is not time-reversible, we have at most five local limit cycles which bifurcate out of the origin.*

For $k = 7$ and $q_7(x,y) = a_1x^7 + a_2x^6y + a_3x^5y^2 + a_4x^4y^3 + a_5x^3y^4 + a_6x^2y^5 + a_7xy^6 + a_8y^7$, doing the following change of variables

$$\begin{aligned} a_1 &= \tfrac{b_1+b_2+b_3+b_4}{64}\,, & a_5 &= \tfrac{3b_1-5b_2-5b_3+35b_4}{64}\,,\\ a_2 &= \tfrac{b_5+3b_6+5b_7+7b_8}{64}\,, & a_6 &= \tfrac{3b_5+b_6-9b_7+21b_8}{64}\,,\\ a_3 &= \tfrac{3b_1-b_2-9b_3-21b_4}{64}\,, & a_7 &= \tfrac{b_1-3b_2+5b_3-7b_4}{64}\,,\\ a_4 &= \tfrac{3b_5+5b_6-5b_7-35b_8}{64}\,, & a_8 &= \tfrac{b_5-b_6+b_7-b_8}{64}\,, \end{aligned}$$

in order to simplify the computations. We have the following results

Theorem 3. *For a lopsided system of degree seven, suppose that $b_6 = b_3 = 0$, we have:* (i) *The origin is a centre if and only if the system is time-reversible.* (ii) *If the system is not time-reversible, then no more than seven local limit cycles can bifurcate out of the origin.*

In general, when $k$ is odd, one conjecture the following

Conjecture. When $k$ is odd, the origin of system (4) is a centre if and only if the system is time-reversible.

## 2. The construction of a focal values

We now describe the procedure for determining the focal values $\eta_{2k}$ and the Liapunov quantities $L(k)$ (see [15]) and write $V(x,y) = V_2(x,y) + V_3(x,y) + \cdots + V_k(x,y) + \cdots$, $(V_2(x,y) = \frac{1}{2}(x^2+y^2))$, where $V_k$ is a homogeneous polynomials of degree $k$; let for $k \geq 2$

$$V_k = \sum_{i=0}^{k-i} V_{k-i,i}\, x^{k-i} y^i\,;$$

for convenience, we say that $V_{i,j}$ is an even or odd coefficient according to whether $i$ is even or odd. Now $\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial t}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}$, the function $V$ in a neighbourhood of the origin is such that its rate of change along orbits is of the form $\dot{V} = \eta_2 r^2 + \eta_4 r^4 + \cdots + \eta_{2k} r^{2k} + \cdots$, where $r^2 = x^2 + y^2$. Let $D_k$ denote the terms of degree $k$ in $\dot{V}$, by direct substitution in the system (2), we get $D_k = y(V_k)_x - x(V_k)_y + R_k(x,y)$, where $R_k(x,y) = (V_{k-1})_x p_2 + (V_{k-1})_y q_2 + \cdots + xp_{k-1} + yq_{k-1}$ and the subscripts $x$ and $y$ denotes partial differentiation with respect to $x$ and $y$ respectively. The idea is to choose the coefficients $V_{i,j}$ and the quantities $\eta_k$ so that $D_k = 0$ if $k$ is odd and $D_k = \eta_k(x^2+y^2)^{k/2}$ if $k$ is even.

Suppose first that $k$ is odd, $k = 2m+1$, the requirement $D_k = 0$ is equivalent to solving a set of $2m+2$ unknowns $V_{i,j}$ with $i+j < k$ and the coefficients arising in the original differential equations. These $2m+2$ equations divide into two sets of $m+1$ linear equations, one set determining the odd coefficients of $V_k$ and the other determining the even coefficients.

When $k$ is even, $k = 2m$, the condition $D_k = \eta_{2m}(x^2+y^2)^m$ gives as $2m+1$ linear equations for $\eta_{2m}$ and the $2m+1$ coefficients of $V_k$. These equations divide into two sets: $m+1$ equations for $\eta_{2m}$ and $m$ odd coefficients of $V_k$, and $m$ equations for the $m+1$ even coefficients. To obtain unique values for the even coefficients of $V_k$, we introduce conditions $V_{m,m} = 0$ if $m$ is even and $V_{m+1,m-1} + V_{m-1,m+1} = 0$ if $m$ is odd. Then the even coefficients of $V_k$ are uniquely determined (for details see [11]).

## 3. The main results

We now consider the lopsided quintic system

$$\dot{x} = \lambda x + y\,, \qquad \dot{y} = -x + \lambda y + q_5(x,y)\,, \tag{5}$$

where $q_5(x,y) = a_1x^5 + a_2x^4y + a_3x^3y^2 + a_4x^2y^3 + a_5xy^4 + a_6y^5$. For a lopsided quartic system it was shown in [1] that $L(k)$ is derived from $\eta_{6k+2}$ in each case.

Now we have $\dot{V} = (x + (V_3)_x + (V_4)_x + \dots)\,(\lambda x + y) + (y + (V_3)_y + (V_4)_y + \dots)(-x + \lambda y + q_5)$, so $\dot{V} = \lambda r^2 + o(r^2)$ as $r \to 0$ where $r^2 = x^2 + y^2$, we have also $\dot{V} = \eta_2 r^2 + o(r^2)$ as $r \to 0$. Then $L(0) = \eta_2 = \lambda$, we set $\lambda = 0$ to compute more focal values. Now

$$\begin{aligned}\dot{V} &= (x + (V_3)_x + (V_4)_x + \dots)(y) + (y + (V_3)_y + (V_4)_y + \dots)(-x + q_5)\\ &= (y(V_3)_x - x(V_3)_y) + (y(V_4)_x - x(V_4)_y + (y(V_5)_x - x(V_5)_y)\\ &\quad + (y(V_6)_x - x(V_6)_y + yq_5) + (y(V_7)_x - x(V_7)_y + (V_3)_y q_5)\\ &\quad + \dots + (y(V_k)_x - x(V_k)_y + (V_{k-4})_y q_5)\,.\end{aligned}$$

Let $D_k$ denote terms of degree $k$ in $\dot{V}$, then $D_k = y(V_k)_x - x(V_k)_y + (V_{k-4})_y q_5$. The condition $D_3 = 0$ gives two sets of equations $3V_{3,0} - 2V_{1,2} = 0$, $V_{1,2} = 0$; $2V_{2,1} - 3V_{0,3} = 0$, $-V_{2,1} = 0$; from theses two sets it follows that $V_3 = 0$.

