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Fix-Finite Approximation Property in Normed Vector Spaces

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1. Introduction

Let D and A be two nonempty subsets in a metric space. We say that the pair (D, A) satisfies the fix-finite approximation property (in short F.F.A.P.) for a family $\mathcal{F}$ of maps (or multifunctions) from D to A, if for every $f \in \mathcal{F}$ and all $\varepsilon > 0$ there exists $g \in \mathcal{F}$ which is ε-near to f and has only a finite number of fixed points. In the particular case where $D = A$, we say that A satisfies the F.F.A.P. for $\mathcal{F}$.

H. Hopf [4] proved by a special construction that any finite polyhedron which is connected and which dimension is greater than one satisfies the F.F.A.P. for any continuous self-map. Later H. Schirmer [5] extended this result to any continuous n-valued multifunction. After this J.B. Baillon and N.E. Rallis showed in [1] that any finite-union of closed convex subsets of a Banach space satisfies the F.F.A.P. for any compact self-map.

In this paper we study the fix-finite approximation property in normed vector spaces. We work with the pair (D, A) such that A satisfies the Schauder condition.

If x is a point of a normed space X and $r > 0$, then we denote by $B(x, r)$ the open ball of radius r and center x. A subset K of X is said to be relatively compact if its closure $\overline{K}$ is compact. The convex hull of a subset $\{x_1, \ldots, x_n\}$ of X is defined by

$$\operatorname{conv}\{x_1, \ldots, x_n\} = \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in [0,1] \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

A subset A of a normed space X is said to enjoy the Schauder condition if for any nonempty relatively compact subset K of A and every $\varepsilon > 0$ there exists a finite cover $\{B(x_i, \eta_{x_i}) : x_i \in A, 0 < \eta_{x_i} < \varepsilon, i = 1, \ldots, n\}$ of K such that for any subset $\{x_{i_l}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_n\}$ with

$$\bigcap_{j=l}^{k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset$$

the convex hull of $\{x_{i_l}, \ldots, x_{i_k}\}$ is contained in A.

For example, any nonempty convex subset of a normed space X and any open subset of X satisfies the Schauder condition (see [6]). Also, all finite-union of closed convex subsets of a Banach space satisfies the Schauder condition (see [1]).

In the present work we first establish the following result (Theorem 3.1): if A is a nonempty subset of a normed space X satisfying the Schauder condition and D is a compact subset of X containing A, then the pair (D, A) satisfies the F.F.A.P. for any n-function.

Secondly we prove (Theorem 3.2): if A is a nonempty subset of a normed space X satisfying the Schauder condition and D is a path and simply connected compact subset of X containing A, then the pair (D, A) satisfies the F.F.A.P. for any n-valued continuous multifunction. As consequence we obtain a generalization of the Schrimer's result [5, Theorem 4.6].

2. Preliminaries

In this section we recall some definitions for subsequent use.

Let X and Y be two Hausdorff topological spaces. A multifunction $F : X \to Y$ is a map from X into the set 2^Y of nonempty subsets of Y. The range of F is $F(X) = \cup_{x \in X} F(x)$.

The multifunction $F : X \to Y$ is said to be upper semi-continuous (usc) if for each open subset V of Y with $F(x) \subset V$ there exists an open subset U of X with $x \in U$ and $F(U) \subset V$.

The multifunction $F : X \to Y$ is called lower semi-continuous (lsc) if for every $x \in X$ and open subset V of Y with $F(x) \cap V \neq \emptyset$ there exists an open subset U of X with $x \in U$ and $F(x') \cap V \neq \emptyset$ for all $x' \in U$.

The multifunction $F : X \to Y$ is continuous if it is both upper semi-continuous and lower semi-continuous.

The multifunction F is compact if it is continuous and the closure of its range $\overline{F(X)}$ is a compact subset of Y.

A point x of X is said to be a fixed point of a multifunction $F : X \to Y$ if $x \in F(x)$. We denote by $\mathrm{Fix}(F)$ the set of all fixed points of F.

Let X and Y be two normed spaces. We denote by $C(X)$ the set of nonempty compact subsets of X. Let A and B be two elements of $C(X)$. The Hausdorff distance between A and B, $d_H(A,B)$, is defined by setting:

$$d_H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$$

where

$$\rho(A,B) = \sup\{d(x,B) : x \in A\},$$

$$\rho(B,A) = \sup\{d(y,A) : y \in B\}$$

and

$$d(x,B) = \inf\{\|y-x\| : y \in B\}.$$

Let F and G be two compact multifunctions from X to Y. We define the Hausdorff distance between F and G by setting:

$$d_H(F,G) = \sup\{d_H(F(x), G(x)) : x \in X\}.$$

Let $\varepsilon > 0$ and F and G be two compact multifunctions from X to Y. We say that F and G are ε-near if $d_H(F,G) < \varepsilon$.

