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# A Lower Bound of the Norm of the Operator $\mathbf{X \longrightarrow AXB + BXA}$

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For two elements $a$ and $b$ of a ring $\mathcal{A}$, we understand by $M_{a,b}$ the two-sided multiplication induced by $a$ and $b$. In the case where $\mathcal{A}$ is a prime $\mathbb{C}^*$-algebra, the question of how to determine the lower bound of the norm of $M_{a,b}+M_{b,a}$ is stated in [4] as an open problem: Does the inequality $\|M_{a,b}+M_{b,a}\| \geq \|a\|\|b\|$ holds for any two elements $a$ and $b$ in a prime $\mathbb{C}^*$-algebra $\mathcal{A}$?.

As a continuation to [4], M. Mathieu [5] proved that $\|M_{a,b} + M_{b,a}\| \geq (2/3)\|a\|\|b\|$ for prime $\mathbb{C}^*$-algebras. Cabrera and Rodriguez [1] proved that for prime $JB^*$-algebras we have the lower estimate $\|M_{a,b}+M_{b,a}\| \geq (1/20412)\cdot \|a\|\|b\|$. Stacho and Zalar [6] proved that for standard operator algebras on a Hilbert space we have $\|M_{a,b}+M_{b,a}\| \geq 2(\sqrt{2}-1)\|a\|\|b\|$, and recently [7] they proved that $\|M_{a,b} + M_{b,a}\| \geq \|a\|\|b\|$ for the algebra of symmetric operators acting on a Hilbert space.

Note that $\|M_{a,b}\| = \|a\|\|b\|$ if and only if $\mathcal{A}$ is a prime $\mathbb{C}^*$-algebra [3] and that the upper estimate $\|M_{a,b} + M_{b,a}\| \leq 2\|a\|\|b\|$ is trivial.

In this paper, we consider the case where $\mathcal{A} = \mathcal{L}(H)$ the algebra of bounded linear operators on a complex Hilbert space $H$. We shall prove that for two operators $A$ and $B$ such that $\inf_{\lambda\in\mathbb{C}} \|B-\lambda A\| = \|B\|$ or $\inf_{\lambda\in\mathbb{C}} \|A-\lambda B\| = \|A\|$ we have $\|M_{A,B} + M_{B,A}\| \geq \|A\|\|B\|$.

Our proof is based on the concept of the numerical range of $A^*B$ relative to $B$ introduced by B. Magajna in [2]:

$$W_B(A^*B) = \{\lambda \in \mathbb{C} : \exists e_n \in H,\ \|e_n\| = 1,\ \lim\langle A^*Be_n, e_n\rangle = \lambda,\ \lim \|Be_n\| = \|B\|\}.$$

In the case $A = I$ this reduces to the Stampfli maximal numerical range of $B$ see [8]. The most interesting properties of $W_B(A^*B)$ are [2]:

(i) $W_B(A^*B)$ is a closed convex subset of the complex plane $\mathbb{C}$ for each $A, B \in \mathcal{L}(H)$.

(ii) The relation $\|B\| = \inf_{\lambda \in \mathbb{C}} \|B - \lambda A\|$ holds if and only if $0 \in W_B(A^*B)$.

For any $x, y \in H$, define the rank-one operator $x \otimes y \in \mathcal{L}(H)$ by the equation

$$(x \otimes y)(z) = \langle z, y \rangle x, \quad \forall z \in H.$$

Our main result is the following:

THEOREM 1. *Let $A, B \in \mathcal{L}(H)$ with $B \neq 0$. Then*

$$\|M_{A,B} + M_{B,A}\| \geq \sup_{\lambda \in W_B(A^*B)} \left\| \|B\|A + \frac{\overline{\lambda}}{\|B\|}B \right\|.$$

*Proof.* Let $\lambda \in W_B(A^*B)$. Then there exists a sequence $\{e_n\}_{n \geq 1}$ of unit vectors in $H$ such that $\lim_n \langle A^*Be_n, e_n \rangle = \lambda$ and $\lim_n \|Be_n\| = \|B\|$. Consider a unit vector $y \in H$. For each $n \geq 1$, we have

$$\|(M_{A,B} + M_{B,A})(y \otimes Be_n)(e_n)\| = \big\| \|Be_n\|^2 Ay + \langle e_n, A^*Be_n \rangle By \big\|.$$

Hence

$$\|M_{A,B} + M_{B,A}\| \geq \frac{1}{\|B\|} \big\| \|Be_n\|^2 Ay + \langle e_n, A^*Be_n \rangle By \big\|.$$

Letting $n \to \infty$, we obtain

$$\|M_{A,B} + M_{B,A}\| \geq \left\| \|B\|Ay + \frac{\overline{\lambda}}{\|B\|}By \right\|.$$

Since $\lambda$ and $y$ are arbitrary in $W_B(A^*B)$ and $H$ respectively, we get

$$\begin{aligned}\|M_{A,B} + M_{B,A}\| &\geq \sup_{\lambda \in W_B(A^*B)} \left( \sup_{\|y\|=1} \left( \left\| \|B\|Ay + \frac{\overline{\lambda}}{\|B\|}By \right\| \right) \right) \\ &= \sup_{\lambda \in W_B(A^*B)} \left\| \|B\|A + \frac{\overline{\lambda}}{\|B\|}B \right\|,\end{aligned}$$

which completes the proof. ∎

An immediate consequence of Theorem 1 is the following corollary:

Corollary 2. *If* $0 \in W_B(A^*B) \cup W_A(B^*A)$*, then*

$$\|M_{A,B} + M_{B,A}\| \geq \|A\|\|B\|.$$

*Remark.* (i) Corollary 2 answers partially the problem mentioned above.

