

Cauchy-Neumann Problem for a Kind of Nonlinear Parabolic Operators

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1. INTRODUCTION

We are interested on strong solvability of the Cauchy-Neumann problem for a class of nonlinear parabolic operators

$$(*) \quad a(x, t, u, u_x, u_{xx}) - u_t = f(x, t, u, u_x)$$

in rectangle $Q = \{(x, t) \in (0, X) \times (0, T)\}$. The data $a(x, t, z, p, \xi)$ and $f(x, t, z, p)$ are supposed to be Carathéodory's functions, i.e, they are measurable in $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and continuous with respect to the other variables $(z, p, \xi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The operator satisfies the condition of Campanato (see [2]) which ensures its “nearness” to the heat equation $u_{xx} - u_t$ both considered as mappings in the Sobolev space $W_2^{2,1}(Q)$. Concerning the right-hand side $f(x, t, z, p)$ we suppose the optimal (quadratic) growth with respect to p .

Strong solvability results for initial-boundary value problems for linear and quasilinear operators have been obtained in [6], [8], [12] supposing *subquadratic* growth of f with respect to p . In [13], [14] and [9] there are considered other kinds of boundary problems for elliptic and parabolic Campanato's operators supposing the optimal growth of the right-hand side with respect to the gradient.

It is worth noting that the Campanato's condition is a nonlinear variant of the famous linear Cordes condition (see [4], [5]) which ensures isomorphic properties of a linear elliptic operator with *solely* bounded and measurable

coefficients. In fact, the both conditions coincide if $a(x, t, z, p, \xi)$ would be linear with respect to (z, p, ξ) . We refer the reader to the recent monograph [9] where detailed study on elliptic and parabolic operators with various kinds of discontinuous coefficients is presented.

General second order parabolic equations, like the ones discussed here, describe in physical applications the time-evolution of the density of some quantity u (say chemical concentration, temperature, electrostatic potential). The principal part $a(x, t, u, u_x, u_{xx})$ describes analytically a relation between nonlinear characteristic of the medium at the point x and the moment t , the diffusion of the quantity u_{xx} , its linear transport u_x , and creation or depletion of the quantity u .

The existence of a $W_2^{2,1}(Q)$ solution to the Cauchy-Neumann problem for the operator (*) is established by the Leray-Schauder fixed point theorem. It is reduced to deriving of a priori estimates for the L^∞ -norm of the solution and for the L^4 -norm of its spatial derivative. The first one is reached by applying the maximum principle due to Nazarov [10]. The second one is obtained making use of a technique developed by Tomi [15] and Von Wahl [16] (see also [1]).

In our considerations we essentially exploit the *near mappings theory* (see [2]) which restricts us to the Sobolev space $W_2^{2,1}(Q)$. On the other hand, the maximum principle has to do with solutions belonging to $W_{n+1}^{2,1}(Q)$ where n is the dimension of the spatial variable x . That is why we are forced to consider our problem for one dimensional spatial variable.

The unicity of the solution follows by the maximum principle imposing additional structure conditions on the data.

2. STATEMENT OF THE PROBLEM

We are interested of the next Cauchy-Neumann problem

$$(1) \quad \begin{cases} a(x, t, u, u_x, u_{xx}) - u_t = f(x, t, u, u_x) & \text{a.e. in } Q, \\ u(x, 0) = \varphi(x) & x \in (0, X), \\ u_x(0, t) + \alpha u(0, t) = \psi_1(t) \\ u_x(X, t) + \alpha u(X, t) = \psi_2(t) & t \in (0, T). \end{cases}$$

defined in the rectangle $Q = \{(x, t) \in (0, X) \times (0, T)\}$. Under a *strong solution* of (1) we mean a function u lying in the Sobolev space

$$W_2^{2,1}(Q) = \left\{ u \in L^2(0, T; H^2(0, X)) : \frac{\partial u}{\partial t} \in L^2(Q) \right\},$$

satisfying the equation above almost everywhere in Q and the conditions on the boundary hold in the trace sense. The space $W_2^{2,1}(Q)$ is equipped with the norm

$$\|u\|_{W_2^{2,1}(Q)} = \left(\int_Q (|u|^2 + |u_{xx}|^2 + |u_t|^2) dxdt \right)^{1/2}.$$