3. Fix-finite approximation property

3.1. Fix-finite approximation property for n-functions.

In this subsection we study the fix-finite approximation property for n-functions. First, we recall the definition of an n-function.

Definition 3.1. Let X and Y be two Hausdorff topological spaces. A multifunction $F : X \to Y$ is said to be an n-function if there exist n continuous maps $f_i : X \to Y$, where $i = 1, \dots, n$, such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in X$ and $f_i(x) \neq f_j(x)$ for all $x \in X$ and $i,j = 1, \dots, n$ with $i \neq j$.

In this subsection we shall prove the following:

Theorem 3.1. *Let A be a nonempty subset of a normed space X satisfying the Schauder condition. If D is a compact subset of X containing A, then the pair (D,A) satisfies the F.F.A.P. for any n-function $F : D \to A$.*

In order to prove Theorem 3.1, we shall need the following lemmas.

LEMMA 3.1. *If a nonempty subset A of a normed space X satisfies the Schauder condition, then for any relatively compact subset K of A and every $\varepsilon > 0$ there exist a finite polyhedron P contained in A and a continuous map $\pi : K \to P$ such that $\|\pi(x) - x\| < \varepsilon$ for all $x \in K$.*

Proof. Let $\varepsilon > 0$ and K be a nonempty relatively compact subset of A. Since A satisfies the Schauder condition, then there exists a finite cover

$$\{B(x_i, \eta_{x_i}) : x_i \in A, 0 < \eta_{x_i} < \varepsilon,\ i = 1, \ldots, n\}$$

of K such that for all subset $\{x_{i_l}, \ldots, x_{i_k}\}$ of $\{x_1, \ldots, x_n\}$ with $\cap_{j=l}^{k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset$ the convex hull of $\{x_{i_l}, \ldots, x_{i_k}\}$ is contained in A.

For all $i = 1, \ldots, n$, let μ_i be the continuous function defined by $\mu_i(x) = \max(0, \eta_{x_i} - \|x - x_i\|)$, for all $x \in K$. Since for all $x \in K$ there exists $i \in \{1, \ldots, n\}$ such that $\|x - x_i\| < \eta_{x_i}$, then $\sum_{i=1}^{n} \mu_i(x) > 0$. Now we can define a continuous function α_i on K by setting:

$$\alpha_i(x) = \frac{\mu_i(x)}{\sum_{i=1}^{n} \mu_i(x)},\ i = 1, \ldots, n,\ \text{for all } x \in K.$$

Let

$$Q = \left\{ \{x_{i_l}, \ldots, x_{i_k}\} \subset \{x_1, \ldots, x_n\} : \cap_{j=l}^{k} B(x_{i_j}, \eta_{x_{i_j}}) \cap K \neq \emptyset \right\}$$

and

$$P = \cup_{\{x_{i_l}, \ldots, x_{i_k}\} \in Q} \operatorname{conv} \{x_{i_l}, \ldots, x_{i_k}\}.$$

Let π be the map from K to P defined by $\pi(x) = \sum_{i=1}^{n} \alpha_i(x) x_i$, for all $x \in K$. Then, the map π is continuous and satisfies the property $\|\pi(x) - x\| < \varepsilon$ for all $x \in K$. ∎

In [6] we introduced the notion of Hopf spaces. These are metric spaces satisfying the F.F.A.P. for any compact self-map. By using [6, Theorem 1.3] and the Schauder condition we obtain the following lemma.

LEMMA 3.2. *Let A be a nonempty subset of a normed space X satisfying the Schauder condition. If D is a compact subset of X containing A, then for all continuous map $f : D \to A$ and for every $\varepsilon > 0$, there exist a finite polyhedron P contained in A and a continuous map $g : D \to P$ which is ε-near to f and has only a finite number of fixed points. In particular every nonempty compact subset of a normed space satisfying the Schauder condition is a Hopf space.*

Proof. Since $f(D)$ is a relatively compact subset of A, then by Lemma 3.1 for a given $\varepsilon > 0$, there exist a finite polyhedron P contained in A and a continuous map $\pi_\varepsilon : f(D) \to P$ such that $\|\pi_\varepsilon(y) - y\| < \frac{1}{2}\varepsilon$, for all $y \in f(D)$. Set $f_\varepsilon = \pi_\varepsilon \circ f$, then the map $f_\varepsilon : D \to P$ is continuous and satisfies $\|f_\varepsilon(x) - f(x)\| < \frac{1}{2}\varepsilon$, for all $x \in D$.