Now $D_4 = \eta_4(x^2 + y^2)^2$ gives two sets of equations $4V_{4,0} - 2V_{2,2} = 0$, $2V_{2,2} - 4V_{0,4} = 0$; $-\eta_4 - V_{3,1} = 0$, $-2\eta_4 + 3V_{3,1} - 3V_{1,3} = 0$, $-\eta_4 + V_{1,3} = 0$; from these two sets with the condition $V_{2,2} = 0$, we get $V_4 = 0$ and $\eta_4 = 0$, so $\eta_4$ does not contribute $L(1)$.

From $D_5 = 0$, we get two sets of equations $5V_{5,0} - 2V_{3,0} = 0$, $3V_{3,2} - 4V_{1,4} = 0$, $V_{1,4} = 0$; $-V_{4,1} = 0$, $4V_{4,1} - 3V_{2,3} = 0$, $2V_{2,3} - 5V_{0,5} = 0$; so we get $V_5 = 0$.

Now $D_6 = \eta_6(x^2 + y^2)^3$ gives two sets of equations $6V_{6,0} - 2V_{4,2} + a_1 = 0$, $4V_{4,2} - 4V_{2,4} + a_3 = 0$, $2V_{2,4} - 6V_{0,6} + a_5 = 0$; $-\eta_6 - V_{5,1} = 0$, $-3\eta_6 + 5V_{5,1} - 3V_{3,3} + a_2 = 0$, $-3\eta_6 + 3V_{3,3} - 5V_{1,5} + a_4 = 0$, $-\eta_6 + V_{1,5} + a_6 = 0$. From the second set one can get $\eta_6 = \frac{1}{16}(5a_6 + a_2 + a_4)$, so $L(1) = 5a_6 + a_2 + a_4$; from these two sets of equations with the condition $V_{4,2} + V_{2,4} = 0$, and after some calculations we get

$$\begin{aligned}V_6 = &\tfrac{-1}{6}(\tfrac{a_3}{4} + a_1)x^6 - \tfrac{1}{16}(a_2 + a_4 + 5a_6)x^5 y - \tfrac{1}{8}a_3 x^4 y^2\\ &+ \tfrac{1}{6}(a_2 - a_4 - 5a_6)x^3 y^3 + \tfrac{1}{8}a_3 x^2 y^4\\ &+ \tfrac{1}{16}(a_2 + a_4 - 11a_6)xy^5 + \tfrac{1}{6}(\tfrac{a_3}{4} + a_5)y^6\,.\end{aligned}$$

By similar calculations, and by using MapleV Release 4, we obtain

$$\begin{array}{llll} V_7 = V_8 = V_9 = 0\,, & V_{10} \neq 0\,, & V_{11} = V_{12} = V_{13} = 0\,, & V_{14} \neq 0\,,\\ V_{15} = V_{16} = V_{17} = 0\,, & V_{18} \neq 0\,, & V_{19} = V_{20} = V_{21} = 0\,, & V_{22} \neq 0\,,\\ V_{23} = V_{24} = V_{25} = 0\,, & V_{26} \neq 0\,, & & \end{array}$$

$$\eta_8 = \eta_{12} = \eta_{16} = \eta_{20} = \eta_{24} = 0\,, \qquad 0 \notin \{\,\eta_{10}\,, \eta_{14}\,, \eta_{18}\,, \eta_{22}\,, \eta_{26}\,\}\,.$$

So far we have the following

LEMMA 1. *For a lopsided quintic system, the Liapunov quantities* $L(k) = \eta_{4k+2}$ *modulo* $\langle \eta_2, \eta_6, \ldots, \eta_{4k-2} \rangle$ *(the ideal generated by* $\eta_2, \eta_6, \ldots, \eta_{4k-2}$*) are for* $k = 0, 1, \ldots, 6$*:*
$L(0) = \lambda$*;*
$L(1) = 5a_6 + a_2 + a_4$ *modulo* $\langle \lambda \rangle$*;*
$L(2) = \eta_{10}$ *modulo* $\langle \lambda, 5a_6 + a_2 + a_4 \rangle$*;*
$L(3) = \eta_{14}$ *modulo* $\langle \lambda, 5a_6 + a_2 + a_4, \eta_{10} \rangle$*;*
$L(4) = \eta_{18}$ *modulo* $\langle \lambda, 5a_6 + a_2 + a_4, \eta_{10}, \eta_{14} \rangle$*;*
$L(5) = \eta_{22}$ *modulo* $\langle \lambda, 5a_6 + a_2 + a_4, \eta_{10}, \eta_{14}, \eta_{18} \rangle$*;*
$L(6) = \eta_{26}$ *modulo* $\langle \lambda, 5a_6 + a_2 + a_4, \eta_{10}, \eta_{14}, \eta_{18}, \eta_{22} \rangle$*.*

For the expressions of $\eta_{10}, \eta_{14}, \eta_{18}, \eta_{22}$ and $\eta_{26}$ see the Appendix.

*Remark.* (1) The focal values $\eta_{4i+2}$ for $i \geq 2$ are of the form $a_6 P_{i1} + a_4 P_{i2} + a_2 P_{i3}$, where $P_{i1}, P_{i2}$ and $P_{i3}$ are polynomials in $a_1, a_3$ and $a_5$, and for $i = 1$ we have $P_{11} = 5$ and $P_{12} = P_{13} = 1$.
(2) If $a_2 = a_4 = a_6 = 0$ then all the Liapunov quantities are zero, so the origin is a center for the system (5).

Now we shall show that $V_k = 0$ for $k \not\equiv 2 \mod 4$ and $\eta_k = 0$ for $k \not\equiv 2 \mod 4$. That is we consider the following

LEMMA 2. *For the system* (5) *we have*

(i) $V_k = 0$ *if* $k \equiv 3 \mod 4$*;*
(ii) $V_k = 0$ *if* $k \equiv 4 \mod 4$, $\eta_k = 0$ *if* $k \equiv 4 \mod 4$*;*
(iii) $V_k = 0$ *if* $k \equiv 5 \mod 4$.

*Proof.* Note that $D_k = y(V_k)_x - x(V_k)_y + (V_{k-4})_y q_5,\ k \geq 6$.
(i) We have $V_3 = V_7 = 0$, let $k = 4l + 3$, $l = 1, 2, \ldots$, now $D_{4l+3} = y(V_{4l+3})_x - x(V_{4l+3})_y + (V_{4l-1})_y q_5$. For $l = 1$ we have $V_7 = 0$. Suppose that the result is true for $l$ so that $V_{4l+3} = 0$, we show that it holds also for $l + 1$. Now $D_{4l+7} = 0$ gives two sets of equations

$$\left.\begin{aligned} (4l+7)V_{4l+7,0} - 2V_{4l+5,2} &= 0 \\ (4l+5)V_{4l+5,2} - 4V_{4l+3,4} &= 0 \\ &\vdots \\ 3V_{3,4l+4} - (4l+6)V_{1,4l+6} &= 0 \\ V_{1,4l+6} &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} -V_{4l+6,1} &= 0 \\ (4l+6)V_{4l+6,1} - 3V_{4l+4,3} &= 0 \\ (4l+4)V_{4l+4,3} - 5V_{4l+2,5} &= 0 \\ &\vdots \\ 2V_{2,4l+5} - (4l+7)V_{0,4l+7} &= 0 \end{aligned}\right\};$$

from these two sets we get $V_{1,4l+6} = V_{3,4l+4} = \cdots = V_{4l+7,0} = 0$ and $V_{4l+6,1} = V_{4l+4,3} = \cdots = V_{0,4l+7} = 0$. Hence $V_{4l+7}$=0, therefore $V_k = 0$ if $k \equiv 3 \mod 4$.

