EXTRACTA MATHEMATICAE Vol. **16**, Núm. 2, 297 – 301 (2001)

Characteristics of Orders Based on Choice Function Tool

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(Research announcement presented by M. Molina)

AMS *Subject Class.* (1991): 90C99, 90A05, 90A06 *Received February* 20, 2001

Consider a general discrete decision-maker problem under certainty, in which the Decision Maker (DM) is to make a choice from a finite set of alternatives, $\Omega = \{x_1, x_2, \ldots, x_n\}$. Binary relations and choice functions define two different tools in such problems to describe DM's preferences.

Binary relations arise when the DM gives information about his/her preferences in paired comparisons (this tool is widely used since Arrow, 1951), while choice functions are employed when the DM has to choose some alternatives, which form the choice set $C(X)$, from some subsets X of Ω. The function connecting $(X, C(X))$ is called *choice function*.

A choice function C is *reflexive* if $C(\{x\}) = \{x\}$ for all $x \in \Omega$, and C is *binary transitive* if

$$\left.\begin{array}{l} C(\{x,\, y\}) = C(\{x\}) = \{x\} \\ C(\{y,\, z\}) = C(\{y\}) = \{y\} \end{array}\right\} \;\Rightarrow\; C(\{x,\, z\}) = C(\{x\}) = \{x\}$$

for all $x, y, z \in \Omega$.

The link between binary relations and choice functions is formed by normal choice functions. A choice function is *normal* when there is a binary relation R such that $C = C_R$, where

$$C_R(X) = \{x \in X : \forall y \in X,\, xR\,y\}, \quad \forall X \subseteq \Omega.$$

The terminology used for binary relation properties is as employed in [5].

Given a set A we will denote by $|A|$ the cardinality of A (i.e., the number of elements).

Characterizations results for normal choice functions can be obtained in [2] and from [6].

Our main objective in this work is to characterize some orders by using information structured into a normal choice function. To this end, we use a result about the uniqueness of binary relation generating a normal choice function.

Proposition 1. *Given a normal choice function C, the associated binary relation R (i.e., the relation R such that $C = C_R$) is unique if and only if C is reflexive.*

Here, we use reflexive normal choice functions, thus, in each considered case the choice function has an associated unique binary relation.

Normal choice functions are proposed as a tool for characterizing four kind of orders: total orders, partial orders, weak orders and quasi orders. These results will be useful in preference modelling in a context where information about preferences is given in terms of a choice function.

Proposition 2. *R is a total order (i.e., antisymmetric, strongly complete and transitive) if and only if $|C_R(X)| = 1$, for all $X \subseteq \Omega$.*

This result generalizes the one appeared in [6], which is restricted to a particular type of choice functions: those whose range is restricted to unit sets, and which uses a weak order defined from the choice function.

Furthermore, we can easily construct the choice function from a total order without using directly the definition of C_R. We consider the elements of Ω reordered using the order of alternatives $x_{(1)}$, $x_{(2)}, \ldots,$ $x_{(n)}$ such that

$$x_{(i)} R\, x_{(j)}, \quad i = 1, \cdots, n, \quad j = i,\, i+1, \cdots, n.$$

Then, we can obtain C_R as follows,

$$C_R(X) = \{x_{(k)}\},$$

with $k = \min\{i : x_{(i)} \in X\}$, $\forall X \subseteq \Omega$.

Now we give a characterization for orders in which reflexivity replaces the strong completeness condition of total orders, i.e., partial orders.

Proposition 3. *R is a partial order (reflexive, antisymmetric and transitive) if and only if C_R is reflexive, binary transitive and $|C_R(X)| \leq 1$, for all $X \subseteq \Omega$.*

R is a partial order but not a total order if it satisfies the above proposition and there exists a subset $X_1 \subseteq \Omega$ such that $C_R(X_1) = \emptyset$.

PROPOSITION 4. *R is a weak order (strongly complete and transitive) if and only if the following two conditions hold:*

(i) $|C_R(X)| \geq 1$, *for all* $X \subseteq \Omega$,

(ii) $X \cap C_R(X') = C_R(X)$ *or* $\emptyset$, *for all* X, $X' \subseteq \Omega$ *such that* $X \subseteq X'$ *and* $|X| = 2$.

In the case of a quasi order (reflexive and transitive), we were trying to find a similar result to the previous one for weak orders, removing the completeness condition. We prove the following proposition, but only we have a necessary condition for R to be a quasi order.

PROPOSITION 5. *If R is a quasi order then for all X, $X' \subseteq \Omega$ such that $X \subseteq X'$ and $C_R(X) \neq \emptyset$ it follows that $X \cap C_R(X') = C_R(X)$ or $\emptyset$.*

The proofs of the above two propositions are based on the concept of *reduction* of a binary relation which appears in [3].