By [6, Theorem 1.3] there exists a continuous map $g : D \to P$ which is $\frac{1}{2}\varepsilon$-near to f_ε and has only a finite number of fixed points. Then, the map g is ε-near to f because for all $x \in D$, we have:

$$\|f(x) - g(x)\| \leq \|f(x) - f_\varepsilon(x)\| + \|f_\varepsilon(x) - g(x)\| < \varepsilon.$$

∎

Proof of Theorem 3.1. Let $\varepsilon > 0$ and $F : D \to A$ be an n-function. Then, there exist n continuous maps $f_i : D \to A$ such that $F(x) = \{f_1(x), \dots, f_n(x)\}$ for all $x \in D$ and $f_i(x) \neq f_j(x)$ for all $x \in D$ and $i, j = 1, \dots, n$ with $i \neq j$.

For all $i, j = 1, \dots, n$ with $i \neq j$, we define $\delta_{(i,j)}(F) = \min\{\|f_i(x) - f_j(x)\| : x \in D\}$. As each f_i is continuous for all $i = 1, \dots, n$ and D is compact, then for each $i, j = 1, \dots, n$ with $i \neq j$, we have $\delta_{(i,j)}(F) > 0$. Therefore,

$$\delta(F) = \min\{\delta_{(i,j)}(F) : i, j = 1, \dots, n,\ i \neq j\} > 0.$$

For a given $\varepsilon > 0$, we set $\lambda = \min(\frac{1}{2}\delta(F), \frac{1}{2}\varepsilon)$. By Lemma 3.2, for each $i = 1, \dots, n$, there exists a map $g_i : D \to A$ which is λ-near to f_i and has only a finite number of fixed points. Let $G : D \to A$ be the multifunction defined by $G(x) = \{g_1(x), \dots, g_n(x)\}$, for all $x \in D$.

Claim 1. The multifunction G is an n-function. Indeed, if there exists $x_0 \in D$ and $i, j = 1, \dots, n$ with $i \neq j$, such that $g_i(x_0) = g_j(x_0)$, then,

$$\|f_i(x_0) - f_j(x_0)\| \leq \|f_i(x_0) - g_i(x_0)\| + \|f_j(x_0) - g_j(x_0)\| < 2\lambda.$$

Therefore, $\delta_{(i,j)}(F) < \delta(F)$. This is a contradiction and our claim is proved.

Claim 2. The multifunction G is ε-near to F. Indeed, for all $i = 1, \dots, n$ and for every $x \in D$, we have, $\|f_i(x) - g_i(x)\| < \frac{1}{2}\varepsilon$. Then, $d_H(F, G) < \varepsilon$.

Claim 3. The multifunction G has only a finite number of fixed points. Indeed, $\mathrm{Fix}(G) = \cup_{i=1}^{n}\mathrm{Fix}(g_i)$ and for all $i = 1, \dots, n$ the maps g_i has only a finite number of fixed points. ∎

Corollary 3.1. *Let C_i, for $i = 1, \dots, m$, be a finite family of nonempty convex compact subsets of a normed space, then $\cup_{i=1}^{m} C_i$ satisfies the F.F.A.P. for any n-function $F : \cup_{i=1}^{m} C_i \to \cup_{i=1}^{m} C_i$.*

3.2. Fix-finite approximation property for n-valued continuous multifunctions. To start this subsection, we give the definition of a n-valued multifunction.

Definition 3.2. Let X and Y be two Hausdorff topological spaces. A multifunction $F : X \to Y$ is said to be n-valued if for all $x \in X$, the subset $F(x)$ of Y consists of n points.

Now we recall the definition of the gap of a n-valued multifunction. Let X and Y be two Hausdorff topological spaces and let $F : X \to Y$ be a n-valued continuous multifunction. Then, we can write $F(x) = \{y_1, \dots, y_n\}$ for all $x \in X$. We define a real function γ on X by

$$\gamma(x) = \inf\left\{\|y_i - y_j\| : \, y_i, y_j \in F(x),\ i, j = 1, \dots, n,\ i \neq j\right\}, \text{ for all } x \in X,$$

and the gap of F by

$$\gamma(F) = \inf\left\{\gamma(x) : \, x \in X\right\}.$$

Since the multifunction F is continuous then the function γ is also continuous [5, p.76]. If X is compact, then $\gamma(F) > 0$.

