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Convergence of Taylor Series in Hardy and Besov Spaces

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1. Introduction

Let ∂D be the unit circle and dt the Haar measure on ∂D; in [6], the author studied the norm convergence of the Taylor series in the Hardy space $H^p(\partial D)$ and, by using ad-hoc computations, showed that this happens if and only if $1 < p < \infty$. The purpose of this note is to point out the role of the Boyd indices in this process, once observed that the Boyd indices of $L^p(\partial D)$ are non-trivial, if and only if $1 < p < \infty$. We define Hardy spaces $H_E(\partial D)$ associated with an arbitrary rearrangement invariant space $E(\partial D)$ over the measure space $(\partial D, dt)$ and prove that the Taylor series converges in $H_E(\partial D)$, if and only if $E(\partial D)$ has absolutely continuous norm and non-trivial Boyd indices (Theorem 1). We obtain analogous results for analytic Besov spaces on the disk (Theorem 2).

The interest of rearrangement invariant space (r.i.s.) originates from Calderon interpolation theorem for linear operators: if T is a linear operator that is bounded both on L^1 and on L^∞, then T is bounded on E, too. On the other hand, Boyd interpolation theorem gives a sufficient condition for the boundedness of quasilinear operators of joint weak type $(p, p; q, q)$: if T is such an operator, then it is bounded on any r.i.s. E whose Boyd indices $\underline{\alpha}_E$ and $\overline{\alpha}_E$ verify the condition $\frac{1}{q} < \underline{\alpha}_E \leq \overline{\alpha}_E < \frac{1}{p}$. There exists many interesting examples of r.i.s.; among them, we only mention L^p, the Lorenz space $L^{p,q}$ and the Orlicz spaces L^φ. For notations, unexplained definitions and other relevant proprieties of r.i.s. and interpolation theory, the reader is refered to [2].

2. Norm convergence of Taylor series

Let $f \in E(\partial D)$ and $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ be its n^{th} Fourier coefficient. We define $H_E(\partial D) = \{f \in E(\partial D) : \widehat{f}(n) = 0 \ \forall \ n < 0\}$, the Hardy space on the disk associated with $E(\partial D)$; this definition extend the classic one, that is obtained if we take $E(\partial D) = L^p(\partial D)$. We also note that, since $a_n(f) = \widehat{f}(n)$, $n \in Z$, are bounded functionals on $E(\partial D)$ and since $H_E(\partial D) = \bigcap_{n<0} Ker\{a_n\}$, $H_E(\partial D)$ is a Banach space. If we define the Hardy space for the disk $H_E(D) = \{f \in H(D) : \ \sup_{0<r<1} \|f_r\|_{E(\partial D)}, \infty\}$, where $f_r(z) = f(rz)$, then one can see that the Poisson extension maps $H_E(\partial D)$ boundedely and onto $H_E(D)$, as it happens in the classic case.

In this paper, we want to study the convergence of the Taylor series in Hardy spaces. It is quite clear that this happens whenever the trigonometric polynomials are dense and when the operators T_N, given by $T_N(f) = \sum_o^N \widehat{f}(n)e^{int}$, $f \in E(\partial D)$, $N \geq 0$, are uniformly bounded on $H_E(\partial D)$.

Lemma 1. *Let $E(\partial D)$ be a r.i.s. The following condition are equivalent:*

(1) *$E(\partial D)$ has absolutely continuous norm (i.e. for all $f \in E(\partial D)$ and for any sequence of measurable subsets $E_n \longrightarrow \emptyset$ a.e., it follows $\|f\chi_{E_n}\|_{E(\partial D)} \longrightarrow 0$);*