The Banach space $H^k(0, X)$ with a positive integer k consists of all L^2 -functions having weak derivatives up to order k , that are 2^{th} -power summable on $(0, X)$. Because of the trace theorems, the lateral boundary data should belong to the corresponding Besov spaces $H^l(0, T)$ ($0 < l < 1$). Let us recall that $H^l(0, T)$ consists of all L^2 -functions for which the following norm is finite

$$\|g\|_{H^l(0,T)} = \|g\|_{L^2(0,T)} + \left(\int_0^T dt \int_0^T \frac{|g(t) - g(\tau)|^2}{|t - \tau|^{1+2l}} d\tau \right)^{1/2}.$$

The data a and f are supposed to be Carathéodory's functions and together with $\varphi(x)$, $\psi_1(t)$ and $\psi_2(t)$ satisfy the following conditions:

- (A) *Campanato's condition*: there exist positive constants γ and K , $K < 1$, such that

$$\begin{cases} \left| \tau - \gamma[a(x, t, z, p, \xi + \tau) - a(x, t, z, p, \xi)] \right| \leq K|\tau|, \\ a(x, t, z, p, 0) = 0 \end{cases}$$

for a.a. $(x, t) \in Q$ and all $z, p, \xi, \tau \in \mathbb{R}$.

- (B1) *Monotonicity* of f with respect to z :

$$\begin{cases} \text{sign } z \cdot f(x, t, z, p) \geq -\mu_1(x, t)p - \mu_2(x, t)z - \mu_3(x, t) \\ \mu_1, \mu_2 \in L^\infty(Q), \mu_3 \in L^4(Q), \mu_1, \mu_2, \mu_3 \geq 0 \end{cases}$$

for all $z \in \mathbb{R}$ such that $|z| \geq M_0$ for some positive constant M_0 .

- (B2) *Quadratic gradient growth*:

$$\begin{cases} |f(x, t, z, p)| \leq f_1(|z|)[f_2(x, t) + |p|^2] \\ f_1, f_2 \geq 0, f_1 \in C^0(\mathbb{R}^+), f_2 \in L^2(Q), \end{cases}$$

and f_1 is monotone nondecreasing function.

(C) *Regularity of the initial and boundary data and compatibility condition:*

$$\begin{aligned} \varphi(x) &\in H^1(0, X); \quad \psi_1(t), \psi_2(t) \in H^{1/4}(0, T), \\ \varphi_x(0) + \alpha\varphi(0) &= \psi_1(0), \quad \varphi_x(X) + \alpha\varphi(X) = \psi_2(0). \end{aligned}$$

THEOREM. (Existence theorem) *Under the assumptions (A), (B1), (B2) and (C) the problem (1) has a solution $u \in W_2^{2,1}(Q)$.*

It is worth noting that (B1) and (B2) do not ensure uniqueness of this solution. However, as shows the next theorem, some additional requirements on the structure of the functions a and f imply unique strong solvability of (1).

THEOREM. (Uniqueness theorem) *Suppose (A) and (C). Let the function a be independent of z and p . Assume further that for almost all $(x, t) \in Q$ and for each $p, p' \in \mathbb{R}$, $f(x, t, z, p)$ is nondecreasing in z and*

$$|f(x, t, z, p) - f(x, t, z, p')| \leq f_3(x, t, z)|p - p'|$$

where $\sup_{|z| \leq M} f_3(\cdot, z) = f_4(x, t) \in L^\infty(Q)$.

Then, if $u, v \in W_2^{2,1}(Q)$ solve the problem (1) they coincide in Q .