In this subsection we show the following:

Theorem 3.2. *Let A be a nonempty subset of a normed space X satisfying the Schauder condition. If D is a path and simply connected compact subset of X containing A, then the pair (D, A) satisfies the F.F.A.P. for any n-valued continuous multifunction $F : D \to A$.*

We recall the following Lemma due to H. Schrimer [5] which is useful for the proof of our result.

Lemma 3.3. *Let X and Y be two compact Hausdorff topological spaces. If X is path and simply connected and $F : X \to Y$ is a n-valued continuous multifunction, then F is an n-function.*

Proof of Theorem 3.2. Let $\varepsilon > 0$ and $F : D \to A$ be a n-valued continuous multifunction. Then, $\gamma(F) > 0$ and $\lambda = \min(\frac{1}{4}\varepsilon, \frac{1}{2}\gamma(F)) > 0$. By Lemma 3.1 there exist a finite polyhedron P contained in A and a continuous map $\pi : F(D) \to P$ such that $\|\pi(y) - y\| < \lambda$ for all $y \in F(D)$. Now we define a continuous multifunction $G : D \to P$ by $G(x) = (\pi \circ F)(x)$, for all $x \in D$.

Claim 1. The multifunction G is n-valued and $\frac{1}{2}\varepsilon$-near to F. Indeed, if $x \in D$ such that $F(x) = \{y_1, \dots, y_n\}$, then $G(x) = \{\pi(y_1), \dots, \pi(y_n)\}$ with $\|y_i - \pi(y_i)\| < \frac{1}{4}\varepsilon$ for all $i = 1, \dots, n$.

Claim 2. There exists an n-function $L : D \to A$ which is ε-near to F and has only a finite number of fixed points. Indeed, from Lemma 3.3 the multifunction $G : D \to P$ is an n-function and by Theorem 3.1 there exists an n-function $L : D \to P$ which is $\frac{1}{2}\varepsilon$-near to G and has only a finite number of fixed points. Then, the multifunction $L : D \to P$ is ε-near to F and has only a finite number of fixed points. ∎

As a consequence of Theorem 3.1 and Theorem 3.2 we obtain the following:

Corollary 3.2. *Let C_i, for $i = 1, \ldots, m$, be a finite family of nonempty convex compact subsets of a normed space such that $\cap_{i=1}^m C_i \neq \emptyset$ or $C_i \cap C_j = \emptyset$ for $i \neq j$, then $\cup_{i=1}^m C_i$ satisfies the F.F.A.P. for any n-valued continuous multifunction $F : \cup_{i=1}^m C_i \to \cup_{i=1}^m C_i$.*

Proof. Let $\varepsilon > 0$ and $F : \cup_{i=1}^m C_i \to \cup_{i=1}^m C_i$ be a n-valued continuous multifunction. For the proof we distinguish the following two cases.

First Case. $C_i \cap C_j = \emptyset$ for $i, j = 1, \ldots, m$ and $i \neq j$. We have, $F|_{C_i} : C_i \to \cup_{i=1}^m C_i$ is a n-valued continuous multifunction for $i = 1, \ldots, m$. From Lemma 3.3, the multifunction $F|_{C_i}$ is a n-function for $i = 1, \ldots, m$. Therefore, for each $i \in \{1, \ldots, m\}$, there exist n continuous maps $f_{i_j} : C_i \to \cup_{i=1}^m C_i$ such that $F(x) = \{f_{i_1}(x), \ldots, f_{i_n}(x)\}$ for all $x \in C_i$. Now for each $j \in \{1, \ldots, n\}$ we can define a continuous map $h_j : \cup_{i=1}^m C_i \to \cup_{i=1}^m C_i$ by $h_j(x) = f_{i_j}(x)$ if $x \in C_i$. It follows that for all $x \in \cup_{i=1}^m C_i$, we have $F(x) = \{h_1(x), \ldots, h_n(x)\}$. Thus, the multifunction F is an n-function. By Corollary 3.1 there exists a n-multifunction $G : \cup_{i=1}^m C_i \to \cup_{i=1}^m C_i$ which is ε-near to F and has only a finite number of fixed points.

Second Case. $\cap_{i=1}^m C_i \neq \emptyset$. It follows from Theorem 3.2 that $\cup_{i=1}^m C_i$ satisfies the F.F.A.P. for any n-valued continuous multifunction. ∎

As a particular case of Corollary 3.2 we obtain a generalization of the Schirmer's result [5, Theorem 4.6].

Corollary 3.3. *If C_1 and C_2 are two nonempty convex compact subsets of a normed space, then $C_1 \cup C_2$ satisfies the F.F.A.P. for any n-valued continuous multifunction $F : C_1 \cup C_2 \to C_1 \cup C_2$.*

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