(ii) We have $V_4 = V_8 = 0$, let $k = 4l+4$, $l = 1, 2, \ldots$ . For $l = 1$ we have $V_8 = 0$ and the corresponding focal value $\eta_8 = 0$. Suppose that the result is true for $l$, i.e., $V_{4l+4} = 0$, and we have to show that it is still true for $l+1$. Since $D_{4l+8} = \eta_{4l+8}(x^2+y^2)^{\frac{4l+8}{2}}$, we have the following two sets of equations

$$\left.\begin{aligned} (4l+8)V_{4l+8,0} - 2V_{4l+6,2} = 0\\ (4l+6)V_{4l+6,2} - 4V_{4l+4,4} = 0\\ \vdots\\ 4V_{4,4l+4} - (4l+6)V_{2,4l+6} = 0\\ 2V_{2,4l+6} - (4l+8)V_{0,4l+8} = 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} -\eta_{4l+8} - V_{4l+7,1} = 0\\ -\binom{2l+4}{1}\eta_{4l+8} + (4l+7)V_{4l+7,1} - 3V_{4l+5,3} = 0\\ -\binom{2l+4}{2}\eta_{4l+8} + (4l+5)V_{4l+5,3} - 5V_{4l+3,5} = 0\\ \vdots\\ -\binom{2l+4}{2l+3}\eta_{4l+8} + 3V_{3,4l+5} - (4l+7)V_{1,4l+7} = 0\\ -\eta_{4l+8} + V_{1,4l+7} = 0 \end{aligned}\right\};$$

the first set with condition $V_{4l+2,4l+2} = 0$ gives $V_{4l+8,0} = V_{4l+6,2} = \cdots = V_{0,4l+8} = 0$, and the second set gives $\eta_{4l+8} = V_{4l+7,1} = \cdots = V_{1,4l+7} = 0$, hence $V_{4l+8} = 0$. Therefore $V_k = 0$ if $k \equiv 4 \mod 4$ and $\eta_k = 0$ if $k \equiv 4 \mod 4$.

(iii) When $k = 4l+5$, $l = 1, 2, \ldots$, $D_{4l+5} = y(V_{4l+5})_x - x(V_{4l+5})_y + (V_{4l+1})q_5$. When $l = 1$, $V_9 = 0$. Assuming that the result is true for $l$, that is $V_{4l+5} = 0$, we shall show that the result is also true for $l+1$. Now $D_{4l+9} = 0$ gives two sets of equation

$$\left.\begin{aligned} (4l+9)V_{4l+9,0} - 2V_{4l+7,2} = 0\\ (4l+7)V_{4l+7,2} - 4V_{4l+5,4} = 0\\ \vdots\\ 3V_{3,4l+6} - (4l+8)V_{1,4l+8} = 0\\ V1, 4l+8 = 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} -V_{4l+8,1} = 0\\ (4l+8)V_{4l+8,1} - 3V_{4l+6,3} = 0\\ (4l+6)V_{4l+6,3} - 5V_{4l+4,5} = 0\\ \vdots\\ 2V_{2,4l+7} - (4l+9)V_{0,4l+9} = 0 \end{aligned}\right\};$$

from these two sets we get $V_{1,4l+8} = V_{3,4l+6} = \cdots = V_{4l+9,0} = 0$ and $V_{4l+8,1} = V_{4l+6,3} = \cdots = V_{0,4l+9} = 0$, then $V_{4l+9} = 0$. Therefore $V_k = 0$ if $k \equiv 5 \mod 4$. ∎

The above lemma shows that, for a lopsided quintic system, there are some focal values which are identically null; in this case these focal values do not contribute a Liapunov quantities.

*Proof of Theorem* 1. Follows immediately from Lemma 2 and from the fact that the Liapunov quantities $L(k)$ are the non-zero expressions obtained by calculating each non-zero $\eta_{2k+2}$. ∎

*Remark.* The origin of a lopsided quintic system (5) is a fine focus of order $k$ if $\eta_2 = \eta_6 = \cdots = \eta_{4k-2} = 0$, but $\eta_{4k+2} \neq 0$, so the Liapunov quantities $L(0), L(1), \ldots$ are the non-zero expressions obtained by calculating each $\eta_{4k-2}$ under the conditions $\eta_2 = \eta_6 = \cdots = \eta_{4k-6} = 0$.

When the origin is a fine focus of order $k$, no more than $k$ limit cycles can bifurcate from the origin under the perturbation of the system (see [3]), these limit cycles are so-called *small-amplitude limit cycles*. But it is not necessarily true that this maximum number is attained, especially when fewer than $k$ Liapunov quantities are derived from $\eta_2, \ldots, \eta_{2k+2}$, in which case it may be that less than $k$ limit cycles bifurcate out of a fine focus of order $k$. However for a lopsided quintic system this cannot occur because we can find $k$ Liapunov quantities from $\eta_2, \ldots, \eta_{4k+2}$.

Now for a lopsided quintic system, we assume that, for $k \leq 6$, $L(k)$ are those described in Lemma 1.

*Proof of Theorem* 2. (A) First the origin is a centre by the symmetry principle, when the system is time-reversible. Second when the origin is a centre, we have to show that the system is time-reversible.