Remark 1. It is easy to see that the principal part $a(x, t, z, p, \xi)$ is a Lipschitz continuous function with respect to ξ . By virtue of the Rademacher theorem, it follows the existence almost everywhere of the derivative a_ξ and its boundedness. This implies the standard ellipticity condition $\frac{1-K}{\gamma} \leq a_\xi \leq \frac{1+K}{\gamma}$ (see [12]). However we use the condition (A), in order to apply the Campanato theory on near operators (cf. [2], [3], [9]).

Remark 2. By virtue of the Sobolev imbedding theorem [7, Chapter II] the solution of (1) is a Hölder function with an exponent $1/2$.

3. A PRIORI ESTIMATES

We start with a useful construction that reduces the problem (1) to one with zero data on the boundary. It is well known from the linear theory (cf. [7]), that the Cauchy–Neumann problem

$$(2) \quad \begin{cases} y_{xx} - y_t = 0 & \text{a.e. in } Q, \\ y(x, 0) = \varphi(x) & x \in (0, X), \\ y_x(0, t) + \alpha y(0, t) = \psi_1(t) \\ y_x(X, t) + \alpha y(X, t) = \psi_2(t) & t \in (0, T) \end{cases}$$

with initial and boundary data satisfying condition (C) is uniquely solvable in the space $W_2^{2,1}(Q)$. Taking $h = u - y$ we have the following nonlinear problem for the new function h

$$(3) \quad \begin{cases} \tilde{a}(x, t, h, h_x, h_{xx}) - h_t = \tilde{f}(x, t, h, h_x) & \text{a.e. in } Q \\ h(x, 0) = 0 & x \in (0, X), \\ h_x(0, t) + \alpha h(0, t) = 0 \\ h_x(X, t) + \alpha h(X, t) = 0 & t \in (0, T). \end{cases}$$

where $\tilde{a}(x, t, h, h_x, h_{xx}) = a(x, t, h+y, h_x+y_x, h_{xx}+y_{xx}) - y_{xx}$ and $\tilde{f}(x, t, h, h_x) = f(x, t, h+y, h_x+y_x)$. It is easy to see that also the functions \tilde{a} and \tilde{f} satisfy the conditions (A), (B1), and (B2).

Moreover, introducing a new function $v = ue^{\alpha x}$ we reduce the problem (1) to the next one

$$\begin{cases} \hat{a}(x, t, v, v_x, v_{xx}) - v_t = \hat{f}(x, t, v, v_x) & \text{a.e. in } Q, \\ v(x, 0) = 0 & x \in (0, X), \\ v_x(0, t) = v_x(X, t) = 0 & t \in (0, T), \end{cases}$$

where $\hat{a} = e^{\alpha x} \tilde{a}(x, t, ve^{-\alpha x}, v_x e^{-\alpha x} - \alpha v e^{-\alpha x}, v_{xx} e^{-\alpha x} - 2\alpha v_x e^{-\alpha x} + v \alpha^2 e^{-\alpha x})$ and $\hat{f} = e^{\alpha x} \tilde{f}(x, t, ve^{-\alpha x}, v_x e^{-\alpha x} - \alpha v e^{-\alpha x})$. Simple calculations show that the new functions satisfy the conditions (A), (B1) and (B2), respectively.

Thus, without loss of generality, we may concentrate our attention to the problem

$$(4) \quad \begin{cases} a(x, t, u, u_x, u_{xx}) - u_t = f(x, t, u, u_x) & \text{a.e. in } Q, \\ u(x, 0) = 0 & x \in (0, X), \\ u_x(0, t) = u_x(X, t) = 0 & t \in (0, T), \end{cases}$$

instead of (1). The solution that we are looking for will belong to the space

$$\overset{\circ}{W}_2^{2,1}(Q) = \left\{ u \in L^2(0, T; H^2(0, X)) : \frac{\partial u}{\partial t} \in L^2(Q); u|_{t=0} = 0, u_x|_{x=0, X} = 0 \right\}$$

equipped with the norm (cf. [3])

$$\|u\|_{(\beta)} = \|u\|_{\overset{\circ}{W}_2^{2,1}(Q)} = \left(\int_Q (|u_{xx}|^2 + \beta^2 |u_t|^2) dx dt \right)^{1/2} \quad \text{with } \beta > 0.$$

To obtain existence of a strong solution to the problem (4) we are going to apply the Leray–Schauder fixed point theorem. For this goal we need of suitable a priori estimates.

