Norm of a Derivation and Hyponormal Operators

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Let H be a complex Hilbert space and let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on H. Following [3], a (symmetric) norm ideal $(J, \|.\|_J)$ in $\mathcal{L}(H)$ consists of a proper two-sided ideal J together with a norm $\|.\|_J$ satisfying the conditions:

(i) $(J, \|.\|_J)$ is a Banach space;

(ii) $\|AXB\|_J \leq \|A\|\|X\|_J\|B\|$ for all $X \in J$ and all operators A and B in $\mathcal{L}(H)$;

(iii) $\|X\|_J = \|X\|$ for X a rank one operator.

For a complete account of the theory of norm ideals, we refer to [3], [7] and [8]. In the sequel we will be particularly interested in operators belonging to the Hilbert-Schmidt class $\mathcal{C}_2(H)$ which represents a Hilbert space when equipped with the inner product $< X, Y > = \operatorname{tr}(XY^*)$, $(X, Y \in \mathcal{C}_2(H))$ where tr stands for the usual trace functional and Y^* denotes the adjoint of Y. For $A \in \mathcal{L}(H)$, the inner derivation induced by A is the operator δ_A defined on $\mathcal{L}(H)$ by $\delta_A(X) = AX - XA$, $X \in \mathcal{L}(H)$.

The norm of an inner derivation δ_A on H has been computed by J. Stampfli [10]:

$$(*) \qquad\qquad \|\delta_A\| = 2d(A),$$

where $d(A) = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$.

Let $(J, \|.\|_J)$ be a norm ideal and let $A \in \mathcal{L}(H)$. If $X \in J$, then $\delta_A(X) \in J$ and $\|AX - XA\|_J = \|(A - \lambda)X - X(A - \lambda)\|_J \leq 2\|A - \lambda\|\|X\|_J$ for all $\lambda \in \mathbb{C}$. Hence $\|AX - XA\|_J \leq 2d(A)\|X\|_J$. Thus the restriction $\delta_{J,A}$ of δ_A to J is a

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bounded linear operator on $(J, \|.\|_J)$, and $\|\delta_{J,A}\| \leq 2d(A)$. If $J = \mathcal{C}_2(H)$, we simply write $\delta_{J,A} = \delta_{2,A}$.

In order to examine the extent to which the identity $(*)$ applies, L. Fialkow [1] introduced the notion of S-universal operators :

An operator $A \in \mathcal{L}(H)$ is S-universal if $\|\delta_{J,A}\| = 2d(A)$ for each norm ideal J in $\mathcal{L}(H)$.

In [1], the author showed that a subnormal operator is S-universal if and only if the diameter of the spectrum is equal twice the radius of the smallest disk containing it. In what follows, we shall prove that the same conclusion holds true for an arbitrary hyponormal operator. This answers an open question in [1] and [2].

Let K be a nonempty bounded subset of the plane. The diameter of K is defined by $\operatorname{diam}(K) = \sup\{|\alpha - \beta| : \alpha, \beta \in K\}$. If $A \in \mathcal{L}(H)$, we mean by $\sigma(A)$, $W(A)$ and R_A respectively the spectrum, numerical range and radius of the smallest disk containing the spectrum.

Our main result is the following

THEOREM 1. *An hyponormal operator $A \in \mathcal{L}(H)$ is S-universal if and only if $\operatorname{diam}(\sigma(A)) = 2R_A$.*

Remark 1. Let $A \in \mathcal{L}(H)$ be hyponormal. From [5] (see also [9]), it follows that $\overline{W(A)} = \operatorname{co}(\sigma(A))$, where the bar denotes the closure and co stands for the convex hull. On the other hand, it turns out [10, Corollary 1], that $R_A = \inf\{\|A - \lambda\| : \lambda \in \mathbb{C}\}$. Thus Theorem 1 above can be reformulated as: "Let $A \in \mathcal{L}(H)$ be hyponormal. Then A is S-universal if and only if $\operatorname{diam}(W(A)) = 2d(A)$."

To prove Theorem 1, we need the next Theorem due to B.S. Nagy and C. Foias [4].

THEOREM 2. *For every hyponormal operator A on a Hilbert space H there exists a normal operator N and a unitary operator U on some Hilbert space K, and a contraction R of H into K, such that:*

(a) $A = R^*NR$.

(b) $\|N\| = \|A\|$.

(c) $NU = UN = N^*$.

(d) $\|R^*Ug\| \leq \|R^*g\|$ for all $g \in K$.

(e) *The manifolds $L_n = U^n RH (n = 0, 1, \cdots)$ form a non-decreasing sequence and span K.*

(f) *For any complex scalars α, β,*

$$\sigma(\alpha N + \beta N^*) \subseteq \sigma_l(\alpha A + \beta A^*) \quad (\sigma_l : \text{``left spectrum''}).$$

COROLLARY 3. *Let A be a hyponormal operator and let N be a normal operator given by Theorem 2. Then $d(A) = \|A\|$ if and only if $d(N) = \|N\|$.*

Proof. Suppose that $d(A) = \|A\|$. By [10, Theorem 2] there exists a sequence $\{x_n\}_n$ in H with $\|x_n\| = 1$ for each n and such that $< Ax_n, x_n > \to 0$ and $\|Ax_n\| \to \|A\|$, as $n \to \infty$. So $< NRx_n, Rx_n > \to 0$ as $n \to \infty$. Moreover, since

$$\|Ax_n\| = \|R^* NRx_n\| \leq \|R^*\| \|NRx_n\| \leq \|R^*\| \|N\| \|Rx_n\| \leq \|N\|,$$

we conclude that $\|Rx_n\| \to 1$ and $\|NRx_n\| \to \|N\|$ as $n \to \infty$. Using again [10, Theorem 2], we conclude that $d(N) = \|N\|$.