Observe that:

- If R is not a weak order, then there exists $X_1 \subseteq \Omega$ with $|X_1| > 1$ such that $|C_R(X_1)| \leq 1$.
- If R is not a partial order, then there exists $X_1 \subseteq \Omega$ such that $|C_R(X_1)| > 1$.

As our goal was to find a characterization result, we define the concept of pseudo binary transitivity (similar to the binary transitivity concept). This concept is defined as follows: C is *pseudo binary transitive* if for all x, y, $z \in \Omega$, such that $x \in C(\{x, y\})$ and $y \in C(\{y, z\})$, $x \in C(\{x, z\})$ holds. Then, we have proved the following proposition.

PROPOSITION 6. *R is a quasi order if and only if C_R is reflexive and pseudo binary transitive.*

The proofs of the mentioned results will appear in [4].

EXAMPLE. Let us consider choice functions C_1, C_2, C_3 and C_4 on $\Omega = \{x_1, x_2, x_3, x_4\}$ defined in Table 1. This choice functions can be proved to be normal.

The binary relations that generate C_1, C_2, C_3 and C_4 will be denoted by R_1, R_2, R_3 and R_4, respectively. These are unique, because choice functions are reflexive (see Proposition 1).

X	$C_1(X)$	$C_2(X)$	$C_3(X)$	$C_4(X)$
x_1	x_1	x_1	x_1	x_1
x_2	x_2	x_2	x_2	x_2
x_3	x_3	x_3	x_3	x_3
x_4	x_4	x_4	x_4	x_4
x_1, x_2	x_1	x_1	x_1, x_2	x_1, x_2
x_1, x_3	x_1	$\emptyset$	x_1	x_1
x_1, x_4	x_4	x_1	x_1	x_1
x_2, x_3	x_3	x_3	x_2	x_2
x_2, x_4	x_4	x_2	x_2	x_2
x_3, x_4	x_4	x_3	x_4	$\emptyset$
x_1, x_2, x_3	x_1	$\emptyset$	x_1, x_2	x_1, x_2
x_1, x_2, x_4	x_4	x_1	x_1, x_2	x_1, x_2
x_1, x_3, x_4	x_4	$\emptyset$	x_1	x_1
x_2, x_3, x_4	x_4	x_3	x_2	x_2
x_1, x_2, x_3, x_4	x_4	$\emptyset$	x_1, x_2	x_1, x_2

Table 1: C_1, C_2, C_3 and C_4

Clearly, R_1 is a total order by Proposition 2. Moreover, if we consider the way of reordering the alternatives indicated before when we have a total order, we obtained that in this case, the order of the alternatives is as follows,

$$x_{(1)} = x_4, \;\; x_{(2)} = x_1, \;\; x_{(3)} = x_3, \;\; x_{(4)} = x_2$$

According to Propositions 3, we have that R_2 is a partial order.

Taking into account Proposition 4, we get that the binary relation R_3 that generates C_3 is a weak order.

Finally, applying Proposition 6 we have that R_4 is a quasi order.

The graph of these orders are shown in Figure 1.

Furthermore, note that C_4 satisfies the condition of Proposition 5, because R_4 is a quasi order.

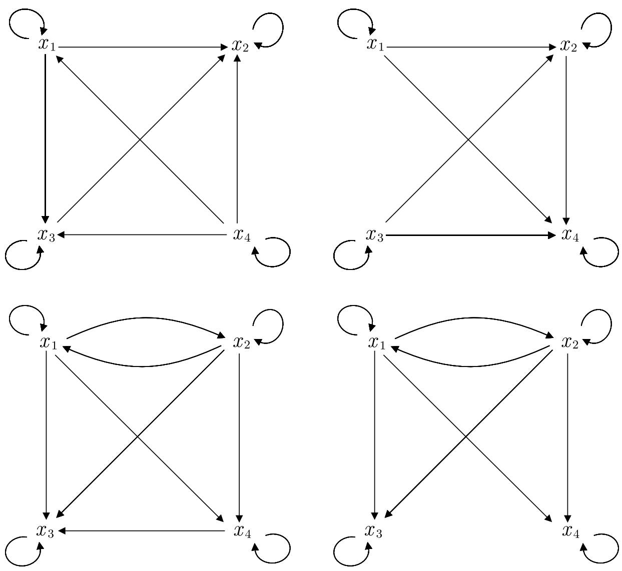

Figure 1: R_1, R_2, R_3 and R_4, resp.

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