LEMMA 1. *If the problem (4) has a strong solution then it is bounded and satisfies*

$$(5) \quad \|u\|_{\infty, Q} \leq X^2 \exp\{2MT\} \sqrt{\gamma/(1-K)} \|\mu_3\|_{4, Q}.$$

Proof. According to Remark 1 we can transform the equation in (4) in a linear one with bounded coefficients

$$A(x, t)u_{xx} - u_t = f(x, t, u, u_x) \quad \text{with} \quad A(x, t) = \int_0^1 a_\xi(x, t, u, u_x, su_{xx}) ds.$$

The assertion follows from the maximum principle due to Nazarov [10]. ■

To derive L^4 estimate for the derivative u_x , we rewrite the equation in (4) in the equivalent form

$$a(x, t, u, u_x, u_{xx}) + b(x, t)u_x^2 - f_2(x, t)u - u_t = F(x, t)$$

where

$$(6) \quad b(x, t) = -\frac{f(x, t, u, u_x)}{f_2(x, t) + u_x^2} \in L^\infty(Q) \quad (\text{see (B2)}),$$

$$(7) \quad F(x, t) = \frac{f(x, t, u, u_x)f_2(x, t)}{f_2(x, t) + u_x^2} - f_2(x, t)u \in L^2(Q).$$

Let $\delta \in [0, 1]$ be a parameter and u be a fixed solution of (4). The problem

$$(8) \quad \begin{cases} a(x, t, u, u_x, v_{xx}) + b(x, t)v_x^2 - f_2(x, t)v - v_t = \delta F(x, t) & \text{a.e. in } Q, \\ v \in \mathring{W}_2^{2,1}(Q) \end{cases}$$

has a trivial solution $v \equiv 0$ for $\delta = 0$ and coincides with (4) for $\delta = 1$. A natural question that arise is the uniqueness of that solution.

LEMMA 2. *Let v_1 and v_2 be two solutions of (8) corresponding to the values $\delta_1 < \delta_2$ of the parameter. Then*

$$\|v_1 - v_2\|_{\infty, Q} \leq (\delta_2 - \delta_1)[f_1(\|u\|_{\infty, Q}) + \|u\|_{\infty, Q}].$$

The proof is based on the maximum principle [10] and is analogous to that of [14, Proposition 3]. The uniqueness of the solution of (8) follows immediately from Lemma 2 taking $\delta_1 = \delta_2$.

LEMMA 3. *If the problem (8) is solvable in $\overset{\circ}{W}_2^{2,1}(Q)$, then its solution has a derivative in L^4 and*

$$(9) \quad \|v_x\|_{4,Q}^2 \leq C(\alpha, K, \|f_1\|_{\infty,Q}, \|f_2\|_{2,Q}, \|u\|_{\infty,Q}).$$

Proof. We consider solutions v' and v'' corresponding to values $\delta' < \delta''$ of the parameter. Hence the difference $w = v' - v''$ is a solution of the problem

$$\begin{cases} a(x, t, u, u_x, (w + v'')_{xx}) - a(x, t, u, u_x, v''_{xx}) - w_t = G(x, t) & \text{a.e. in } Q, \\ w \in \overset{\circ}{W}_2^{2,1}(Q) \end{cases}$$

where

$$G(x, t) = F(x, t)(\delta' - \delta'') - b(x, t)((w_x + v'_x)^2 - v''_x{}^2) + f_2(x, t)w.$$