With $\lambda = 5a_6 + a_2 + a_4 = 0$, we have $L(0) = L(1) = 0$. We suppose that

$$a_2 = -5a_6 - a_4\,. \tag{6}$$

We substitute (6) in $\eta_{10} = 0$, when $14a_6 + a_4 \neq 0$ we get

$$a_3 = -(14a_6 + a_4)^{-1}(7a_1a_4 + 2a_5a_6 - a_5a_4 + 50a_1a_6)\,. \tag{7}$$

If $14a_6 + a_4 = 0$, knowing $a_4 = -14a_6$ and (6) we have $\eta_{10} = -\frac{1}{16}a_6(a_5 - 3a_1)$, the vanishing of $\eta_{10}$ gives two conditions $a_6 = 0$ or $a_5 = 3a_1$. If $a_6 = 0$ we have $a_4 = a_2 = 0$, and we get that the system is invariant by the change of variables $(x, y, t) \mapsto (-x, y, -t)$ and this ensures that the system is time-reversible. If $a_5 = 3a_1$ with $a_6 \neq 0$, knowing $a_4 = -14a_6$ and (6) we have

$$\eta_{14} = a_6(560a_1^2 + 192a_6^2 + 9a_3^2 + 136a_3a_1)\,,$$

$$\eta_{18} = a_6(80112a_3a_1^2 + 250944a_1^3 + 280832a_6^2a_1 + 9420a_1a_3^2 + 405a_3^3 + 32320a_6^2a_3)\,.$$

For the vanishing of $\eta_{14}$ and $\eta_{18}$, we compute the resultant of $\eta_{14}$ with $\eta_{18}$ rapport to $a_6$ which is

$$\Re(\eta_{14}, \eta_{18}, a_6) = (1665a_3 + 13316a_1)^2(8a_1 + a_3)^4\,.$$

If $a_3 = -8a_1$ we have $\eta_{14} = a_6(a_1^2 + 4a_6^2)$, since $a_6 \neq 0$ it is impossible $\eta_{14}$ to be zero. If $a_3 = -\frac{13316}{1665}a_1$ we have $\eta_{14} = a_6(51317a_1^2 + 205350a_6^2)$, and it is not possible for $\eta_{14}$ to be zero for the same reason. So if $14a_6 + a_4 = 0$ the only possibility for the origin to be a centre is $a_6 = a_4 = a_2 = 0$ which gives the system time-reversible.

If $14a_6 + a_4 \neq 0$, knowing $a_2$ from (6) and $a_3$ from (7), for the vanishing of $\eta_{14}, \eta_{18}, \eta_{22}$ and $\eta_{26}$ we compute $\Re(\eta_{14}, \eta_{18}, a_1)$, $\Re(\eta_{14}, \eta_{22}, a_1)$ and $\Re(\eta_{14}, \eta_{26}, a_1)$ which are the resultants of $\eta_{18}$, $\eta_{22}$ and $\eta_{26}$ with $\eta_{14}$ rapport to $a_1$ respectively. We obtain

$$\Re(\eta_{14}, \eta_{18}, a_1) = (a_4 + 10a_6)(a_4 + 2a_6)(a_4^2 - 20a_6a_4 + 100a_6^2 + 16a_5^2)^2\psi_1(a_4, a_5, a_6),$$

$$\Re(\eta_{14}, \eta_{22}, a_1) = (a_4^2 - 20a_6a_4 + 100a_6^2 + 16a_5^2)^2\psi_2(a_4, a_5, a_6),$$

$$\Re(\eta_{14}, \eta_{26}, a_1) = (a_4^2 - 20a_6a_4 + 100a_6^2 + 16a_5^2)^2\psi_3(a_4, a_5, a_6),$$

where $\psi_1, \psi_2$ and $\psi_3$ are polynomials in $a_4, a_5$ and $a_6$.

If $a_4 + 10a_6 = 0$, after the substitution of $a_4 = -10a_6$ we obtain $\eta_{14} = a_6(a_5 - 5a_1)^2$. For $\eta_{14} = 0$ we have two cases $a_6 = 0$ and $a_5 = 5a_1$. If $a_6 = 0$ we get $a_4 = 0$ which is not possible because we have $14a_6 + a_4 \neq 0$. If $a_5 = 5a_1$ with $a_6 \neq 0$ we obtain $\eta_{22} = a_6(a_6^2 + a_1^2)^2$, which is impossible because the origin is centre and $\eta_{22} \neq 0$. Now if $a_4 + 2a_6 = 0$, we substitute $a_4 = -2a_6$ in $\eta_{14}$, we get $\eta_{14} = a_6(a_5 + 3a_1)(7a_5 - 27a_1)$. If $a_6 = 0$ we have $a_4 = 0$ and it is impossible because $14a_6 + a_4 \neq 0$. If $a_5 + 3a_1 = 0$ with $a_6 \neq 0$ we change $a_5 = -3a_1$ in $\eta_{22}$ and we obtain $\eta_{22} = a_6(a_6^2 + a_1^2)^2$, so it is not possible because the origin is a centre and $\eta_{22} \neq 0$. Next if $7a_5 - 27a_1 = 0$ with $a_5 = -\frac{27}{7}a_1$ we have $\eta_{22} = a_6(1152480a_6^4 + 78302980a_1^2a_6^2 + 23426337a_1^4)$ and it is not possible to vanish $\eta_{22}$ since $a_6 \neq 0$.

If $a_4^2 - 20a_6a_4 + 100a_6^2 + 16a_5^2 = (10a_6 - a_4)^2 + 16a_5^2 = 0$, this implies $a_5 = 0$ and $a_4 = 10a_6$, so we have $\eta_{14} = a_6(16a_6^2 + a_1^2)$ which is not possible to vanish it since $14a_6 + a_4 \neq 0$. Finally, in order to vanish the last term of $\Re(\eta_{14}, \eta_{18}, a_1)$, that is $\psi_1(a_4, a_5, a_6)$, we do the changes $a_4 = c\,a_6$ and $a_5 = d\,a_6$, where $c, d \in R$ and $a_6 \neq 0$.

If $a_6 = 0$ we have $\psi_1(a_4, a_5, a_6) = a_5^2a_4^9$. We substitute $a_6 = a_5 = 0$ in $\eta_{14}$ and we get $\eta_{14} = a_4(18a_1^2 + a_4^2)$, and it is impossible to vanish $\eta_{14}$ because we have $14a_6 + a_4 \neq 0$. If $a_6 \neq 0$ after the change mentioned previously, we compute $\Re(\psi_1, \psi_2, d)$ and $\Re(\psi_1, \psi_3, d)$ which are

$$\Re(\psi_1, \psi_2, d) = a_6^{80}(c+10)^{12}\delta_1(c)\,,$$
$$\Re(\psi_1, \psi_3, d) = a_6^{112}(c+10)^{12}(c+8)^2\delta_2(c)\,,$$

where $\delta_1$ and $\delta_2$ are polynomials in the variable $c$ and they have not common roots. Since $a_6 \neq 0$, $a_4 + 10a_6 \neq 0$ and the polynomials $\delta_1$ and $\delta_2$ have not common roots, so the only possibility is $c = -8$, in this case we put $a_4 = -8a_6$ in $\psi_1$ and we obtain $\psi_1 = a_6^9 a_5^2$. We have $\psi_1 = 0$ implies $a_5 = 0$, so we put $a_4 = -8a_6$ and $a_5 = 0$ in $\eta_{14}$, we get $\eta_{14} = a_1^2 a_6$, the vanishing of $\eta_{14}$ yelds $a_1 = 0$. Now we have $\eta_{22} = a_6^5$ and it is impossible to be zero since we have $a_6 \neq 0$.