(2) *the trigonometric polynomials are dense in $E(\partial D)$.*

Proof. First assume that $E(\partial D)$ has absolutely continuous norm. Let $f \in E(\partial D)$ and fix $\epsilon > 0$; since $E(\partial D)$ has absolute continous norm, the bounded simple functions are dense in $E(\partial D)$, by [2], Theorem I 3.11, so there exists a simple function s, finite everywhere, such that $\|f - s\|_{E(\partial D)} < \epsilon/3$; fix $\delta > 0$; by Lusin's theorem, there exists a continuous function g on ∂D, that differs by s on a set A of measure less then δ and such that $\|g\|_\infty \leq \|s\|_\infty$; hence $\|s - g\|_{E(\partial D)} \leq 2\|s\|_\infty \|\chi_A\|_{E(\partial D)}$. Since $E(\partial D)$ has absolutely continuous norm, we may choose δ such that $\|\chi_A\|_{E(\partial D)} < \frac{\epsilon}{6\|s\|_\infty}$, for any measurable set $A \subset \partial D$ of measure $|A| < \delta$, getting $\|s - g\|_{E(\partial D)} < \epsilon/3$. By Weierstrass approximation theorem, there exists a trigonometric polynomial on ∂D such that $\|g - P\|_\infty < \epsilon/3$, hence $\|g - P\|_{E(\partial D)} < \epsilon/3$, since $L^\infty(\partial D) \hookrightarrow E(\partial D)$, continuously and the inclusion can be always be assumed to be of norm one (see [2], Theorem II 6.6); putting all this toghether, we finally get $\|f - P\|_{E(\partial D)} < \epsilon$.

In order to prove the converse, we simply note that, if the trigonometric polynomials are dense in $E(\partial D)$, then clearly $E(\partial D)$ is separable (trigonometric polynomials with rational coefficients form a dense set), hence its norm is absolutely continuous, by [2], Theorem II 5.5. ∎

THEOREM 1. (See [3]) *Let $E(\partial D)$ be a r.i.s. with absolutely continuous norm. The following condition are equivalent:*

(1) *the Taylor series of any function $f \in H_E(\partial D)$ converges to f in the norm of $E(\partial D)$;*

(2) *the Fourier series of any $f \in E(\partial D)$ converges to f in the norm of $E(\partial D)$;*

(3) *$E(\partial D)$ has non trivial Boyd indices.*

Proof. (1) $\Rightarrow$ (2) Let $f \in E(\partial D)$, $\epsilon > 0$; by Lemma 1, there exist a trigonometric polynomial P such that $\|f - P\|_{E(\partial D)} < \epsilon$; then, for all $N \geq \deg P$, we have

$$\|f - S_N f\|_{E(\partial D)} \leq \|f - P\|_{E(\partial D)} + \|S_N P - S_N f\|_{E(\partial D)} ,$$

where $S_N(f) = \sum_{-N}^{N} \widehat{f}(n) e^{int}$ $f \in E(\partial D)$; so it would suffice to show that the operators S_N are uniformly bounded on $E(\partial D)$. Since, by hypothesis, $\lim_{N\to\infty} T_N f = f$, for all $f \in H_E(\partial D)$, then, according to the uniform boundedness principle, there exists a constant $C > 0$, such that $\|T_N f\|_{E(\partial D)} \leq C\|f\|_{E(\partial D)}$, for all $f \in H_E(\partial D)$ and $N \geq 1$. Now let Q be a trigonometric polynomial and let $K = \deg Q$; then $S_N Q = e^{-iKt} T_{2N}(e^{iKt} Q)$, for all $N \geq K$, hence $\|S_N Q\|_{E(\partial D)} \leq \|T_{2N}(e^{iKt} Q)\|_{E(\partial D)} \leq C\|e^{iKt} Q\|_{E(\partial D)}$. This proves that $\|S_N Q\|_{E(\partial D)} \leq C\|Q\|_{E(\partial D)}$ and this, together with the density of the trigonometric polynomials in $E(\partial D)$ (Lemma 1), end the proof.