Conversely, suppose that $d(N) = \|N\|$. We have $R_N = \|N\|$ (see Remark 1). Since $\sigma(N) \subseteq \sigma(A)$ (Theorem 2, (f)), then $R_N \leq R_A$. Using (b) of Theorem 2, we obtain $\|A\| \leq R_A$. Hence $d(A) = \|A\|$, which completes the proof. ∎

Proof of Theorem 1. We adopt the notation of Theorem 2. Since S-universality and hyponormality are preserved under translations, we may assume that $d(A) = \|A\|$ and hence $d(N) = \|N\|$ (Corollary 3).

Suppose that A is S-universal. By (b) of Theorem 2, we have

$$\|\delta_{2,A}\| = \|\delta_A\| = 2\|A\| = 2\|N\| = \|\delta_N\|.$$

So we can find a sequence $\{X_n\}_n$ in $\mathcal{C}_2(H)$ with $\|X_n\|_2 = 1$, for which $\|AX_n - X_nA\|_2 \to 2\|A\|$ as $n \to \infty$. Since

$$\|AX_n - X_nA\|_2 \leq \|AX_n\|_2 + \|X_nA\|_2 \leq \|A\| + \|X_nA\|_2 \leq 2\|A\|,$$

we deduce that

$$\|AX_n\|_2 \to \|A\|.$$

Similarly, we get

$$\|X_nA\|_2 \to \|A\|.$$

Now, from the identity

$$\|AX_n - X_nA\|_2^2 = \|AX_n\|_2^2 + \|X_nA\|_2^2 - 2\Re(< AX_n, X_nA >),$$

we conclude that $-\Re(< AX_n, X_nA >) \to \|A\|^2$ as $n \to \infty$, here $\Re$ denotes the real part.

Consider the operator $RX_nR^* \in \mathcal{L}(K)$. Since $X_n \in \mathcal{C}_2(H)$ and $\|X_n\|_2 = 1$, then $RX_nR^* \in \mathcal{C}_2(K)$ and $\|RX_nR^*\|_2 \leq 1$. Furthermore

$$
\begin{aligned}
< NRX_nR^*, RX_nR^*N > &= \operatorname{tr}(NRX_nR^*(RX_nR^*N)^*) \\
&= < AX_n, X_nA > .
\end{aligned}
$$

Hence

$$\Re(< NRX_nR^*, RX_nR^*N >) \to -\|N\|^2 \text{ as } n \to \infty.$$

Since $|\Re(< NRX_nR^*, RX_nR^*N >)| \leq \|NRX_nR^*\|_2\|RX_nR^*N\|_2 \leq \|N\|^2$, it follows that

$$\|NRX_nR^*\|_2 \to \|N\|, \ \ \|RX_nR^*N\|_2 \to \|N\| \ \ \ \text{as } n \to \infty.$$

Whence we infer

$$\|\delta_{2,N}(RX_nR^*)\|_2 \to 2\|N\| \ \ \ \ \text{as } n \to \infty.$$

That is

$$\|\delta_{2,N}\| = 2\|N\|.$$

Since N is normal, Lemma 4.11 of [1] guarantees that

$$\operatorname{diam}(\sigma(N)) = \|\delta_2(N)\|.$$

On the other hand, by (f) of Theorem 2, we see that

$$\operatorname{diam}(\sigma(N)) \leq \operatorname{diam}(\sigma(A)) \leq \|\delta_{2,A}\| \leq 2\|N\|.$$

Therefore

$$\operatorname{diam}(\sigma(A)) = 2\|N\| = 2\|A\| = 2R_A.$$

The sufficient condition follows from the general fact: For each operator $A \in \mathcal{L}(H)$, we have

(i) $\sigma(\delta_{2,A}) = \sigma(A) - \sigma(A)$ (see[6]);
(ii) $r(\delta_{2,A}) \leq \|\delta_{2,A}\| \leq 2d(A)$. ∎

Remarks 4. (i) The sufficient condition of the above Theorem follows also from [1, Corollary 4.8].

(ii) Theorem 1 answers the question 2.17 of [2] in the affirmative.

We close with a remark. Let A and N as in Theorem 2. From the assertion (f), we see that $\sigma(N) \subseteq \sigma(A)$. So diam$(\sigma(N)) \leq$ diam$(\sigma(A))$. Since diam$(\sigma(N)) = \|\delta_{2,N}\|$ and diam$(\sigma(A)) \leq \|\delta_{2,A}\|$, then one can see that $\|\delta_{2,N}\| \leq \|\delta_{2,A}\|$. Thus the following question seems natural. Does the equality $\|\delta_{2,N}\| = \|\delta_{2,A}\|$ holds true? Note that an affirmative answer to this question would provide an affirmative answer to the question 2.16 of [2].

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