(B) In the above proof (A), it is clear that the system is not time-reversible when we have $\eta_2 = \eta_6 = \eta_{10} = \eta_{14} = \eta_{18} = 0$ and $\eta_{22} \neq 0$. Since we have $L(k)$ is derived from $\eta_{4k+2}$, we have $L(0) = L(1) = L(2) = L(3) = L(4) = 0$ and $L(5) \neq 0$, so the order of the origin is five, then we have at most five local limit cycles which bifurcate out of the origin. ∎

The Liapunov quantities $L(k)$ for the lopsided system of degree seven are available in the following e-mail address: salih@@math.unice.fr.

*Proof of Theorem* 3. (A) If the system is reversibe, by the symmetry principle, the origin is a center. Now we suppose that the origin is a center, we have to show that the system is time-reversible. In order to have a fine focus, we put $L(0) = \lambda = 0$, so the first Liapunov quantities $L(1)$ is $\frac{b_5}{128}$. Knowing $b_5 = 0$ from $L(1) = 0$ we compute the second Liapunov quantities $L(2)$ which is $\frac{b_3b_6 - b_1b_6 - b_2b_7 + b_4b_7 - b_3b_8}{16384}$. Since we have $b_6 = b_3 = 0$, the vanshing of $L(2)$ implies $b_7 = 0$ or $b_4 = b_2$.

(a) If $b_7 = 0$, we have the third Liapunov quantities $L(3) = \frac{b_2b_8(52b_1 - 9b_2)}{33554432}$. $L(3) = 0$ gives $b_2 = 0$ or $b_8 = 0$ or $b_2 = \frac{52}{9}b_1$.

(a.1) If $b_2 = 0$ the fourth Liapunov quantities $L(4) = -\frac{b_8b_1^3}{67108864}$, so $L(4) = 0$ implies $b_1 = 0$ or $b_8 = 0$.

(a.1.1) If $b_1 = 0$ with $b_8 \neq 0$, in this case we have $L(5) = L(6) = 0$ and $L(7) = 1491b_8(b_4^2 + b_8^2)^3$, so it is impossible to vanish $L(7)$ which is a contradiction because we have a center for the origin.

(a.1.2) If $b_8 = 0$ with $b_1 \neq 0$, we have $L(5) = L(6) = L(7) = 0$ and we get that the system is time-reversible.

(a.2) If $b_8 = 0$ the computation gives $L(5) = L(6) = L(7) = 0$ and we obtain another particular case of a time-reversible system.

(a.3) If $b_2 = \frac{52}{9} b_1$ with $b_1 \neq 0$and $b_8 \neq 0$, we have $L(4) = -7b_1 b_8(10859 b_1^2 + 1755 b_4^2 + 1755 b_8^2)$ and in this case it is not possible to vanish $L(4)$ which is a contradiction.

(b) If $b_2 = b_4$ we compute the fourth Liapunov quantities which is $L(3) = 30b_1^2 b_7 - 10b_1 b_4 b_7 + 18b_4^2 b_7 + 52b_1 b_4 b_8 - 9b_4^2 b_8 + 3b_7^2 b_8$.

(b.1) If $52b_1b_4 - 9b_4^2 + 3b_7^2 \neq 0$ the vanishing of $L(3)$ gives $b_8 = \frac{2b_7(5b_1b_4 - 15b_1^2 - 9b_4^2)}{52b_1b_4 - 9b_4^2 + 3b_7^2}$.

(b.1.1) If $b_7 = 0$ we have $b_8 = 0$ which is the case (a.2).

(b.1.2) If $b_7 \neq 0$, we suppose that $b_7 = 1$ and after we compute $L(4)$, $L(5)$ and $L(6)$, for the vanshing of $L(4)$, $L(5)$ and $L(6)$ we calcul $\Re(L(4), L(5), b_1)$ and $\Re(L(4), L(6), b_1)$ which are the resultants of the polynomials $L(5), L(6)$ with $L(4)$ rapport to $b_1$ respectively, obtaining the following polynomials

$$\Re(L(4), L(5), b_1) = b_4^4(45 - 10b_4^2 + 7737b_4^4)^{12} P_1(b_4)\,,$$
$$\Re(L(4), L(6), b_1) = b_4(45 - 10b_4^2 + 7737b_4^4)^{15} P_2(b_4)\,,$$

where $P_1(b_4)$ and $P_2(b_4)$ are polynomials of $b_4$ and of degree 44 and 58 respectively. Moreover, they have no common roots. So the only possibility is $b_4 = 0$ which gives another particular case of a time-reversible system.

(b.2) If $52b_1b_4 - 9b_4^2 + 3b_7^2 = 0$ and $b_4 \neq 0$ we change $b_1 = \frac{9b_4^2 2 - 3b_7^2}{52b_4}$ in $L(3)$ which gives $L(3) = \frac{3b_7(7737b_4^4 - 10b_4^2 b_7^2 + 45b_7^4)}{b_4^2}$. Since $b_4 \neq 0$, knowing $b_7 = 0$ from $L(3) = 0$ we compute $L(4)$ which is $L(4) = -7b_4 b_8(208337b_4^2 + 175760b_8^2)$, so $L(4) = 0$ implies $b_8 = 0$ and is the case (a.2) of a time-reversible system.

(b.3) If $52b_1b_4 - 9b_4^2 + 3b_7^2 = 0$ and $b_4 = 0$ we have $b_7 = 0$, so $L(3) = 0$. The computation gives $L(4) = -b_8 b_1^3$ and $L(5) = b_8 b_1^2(14825 b_1^2 + 4002 b_8^2)$. Both cases $b_1 = 0$ and $b_8 = 0$ gives that the system is time-reversible by the symmetry conditions.