(2) $\Rightarrow$ (1) is obvious, while (2) $\Leftrightarrow$ (3) is implicit in [3]. ∎

In [6], the author studied the problem of norm convergence of Taylor series in $H^p(\partial D)$, $1 < p < \infty$, and solved it by using ad-hoc computations; our approach is completely different of his and since for such p, the space $L^p(\partial D)$ has absolutely continuous norm and non-trivial Boyd indices we reobtain his results as a corollary:

COROLLARY 1. (See [6], Corollary 3) *The Taylor series of a function $f \in H^p(\partial D)$ converges to f in the norm of $H^p(\partial D)$, if and only if $1 < p < \infty$.*

3. Besov spaces

Let $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$ be the Möbius invariant measure on D, $E(d\lambda)$ a r.i.s. over the measure space $(D, d\lambda)$ and $n \geq 2$; the analytic Besov spaces associated with $E(d\lambda)$, consist in those analytic functions on D such that the function $z \to (1- \mid z \mid^2)^n f^{(n)}(z) \in E(d\lambda)$ (see e.g. [4]); when $E(d\lambda)$ has non-trivial Boyd indices, B_E are Möbius invarriant, in the sense of [1] (see [5]). In particular, the polynomials are dense in B_E, by [1], Proposition 2.

Theorem 2. *Let $E(d\lambda)$ be a r.i.s. over $(D, d\lambda)$, with non-trivial Boyd indices; then Taylor series of each function in B_E converge to f (in the norm of B_E).*

Proof. By [6], Corollary 5, there exists a constant $C > 0$ such that for all $N \geq 1$ and for all $1 < p < \infty$, $T_N : B_p \longrightarrow B_p$ is bounded and $\|T_N\| \leq C$ for all $N \geq 1$; the hypothesis on $E(d\lambda)$ and Boyd's theorem for Besov spaces (see [4], Theorem 5) ensure the uniform boundedness of T_N on B_E; as observed earlier, this fact, toghether with the density of polynomials in B_E, end the proof. ∎

4. Examples and comments

The conditions given in Theorem 1 and in Theorem 2 are expressed in terms of Boyd indices and absolutely continuity of the norm; so it is natural to ask if one of this two proprieties implies the other. The answer to this question is positive for important classes of r.i.s.: the Lorenz space $L^{p,q}$ $(1 < p < \infty$, $1 \leq q \leq \infty)$ has absolutely continuous norm if and only if has non trivial Boyd indices; a similar phenomenon ocures with the Orlicz space L^Φ, whose norm is absolutely continuous if and only if its superior Boyd indice is non-trivial.

Nevertheless, there exists relevant examples of r.i.s. that proves that the two proprieties are not related. The following example, that we shall present here, is due to Sharpley (see [2], pag. 285). Let E r.i.s., let $1 \leq q \leq \infty$ and

$$\Lambda_q(E) = \left\{ f \in \mathcal{M}(\partial D, dt) \,:\, \|f\|_{\Lambda_q} = \left(\int_0^\infty [f^{**}(t)\varphi_E(t)]^q \frac{dt}{t} \right)^{1/q} < \infty \right\},$$

where $f^{**}(t) = \int_0^t f^*(s)ds$ is the maximal function, f^* the decreasing rearrangement of f and φ_E the fundamental function of E (if $q = \infty$, then we take sup instead of integral); then, if E has non trivial fundamental indices, then the space $\Lambda_q(E)$ is a r.i.s. whose Boyd and fundamental indices coincide with the fundamental indices of E; further, $\Lambda_q(E)$ has absolutely continuous norm if and only if $1 \leq q < \infty$. So choosing $q = \infty$ and E such that $\widetilde{q}_E < \infty$, we obtain a r.i.s. $\Lambda_\infty(E)$ which has non trivial Boyd indices and has absolutely continuous norm, while if we take $q < \infty$ and E with trivial fundamental indices, then we obtain a r.i.s. $\Lambda_{q(E)}$ whose Boyd indices are trivial, but which has absolutely continuous norm. So the two proprieties are not related.

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