(ii) The estimate in Corollary 2 is, in general the best possible.

(iii) The condition in Corollary 2 is not necessary: take $A = B \neq 0$. Then $\|M_{A,B} + M_{B,A}\| = 2\|A\|^2 > \|A\|^2$, but $0 \notin W_A(A^*A) = \{\|A\|^2\}$.

The following result is a generalization of theorems 1 and 2 of [8].

Theorem 3. *Let* $A, B \in \mathcal{L}(H)$*. Then the following conditions are equivalent:*

(i) $0 \in W_B(A^*B)$,

(ii) $\|B\| \leq \|B + \lambda A\|, \ \lambda \in \mathbb{C}$,

(iii) $\|B\|^2 + |\lambda|^2 m^2(A) \leq \|B + \lambda A\|^2, \ \lambda \in \mathbb{C}$, *where*

$$m(A) = \inf\{\|Ax\| : x \in H, \|x\| = 1\}.$$

*Proof.* The implication (iii) $\Rightarrow$ (ii) is clear and the equivalence (i) $\Leftrightarrow$ (ii) is contained in ([2], p. 519). Next we show that (i) $\Rightarrow$ (iii). Since $0 \in W_B(A^*B)$, there must be a sequence of unit vectors $\{e_n\}_{n\geq 1}$ such that $\lim_n \langle A^*Be_n, e_n\rangle = 0$ and $\lim_n \|Be_n\| = \|B\|$. Let $\lambda \in \mathbb{C}$. For each $n \geq 1$, we have

$$\begin{aligned}\|(B + \lambda A)e_n\|^2 =& \|Be_n\|^2 + |\lambda|^2\|Ae_n\|^2 + 2\,\mathrm{Re}(\overline{\lambda}\langle A^*Be_n, e_n\rangle)\\ \geq& \|Be_n\|^2 + |\lambda|^2 m^2(A) + 2\,\mathrm{Re}(\overline{\lambda}\langle A^*Be_n, e_n\rangle),\end{aligned}$$

where "Re" denotes the real part. Letting $n \to \infty$, we get

$$\|B + \lambda A\|^2 \geq \|B\|^2 + |\lambda|^2 m^2(A)$$

and this proves the theorem. ∎

The next corollary is proved in [8] in the case $A = I$, but the same reasoning applies to the general situation considered here.

Corollary 4. *Let* $A, B \in \mathcal{L}(H)$ *such that* $m(A) \neq 0$*. Then there exists a unique* $z_0 \in \mathbb{C}$ *such that*

$$\|B - z_o A\|^2 + |\lambda|^2 m^2(A) \leq \|(B - z_0 A) + \lambda A\|^2$$

*for all* $\lambda \in \mathbb{C}$*. Moreover,* $0 \in W_{B-\lambda A}(A^*(B - \lambda A))$ *if and only if* $\lambda = z_0$*.*

*Proof.* The function $\lambda \to \|B - \lambda A\|$ is continuous with $\lim_{|\lambda|\to\infty} \|B - \lambda A\| = \infty$. So by a compactness argument, there exists $z_0 \in \mathbb{C}$ such that $\|B - z_0 A\|^2 \leq \|(B - z_0 A) + \lambda A\|^2$ for all $\lambda \in \mathbb{C}$. The rest of the proof follows easily from Theorem 2. ∎

PROPOSITION 5. *If* $\|A\|\|B\| \in W_A(B^*A) \cap W_{A^*}(BA^*)$, *then*

$$\|M_{A,B} + M_{B,A}\| = \|M_{A,B}\| + \|M_{B,A}\| = 2\|A\|\|B\|.$$

*Proof.* Suppose $\|A\|\|B\| \in W_A(B^*A) \cap W_{A^*}(BA^*)$. Then we can find two unit sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ of vectors in $H$ such that

$$\lim_n \langle B^*Ax_n, x_n\rangle = \|A\|\|B\|, \quad \lim_n \|Ax_n\| = \|A\|,$$

$$\lim_n \langle BA^*y_n, y_n\rangle = \|A\|\|B\|, \quad \lim_n \|A^*y_n\| = \|A\|.$$

Since $|\langle B^*Ax_n, x_n\rangle| \leq \|Ax_n\|\|Bx_n\|$ and $|\langle BA^*y_n, y_n\rangle| \leq \|A^*y_n\|\|B^*y_n\|$, then $\lim_n \|Bx_n\| = \|B\|$ and $\lim_n \|B^*y_n\| = \|B\|$. For each $n \geq 1$, we have

$$\begin{aligned}\|(M_{A,B} + M_{B,A})(x_n \otimes y_n)(B^*y_n)\|^2 \\ = \|B^*y_n\|^4\|Ax_n\|^2 + |\langle AB^*y_n, y_n\rangle|^2\|Bx_n\|^2 \\ + 2\|B^*y_n\|^2 Re(\langle y_n, AB^*y_n\rangle\langle B^*Ax_n, x_n\rangle.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\|M_{A,B} + M_{B,A}\| \geq 2\|A\|\|B\|.$$

Whence

$$\|M_{A,B} + M_{B,A}\| = 2\|A\|\|B\|.$$

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