(B) In the above (a.1.1), the system is not time-reversible when we have $L(0) = L(1) = L(2) = L(3) = L(4) = L(5) = L(6) = 0$ and $L(7) = 1491b_8(b_4^2 + b_8^2)^3 \neq 0$, so the order of the origin is seven then no more than seven local limit cycles can bifurcate out of the origin. ∎

## Appendix

The focal values $\eta_{10}$, $\eta_{14}$, $\eta_{18}$, $\eta_{22}$ and $\eta_{26}$ are of the following forms:

$$\eta_{10} = a_6(61a_5 - 3a_3 - 95a_1) + a_4(15a_5 + 3a_3 - 13a_1) + a_2(13a_5 + a_1 + 5a_3)\,,$$

$$\begin{aligned}\eta_{14} = {} & a_6(6490a_2a_4 - 1016a_3^2 - 24380a_1^2 - 5043a_4^2 + 4293a_2^2 + 4724a_5^2 - 1303a_2a_6 \\ & - 67407a_4a_6 - 17120a_3a_1 - 848a_3a_5 - 34040a_1a_5 - 176705a_6^2) + a_4(1332a_5^2 \\ & - 6296a_1a_5 + 665a_2^2 + 1401a_2a_4 - 88a_3^2 - 5212a_1^2 + 480a_3a_5 - 3664a_1a_3 + 251a_4^2) \\ & + a_2(932a_5^2 + 72a_3^2 - 2828a_1^2 - 2368a_1a_3 + 848a_3a_5 - 3032a_1a_5 - 421a_2^2)\,,\end{aligned}$$

$$\begin{aligned}\eta_{18} = {} & a_6(1368031a_1a_2^2 + 647766a_3a_2^2 - 2266941a_1a_4^2 - 798715a_1a_3^2 - 1534750a_3a_1^2 \\ & - 156825a_5a_3^2 - 81988625a_5a_6^2 - 20876950a_3a_6^2 - 282915a_5^3 - 10030a_3^3 + 4611525a_1a_6^2 \\ & - 3445435a_1a_5^2 - 4052465a_1a_2a_6 - 5443150a_2a_3a_6 - 454610a_3a_5^2 - 4075525a_1^2a_5 \\ & - 5968807a_5a_4^2 - 2411346a_3a_4^2 - 14434500a_3a_4a_6 - 5905275a_1a_4a_6 - 40930405a_4a_5a_6 \\ & + 972621a_2^2a_5a_6 - 13780795a_2a_5a_6 - 4018125a_1^3 - 1371430a_1a_2a_4 - 3511640a_1a_3a_5 \\ & - 2184426a_2a_4a_5 - 839360a_2a_3a_4) + a_4(60997a_1a_2^2 + 420459a_2^2a_5 - 331337a_1a_3^2 \\ & - 736880a_1^2a_3 - 69945a_5^3 - 219299a_5a_4^2 - 188397a_1a_4^2 - 78308a_3a_4^2 - 882437a_1a_5^2 \\ & - 127636a_3a_5^2 - 1286855a_1^2a_5 - 75375a_3^2a_5 + 158113a_2a_4a_5 - 1117275a_1^3 \\ & + 242068a_2^2a_3 - 141741a_1a_2a_4 + 134034a_2a_3a_4 - 17936a_3^3 - 1087900a_1a_3a_5) \\ & + a_2(78327a_2^2a_5 - 65625a_3^2a_5 - 72795a_5^3 - 124139a_1a_2^2 + 12686a_3a_2^2 - 21386a_3^3 \\ & - 1368605a_1^2a_5 - 403667a_1a_3^2 - 104206a_3a_5^2 + 647766a_2a_3a_6 - 1182525a_1^3 \\ & - 1058080a_1a_3a_5 - 1034090a_1^2a_3 - 704027a_1a_5^2)\,,\end{aligned}$$

$$\begin{aligned}\eta_{22} = {} & a_6(18365025280a_1^3a_3 - 229652191920a_3a_5a_6^2 + 367415040a_3^2a_5^2 + 14954928960a_1^2a_5^2 \\ & + 7348134262a_2^3a_6 - 59366026702a_2^2a_6^2 - 1753788287a_2^4 + 443058852925a_2a_6^3 \\ & + 343126337a_4^4 + 644392962866a_4^2a_6^2 + 50543927274a_4^3a_6 + 3000453114445a_4a_6^3 \\ & - 495804061320a_5^2a_6^2 - 29266290400a_3^2a_6^2 + 210883952a_3^4 + 207774673400a_1^2a_6^2 \\ & - 11539712800a_1^4 - 803864096a_5^4 - 39636377600a_3^2a_4a_6 + 54928890280a_1^2a_4a_6 \\ & - 51553494178a_2a_4^2a_6 + 692215808a_3^3a_5 - 42192981400a_1^2a_2a_6745985280a_3a_5^3 \\ & + 4020073456a_1a_2^2a_5 + 9269588648a_1^2a_2^2 - 45093552450a_2^2a_4a_6 - 183469856a_2^2a_3^2 \\ & - 3937648344a_2^2a_5^2 - 6891973242a_2^2a_4^2 - 2869515516a_2^3a_4 - 8938261628a_2a_4^3 \\ & + 26715614436a_2a_4a_6^2 + 953236592a_1a_2^2a_3 - 1577147792a_2^2a_3a_5 + 4795803337725a_6^4 \\ & - 151407450392a_2a_5^2a_6 - 30016606976a_2a_3^2a_6 - 4651858568a_1^2a_4^2 - 59785062408a_4^2a_5^2 \\ & - 10953900944a_3^2a_4^2 - 307010757592a_4a_5^2a_6 + 639750967760a_1a_5a_6^2 \\ & + 297573354320a_1a_3a_6^2 - 124613041424a_2a_3a_5a_6 - 13627892752a_1a_2a_5a_6 \\ & - 18127553520a_1^2a_2a_4 - 25251198224a_1a_2a_3a_6 - 43369822112a_2a_3a_4a_5\end{aligned}$$

$$
\begin{aligned}
&- 22049322912a_1a_2a_3a_4 - 25438955168a_1a_2a_4a_5 - 50088734960a_2a_4a_5^2\\
&- 11175315200a_2a_3^2a_4 - 479696091a_3a_4^2a_5 + 13580240640a_1^2a_3^2 + 14042528640a_1^3a_5\\
&+ 1799962624a_1a_3^3 + 4293665152a_1a_5^3 - 603963824a_1a_3a_4^2 + 3085241488a_1a_4^2a_5\\
&- 204625874320a_3a_4a_5a_6 + 183482955632a_1a_4a_5a_6 + 92214965104a_1a_3a_4a_6\\
&+ 28761265920a_1^2a_3a_5 + 6446897664a_1a_3a_5^2 + 5090899072a_1a_3^2a_5) + a_4(82565504a_1a_5^3\\
&+ 939443520a_2^2a_3^2 + 366048986a_2^3a_4 + 283330525a_2^4 - 236989555a_2a_4^3 + 156242746a_2^2a_4^2\\
&- 3598154744a_4^2a_5^2 - 730655920a_3^2a_4^2 - 2083280a_3^4 - 1305374584a_1^2a_4^2 - 5249595680a_1^4\\
&- 281377824a_5^4 - 40757248a_1^3a_3 + 591080256a_1^2a_5^2 - 78058752a_3^3a_5\\
&- 1978103432a_1^2a_2a_4 + 859328256a_1^2a_3^2 + 2128280104a_1^2a_2^2 + 1220339368a_2^2a_5^2\\
&+ 2096839536a_1a_2^2a_5 + 1041265008a_1a_2^2a_3 + 2439999088a_2^2a_3a_5 - 61794503a_4^4\\
&- 3320323080a_2a_5^2a_4 - 459155312a_2a_3^2a_4 - 3112823888a_3a_4^2a_5 - 1414781840a_1a_4^2a_5\\
&- 1131466576a_1a_3a_4^2 - 1324717184a_1^3a_5 - 343726848a_3^2a_5^2 - 131394304a_1a_3^3\\
&- 524678400a_3a_5^3 - 2397631152a_2a_3a_4a_5 - 3075959792a_1a_2a_4a_5\\
&- 2642359984a_1a_2a_3a_4 + 1321310976a_1^2a_3a_5 - 747591168a_1a_3a_5^2 - 752934784a_1a_3^2a_5)\\
&+ a_2(1563924176a_2^2a_3a_5 + 763676984a_1^2a_2^2 - 37472592a_3^4 + 508014080a_2^2a_3^2\\
&+ 1240789688a_2^2a_5^2 - 10569385120a_1^4 - 153042848a_5^4 - 458829568a_3a_5^3\\
&- 2821853632a_1^2a_5^2 - 196518912a_3^3a_5 + 1190313680a_1a_2^2a_3 + 1886015440a_1a_2^2a_5\\
&- 2503069440a_1^2a_3^2 + 233938921a_2^4 - 8527134336a_1^3a_5 - 446551808a_3^2a_5^2\\
&- 723067904a_1a_3^3 - 7291479552a_1^3a_3 - 299255936a_1a_5^3 - 5839216384a_1^2a_3a_5\\
&- 2179015680a_1a_3a_5^2 - 2452349312a_1a_3^2a_5)\,,
\end{aligned}
$$