Having in mind condition (A) and Young's inequality, we obtain

$$|w_{xx} - \gamma w_t|^2 \leq K^2(1 + \varepsilon)|w_{xx}|^2 + \left(\gamma^2 + \frac{\gamma^2}{\varepsilon}\right) |G(x, t)|^2$$

where $w = v' - v''$. On the other hand, Lemma 2.3 in [3] yields

$$\|w\|_{(\gamma)}^2 \leq \int_Q K^2(1 + \varepsilon)|w_{xx}|^2 dxdt + \int_Q \left(\gamma^2 + \frac{\gamma^2}{\varepsilon}\right) |G(x, t)|^2 dxdt.$$

Choosing ε so small that $K^2(1 + \varepsilon) < 1$ and making use of Lemma 2, we get

$$\begin{aligned} \|w\|_{(\beta)} &\leq C\|G(x, t)\|_{2,Q} \leq C\left[1 + \|v'_x\|_{L^4(Q)}^2 + \|v''_x\|_{L^4(Q)}^2\right] \\ &\leq C\left[1 + \|v'_x\|_{4,Q}^2 + \|w_x\|_{4,Q}^2\right], \end{aligned}$$

where $\beta^2 = \gamma^2 / (1 - K^2(1 + \varepsilon))$. From the Gagliardo-Nirenberg interpolation inequality (cf. [7], [11]) follows

$$\|w_{xx}\|_{2,Q} \leq \|w\|_{(\beta)} \leq C\left\{1 + \|v'_x\|_{4,Q}^2 + M\|w_{xx}\|_{2,Q}\right\}$$

with $M = N(\delta'' - \delta') [f_1(\|u\|_{\infty,Q}) + \|u\|_{\infty,Q}]$ and N is a constant which depends only on Q .

If $\delta'' - \delta' = \tau$ is so small that $CN\tau [f_1(\|u\|_{\infty,Q}) + \|u\|_{\infty,Q}] < 1$ we can write

$$\|w_{xx}\|_{2,Q} \leq C(1 + \|v'_x\|_{4,Q}^2).$$

Hence

$$(10) \quad \|v''_x\|_{4,Q}^2 \leq C' + C'' \|v'_x\|_{4,Q}^2.$$

In the case of $\delta' = 0$ and $\delta'' = \tau$ we have $v' = 0$ and $v'' = v^\tau$ and it follows immediately an estimate for the L^4 norm of the derivative of v^τ . To complete the proof of Lemma 3 we separate the interval $[0, 1]$ of m subintervals each of length less than or equal to τ and repeat the above procedure m times. ■

LEMMA 4. *Let the conditions (A), (B1) and (B2) be fulfilled. Then for any τ small enough, the problem (8) with $\delta = \tau$ has a strong solution $v^\tau \in \mathring{W}_2^{2,1}(Q)$.*

Proof. In order to prove our assertion we are going to exploit the Leray-Schauder fixed point theorem. Define the space

$$\mathcal{S} = \{y: y \in L^\infty(Q), y_x \in L^4(Q), \|y\|_{\mathcal{S}} = \|y\|_{\infty,Q} + \|y_x\|_{4,Q}\},$$

and consider the operator $\mathcal{M}: [0, 1] \times \mathcal{S} \rightarrow \mathring{W}_2^{2,1}(Q)$ defined by the problem

$$(11) \quad \begin{cases} a(x, t, u, u_x, z_{xx}) - z_t = \\ \quad \sigma [\tau F(x, t) - b(x, t)|y_x|^2 + f_2(x, t)y] \quad \text{a.e. in } Q, \\ z \in \mathring{W}_2^{2,1}(Q). \end{cases}$$

From conditions (B2), (6) and the definition of \mathcal{S} follows that the right-hand side is L^2 function. On the other hand the condition (A) and [2, Theorem 9] mean “nearness” between the operators $a(x, t, u, u_x, z_{xx}) - z_t$ and $z_{xx} - \gamma z_t$ which gives that the problem (11) has a solution if and only if the next problem is solvable

$$(12) \quad \begin{cases} z_{xx} - \gamma z_t = \sigma [\tau F(x, t) - b(x, t)|y_x|^2 + f_2(x, t)y] \quad \text{a.e. in } Q, \\ z \in \mathring{W}_2^{2,1}. \end{cases}$$

The linear theory (see [7]) asserts strong solvability of (12) and hence also of (11). In such a way, the operator \mathcal{M} is well defined. Further on, having in mind the imbedding theorem ([7]), we get $\mathring{W}_2^{2,1}(Q) \subset \mathcal{S}$ compactly and thus we can consider \mathcal{M} to act from $[0, 1] \times \mathcal{S}$ into \mathcal{S} .