$$
\begin{aligned}
\eta_{26} = {}& a_6(330066419200a_3^5 + 7333942231560a_5^5 + 67820798325000a_1^5 + 58279519566240a_1^2a_3^3\\
&+ 197072391042800a_1^3a_3^2 + 5809865036640a_3^3a_5^2 + 42996674880040a_1a_5^4\\
&+ 12150099552080a_3^2a_5^3 + 14779209528440a_3a_5^4 + 6709637090240a_1a_3^4\\
&+ 1890474416320a_5a_3^4 + 116603419747920a_1^2a_5^3 + 293563779675600a_1^3a_5^2\\
&+ 92700118782880a_1a_3a_5^3 + 35317104663680a_1a_5a_3^3 + 81147959257680a_1a_3^2a_5^2\\
&+ 213578571949680a_1^2a_5a_3^2 + 485118933034400a_5a_3a_1^3 + 267141060931920a_1^2a_3a_5^2\\
&+ 282315650827000a_3a_1^4 + 322579640761000a_1^4a_5 - 130522020749255a_5^3a_4^2\\
&- 8387828245524a_2^2a_3^3 + 56014253028768a_3a_4^4 - 49549361352443a_5^3a_2^2\\
&- 28060886366153a_1a_2^4 + 95854010972159a_1a_4^4 - 7817615541820a_3a_2^4\\
&- 10086080674055a_5a_2^4 - 11381230521984a_4^2a_3^3 + 133069714030937a_5a_4^4\\
&+ 6500845961055a_2^2a_1^3 + 133465709269875a_4^2a_1^3 - 14830798208168a_4a_3a_2^3\\
&- 67641038288284a_2^2a_3a_4^2 - 257720854414668a_2a_4a_3a_5^2 - 27251007122988a_2a_4a_3^3
\end{aligned}
$$

$$
\begin{aligned}
&- 318107333416148a_2a_5a_4^3 - 138004949154714a_5a_2^2a_4^2 - 73027244971276a_2^2a_3a_5^2\\
&- 70516829898821a_5a_4^2a_3^2 - 153546931763512a_4^2a_3a_5^2 - 28276671949808a_2a_3a_4^3\\
&- 27027054947708a_1a_2^3a_4 + 502987179492749a_1a_4^2a_5^2 - 15022156474575a_1a_2^2a_5^2\\
&+ 306370174664148a_4^2a_3a_1^2 + 544201331803239a_4^2a_1^2a_5 - 49923322520789a_2^2a_1^2a_5\\
&- 60228453854084a_1a_2^2a_3a_5 + 147642070273328a_1a_2a_4a_3a_5 + 229586582693974a_1a_2a_4a_5^2\\
&+ 8399047565582a_1a_2a_4a_3^2 + 520523114076932a_1a_4^2a_3a_5 + 143701551491853a_1a_4^2a_3^2\\
&+ 159637616472330a_2a_4a_1^2a_5 - 60210129497830a_2a_4a_1^3 + 26313051508940a_2a_4a_3a_1^2\\
&- 64803381784128a_2^2a_3a_1^2 - 39233416133129a_5a_2^2a_3^2 + 51382401414996a_1a_2a_4^3\\
&- 36105030545278a_1a_2^2a_4^2 - 35422970864151a_1a_2^2a_3^2 - 37443094819284a_5a_2^3a_4\\
&- 174394268779098a_2a_5^3a_4 - 138681510382870a_2a_5a_4a_3^2 - 323453706772505a_6^2a_5^3\\
&+ 2006894331495125a_6^2a_1^3 + 10166013183326272 5a_6^4a_5 + 40953518382860a_6^2a_3^3\\
&+ 31572410771915700a_6^4a_3 + 18413071754336875a_6^4a_1 - 39518813095086a_6^2a_5a_2^2\\
&+ 8498636675271852a_6^2a_3a_4^2 + 1016166835408125a_6a_4a_1^3 - 72234826512472a_6^2a_3a_2^2\\
&+ 2612132948416710a_6a_5a_4^3 - 37205425210072a_6a_2a_3^3 + 21085164522029186a_6^2a_5a_4^2\\
&+ 8852783125929800a_6^3a_2a_3 - 20713196322a_6a_4a_3^3 + 1378427874429338a_6a_1a_4^3\\
&- 130837116197170a_6^2a_1a_2^2 + 6928088238675375a_6^3a_1a_2 + 8466497447057590a_6^2a_1a_4^2\\
&+ 6966818500367675a_6^2a_1a_5^2 - 56673097331568a_6a_3a_2^3 - 130566017124078a_6a_5a_2^3\\
&+ 1103034199853016a_6a_3a_4^3 - 303452292331339a_6a_2a_5^3 + 1723639393162675a_6^2a_1a_3^2\\
&- 50049684590025a_6a_2a_1^3 - 415339262013349a_6a_4a_5^3 + 22904797744684825a_6^3a_2a_5\\
&+ 212899251534180a_6^2a_3a_5^2 + 1813000844858885a_6^2a_5a_3^2 + 76083429863625955a_6^3a_4a_5\\
&+ 27849773118111510a_6^3a_4a_3 + 22135200295510825a_6^3a_1a_4 + 6375559257922825a_6^2a_1^2a_5\\
&+ 3370280456955400a_6^2a_3a_1^2 - 1518892245634a_6a_1a_2^3 + 274669111680665a_6a_1a_2a_3^2\\
&+ 3964734175174840a_6^2a_2a_3a_4 - 402319453773360a_6a_2a_3a_5^2\\
&+ 9914390989223980a_6^2a_2a_5a_4 + 1043745945018758a_6a_2a_5a_4^2 - 66870212534795a_6a_4a_5a_3^2\\
&- 204662256107425a_6a_2a_5a_3^2 - 300136836224044a_6a_2^2a_3a_4 - 544740615505390a_6a_5a_2^2a_4\\
&- 275779330237490a_6a_4a_3a_5^2 + 3456709588956075a_6a_1a_4a_5^2 + 873331938528114a_6a_1a_2a_4^2\\
&+ 1197424498149545a_6a_1a_2a_5^2 + 998468005582095a_6a_1a_4a_3^2 - 99686917462378a_6a_1a_2^2a_4\\
&+ 3537725263924845a_6a_4a_1^2a_5 + 2004534621313290a_6a_4a_3a_1^2\\
&+ 6659770566880900a_6^2a_1a_3a_5 + 1145536264012260a_6a_1a_2a_3a_5\\
&+ 3571048856641520a_6a_1a_4a_3a_5 + 4468409448665540a_6^2a_1a_2a_4\\
&+ 441976380212300a_6a_2a_3a_1^2 + 1006298868243275a_6a_2a_1^2a_5 + 352111914681660a_6a_2a_3a_4^2)\\
&+ a_4(120061175360a_3^5 + 1911973192920a_5^5 + 1558131351000a_1^5 + 14436455920320a_1^2a_3^3\\
&+ 42414502817680a_1^3a_3^2 + 2300236041600a_3^3a_5^2 + 12315023261240a_1a_5^4
\end{aligned}
$$

$$
\begin{aligned}
&+ 4201675930480a_3^2a_5^3 + 4359474042280a_3a_5^4 + 1944445350400a_1a_3^4 + 759695511680a_5a_3^4 \\
&+ 33339113416560a_1^2a_5^3 + 67757811271920a_1^3a_5^2 + 27939438312800a_1a_3a_5^3 \\
&+ 10625049547840a_1a_5a_3^3 + 24824284697520a_1a_3^2a_5^2 + 54444742996560a_1^2a_5a_3^2 \\
&+ 106555966407520a_5a_3a_1^3 + 71853838638960a_1^2a_3a_5^2 + 49427955420200a_3a_1^4 \\
&+ 58928394236600a_1^4a_5 - 11598193940971a_5^3a_4^2 + 104283170966a_2^2a_3^3 + 611044081362a_3a_4^4 \\
&- 10165897154629a_5^3a_2^2 + 1100788237193a_1a_2^4 + 2237055021725a_1a_4^4 \\
&+ 42337525798a_3a_2^4 - 1597464772509a_5a_2^4 - 1602698722466a_4^2a_3^3 + 1610401969399a_5a_4^4 \\
&+ 26573018501677a_2^2a_1^3 + 3654320624923a_4^2a_1^3 + 1854114140180a_4a_3a_2^3 \\
&- 1464085628280a_2^2a_3a_4^2 - 32111724329668a_2a_4a_3a_5^2 - 3190730163692a_2a_4a_3^3 \\
&- 5737585472999a_2a_5a_4^3 - 6436384720250a_5a_2^2a_4^2 - 10662240312762a_2^2a_3a_5^2 \\
&- 8723633391437a_5a_4^2a_3^2 - 16612839169938a_4^2a_3a_5^2 - 2655256731292a_2a_3a_4^3 \\
&+ 350332548294a_1a_2^3a_4 + 20248648034189a_1a_4^2a_5^2 - 2039732742629a_1a_2^2a_5^2 \\
&+ 10237879734242a_4^2a_3a_1^2 + 21602819201267a_4^2a_1^2a_5 + 4853511829933a_2^2a_1^2a_5 \\
&- 6626937983168a_1a_2^2a_3a_5 - 10828166863364a_1a_2a_4a_3a_5 + 3407702770837a_1a_2a_4a_5^2 \\
&- 8298438084355a_1a_2a_4a_3^2 + 18264727325368a_1a_4^2a_3a_5 + 3877060636961a_1a_4^2a_3^2 \\
&- 3169355199641a_2a_4a_1^2a_5 - 3171638173733a_2a_4a_1^3 - 9814404975784a_2a_4a_3a_1^2 \\
&+ 4228103267298a_2^2a_3a_1^2 - 2416094627003a_5a_2^2a_3^2 + 479726420311a_1a_2a_4^3 \\
&- 1226586740358a_1a_2^2a_4^2 - 3521208052881a_1a_2^2a_3^2 - 93132260382a_5a_2^3a_4 \\
&- 21242870961367a_2a_5^3a_4 - 16908577818157a_2a_5a_4a_3^2) + a_2(103806066560a_3^5 \\
&+ 1731877036680a_5^5 - 47232284379000a_1^5 + 10269337354080a_1^2a_3^3 + 21841273872880a_1^3a_3^2 \\
&+ 2142413552160a_3^3a_5^2 + 11639072490920a_1a_5^4 + 3977951120080a_3^2a_5^3 \\
&+ 4068544257400a_3a_5^4 + 1541479280320a_1a_3^4 + 680161334720a_5a_3^4 \\
&+ 32744138080080a_1^2a_5^3 + 48925728507600a_1^3a_5^2 + 27304751388320a_1a_3a_5^3 \\
&+ 9217698058240a_1a_5a_3^3 + 23505927669840a_1a_3^2a_5^2 + 41884303008240a_1^2a_5a_3^2 \\
&+ 63062441226400a_5a_3a_1^3 + 62755399924560a_1^2a_3a_5^2 - 2255620347400a_3a_1^4 \\
&+ 6224629884200a_1^4a_5 + 1436734644752a_2^2a_3^3 + 121342060727a_5^3a_2^2 \\
&+ 2641164451395a_1a_2^4 + 126522097992a_3a_2^4 - 990970624859a_5a_2^4 \\
&+ 23939881405693a_2^2a_1^3 + 4349883335368a_2^2a_3a_5^2 + 7665519051203a_1a_2^2a_5^2 \\
&+ 22039194611561a_2^2a_1^2a_5 + 14337209801884a_1a_2^2a_3a_5 + 20546899565244a_2^2a_3a_1^2 \\
&+ 4830698633877a_5a_2^2a_3^2 + 6719813849475a_1a_2^2a_3^2).
\end{aligned}
$$

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