The condition $a(x, t, u, u_x, 0) = 0$, required in (A), shows that $\mathcal{M}(0, y) = 0$ for each $y \in \mathcal{S}$.

The continuity of the operator \mathcal{M} with respect to y follows by standard arguments (see [8], [12]).

Finally, the a priori estimate (10) provides a uniform with respect to σ and y bound for each solution to the equation $\mathcal{M}(\sigma, y) = y$. Therefore, the Leray-Schauder theorem implies existence of a fixed point of the mapping $\mathcal{M}(1, \cdot)$ which, in view of the definition of \mathcal{M} , becomes a solution of the problem (8) with $\delta = \tau$. ■

4. PROOF OF THE EXISTENCE THEOREM

Proof. We are going to use ones again the Leray-Schauder theorem. For this goal we take $u \in \mathcal{S}$ and $\sigma \in [0, 1]$ and consider

$$(13) \quad \begin{cases} a(x, t, u, u_x, y_{xx}) - y_t = \sigma f(x, t, u, u_x) & \text{a.e. in } Q, \\ y \in \mathring{W}_1^{2,1}(Q). \end{cases}$$

Since the operator above is “near” to the heat operator and $f \in L^2(Q)$ in view of (B2) the problem (13) admits a unique solution $y \in \mathring{W}_2^{2,1}(Q)$. Define now the operator

$$\mathcal{N}(\sigma, u): [0, 1] \times \mathcal{S} \rightarrow \mathring{W}_2^{2,1}(Q) \hookrightarrow \mathcal{S}.$$

As before \mathcal{N} is a continuous and compact one and from (5) and (9) follows $\|u\|_{\mathcal{S}} \leq C$ for each solution u of the problem $\mathcal{N}(\sigma, u) = u$ with C independent of u and σ . Thus the Leray-Schauder theorem asserts existence of a fixed point of the mapping $\mathcal{N}(1, \cdot)$ which is the desired solution of (4). ■

5. PROOF OF THE UNIQUENESS THEOREM

Proof. Let u and v be two solutions of (4). Introduce the function $w = u - v \in \mathring{W}_2^{2,1}(Q)$. Since $f(x, t, z, p)$ is nondecreasing in z , we have

$$-f(x, t, u, v_x) + f(x, t, v, v_x) \leq 0 \quad \text{a.e. in } Q_+,$$

where $Q_+ = \{x \in Q : w(x, t) = u(x, t) - v(x, t) > 0\}$.

On the other hand

$$a(x, t, u_{xx}) - a(x, t, v_{xx}) = w_{xx} \int_0^1 a_\xi(x, t, sw_{xx} + v_{xx}) ds = A(x, t)w_{xx}.$$

Moreover, we can linearize in the same manner the function f , i.e.

$$f(x, t, u, v_x) - f(x, t, u, u_x) = -w_x \int_0^1 f_p(x, t, u, sw_x + v_x) ds = w_x F(x, t).$$

Note that f is Lipschitz continuous with respect to p and hence the derivative f_p exists almost everywhere and is bounded. Then

$$\begin{cases} A(x, t)w_{xx} + F(x, t)w_x - w_t \geq 0 & \text{a.e. in } Q_+, \\ w \in \overset{\circ}{W}_2^{2,1}(Q), \end{cases}$$

with $A, F \in L^\infty(Q)$. The Nazarov maximum principle (cf. [10]) implies that $\sup_{Q_+} w \leq 0$ whence $w \leq 0$ almost everywhere in Q . Analogously $\inf_{Q_-} w \geq 0$, whence $w \geq 0$ almost everywhere in Q and this completes the proof of Theorem 2. ■

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