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Higher-Order Differential Equations Represented by Connections on Prolongations of a Fibered Manifold*

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1. Introduction

The main goal of the presented work is a generalization of the ideas, constructions an results from the first and second-order situation, studied in [63], [64], to that of an arbitrary finite-order one. Moreover, the investigation extends the ideas of [65] from the one-dimensional base X corresponding to O.D.E.

First, all the basic underlying structures and notions used are recalled in Sections 2-6 in accordance with [56], [44], [45]. Moreover, a comprehensive description of higher-order connections is presented in Sections 7-8, with the same references.

In Section 9, the equations represented by higher-order connections are described in details. Here, the formalism and ideas follow and combine those of [1], [7], [26], [48], [49], [53], [56], [58], and also [67]. In particular, the integrability is discussed in terms of the corresponding horizontal distributions.

The *r-th jet prolongation* of the equations in question is the object of the study of Section 10. In general, the prolongation of an equation carries the information on the equation together with a given number of 'consequences', obtained by differentiating the original equation. In case of connections, the

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construction of the prolongation in terms of the prolongations of corresponding morphisms results in a very transparent characterization, which follows the definition of a *field of paths* as a local lower-order connection representing the order-reduction of the initial equations (we refer to [39] for motivations). For related ideas, [26], [56] and [67] are good examples.

Section 11 is devoted to the classification of the *symmetries* in sense of [64] according to [1], [2],[48], [67] and mainly [25], [43], [56], now for the higher-order connections. Infinitesimal symmetries as the generators of invariant transformations are studied in terms of the corresponding decompositions on tangent bundles, and both sets of *characteristic* and *shuffling* symmetries are described in accordance with [25]. The use of the *vertical prolongation* $\mathcal{V}\Gamma^{(k+1)}$ finds its application within the 2-fibered manifold $V_{\pi_k}J^k\pi \overset{\tau_{J^k\pi}}{\to} J^k\pi \overset{\pi_k}{\to} X$, where a linear connection on $\tau_{J^k\pi}$ whose integral sections are the symmetries is found. Finally, the relations between symmetries for a connection and its field of paths are derived, again in terms of vertical prolongations. In Section 13 we recall all the necessary concepts of the theory of 2-*fibered manifolds*, adopted to our purposes. The formalism in fact comes from [28]; nevertheless, it has been implemented into that of [56] in [23].

The most interesting part of the presented theory is that having to do with the interrelations between equations represented by connections on various fibrations, which starts in Section 14. First, we give a summary of the notions characterizing connections on the affine bundle $\pi_{k+1,k}\colon J^{k+1}\pi \to J^k\pi$. The point is that such connections represent first-order equations for (local) $(k+1)$-connections on π. In Section 15, the concept of the *characterizability* is introduced. A connection Ξ on $\pi_{k+1,k}$ is characterizable if it uniquely determines the $(k+2)$-connection on π by the intersection with the Cartan distribution. In fact, the construction generalizes that of the *associated semispray* to a given *dynamical connection* (cf. [16], [19], [9], [17], [46], [3], [11] etc.) and it results in the *method of characteristics* for Ξ, discussed in Section 16. As regards both the name and the meaning, the approach is quite near to the ideas dealing with Pfaffian systems in [53] and particularly [57]. Reaping the benefit of the fact that each integral section of Ξ is the field of paths of $\Gamma^{(k+2)}$, the integral 'surfaces' of Ξ are foliated by $(k+1)$-jets of integral 'curves' of $\Gamma^{(k+2)}$ (=*characteristics*). This was first studied in [42]. The relation between the equations studied can be roughly (and non-geometrically) expressed as follows (suppose $k=0$): if the equations for Ξ are given by

$$dy_i^\sigma = \Xi_{ij}^\sigma\, dx^j + \Xi_{i\lambda}^\sigma\, dy^\lambda,$$

then those for its characteristic $\Gamma^{(2)}$ are

$$y^\sigma_{ij} = \frac{dy^\sigma_i}{dx^j} = \Xi^\sigma_{ij}\frac{dx^j}{dx^j} + \Xi^\sigma_{i\lambda}\frac{dy^\lambda}{dx^j} = \Xi^\sigma_{ij} + \Xi^\sigma_{i\lambda} y^\lambda_j.$$

In the rest of the Section, the *formal mixed curvature* is studied in accordance with Section 13 and [29], having to do with relations between the characterizability and integrability.

A dual *method of fields of paths* is introduced in Section 17, where the *integral* of an integrable $\Gamma^{(k+2)}$ is an integrable Ξ on $\pi_{k+1,k}$ whose characteristic connection is just $\Gamma^{(k+2)}$. The existence of such an integral allows the order-reduction of $\Gamma^{(k+2)}$ to (local) integral sections of Ξ. In this respect, the existence of both local and global integrals is discussed. It is necessary to admit that just this part contains the largest gap of the theory; namely, the construction of a *global* connection Ξ on $\pi_{k+1,k}$ *associated* to $\Gamma^{(k+2)}$ is not yet clear in general. The case of $k = 0$ and arbitrary $\dim X$ is recalled according to [22], while that of $\dim X = 1$ and arbitrary k, is due to [62].

It turns out that the complement to the Cartan distribution $C^{\Gamma^{(k+1)}}$ in the decomposition of $T\Gamma^{(k+1)}(J^k\pi) \subset TJ^{k+1}\pi$, expressed in terms of the vertical prolongation $\mathcal{V}\Gamma^{(k+1)}$ in Section 11, has a global counterpart in the so-called *reduced connection* (partially motivated by [67]) associated to any characterizable connection on $\pi_{k+1,k}$ in terms of the corresponding $f(3,-1)$-*structure* (for this notion, see e.g. [19]). These concepts are studied in Section 18, the observations of which can be summarized by saying that *each characterizable connection on $\pi_{k+1,k}$ splits into the direct sum of its characteristic connection and the corresponding reduced connection.*

Section 20 is completely motivated by [39]. The background is the 2-fibered manifold

$$J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X,$$

the formalism of which leads to a geometric description of a generalization of the method of fields of path, presented in Section 17. If $\Gamma^{(k+r+1)}$ is a $(k+r+1)$-connection on π, then the method gives a $(k+1)$-connection $\Gamma^{(k+1)}$ on π representing the order-reduction of the equations represented by $\Gamma^{(k+r+1)}$, all for $r \geq 2$. In fact, this is obtained by means of looking for the prolongation of $\Gamma^{(k+1)}$, which is a section of $\pi_{k+r,k}$ (a *jet field*). In this respect, the connections on $\pi_{k+r,k}$ are studied, as well, which results in the definition of the $\pi_{k+r,k}$-*integral* of $\Gamma^{(k+r+1)}$. It should be mentioned that for $r \geq 2$, $\pi_{k+r,k}$ is *not* an affine bundle, hence the ideas on formal 'curvature-like' concepts are not repeated. In [65], the above formalism was exampled for a description of *regular variational equations* in sense of [39]. Section 21 recalls the well-known identifications and canonical morphisms for k-jet manifolds and related structures, all in accordance with [9], [11], [12], [14], [16], [17], [19], [32], [55], [56], [61], [63]. The *natural vector-valued one-forms* are presented due to [16], [17], [20] and [62].

Following the particular type of fibration, the (first-order) *connections* on $\pi : \mathbb{R} \times M \longrightarrow \mathbb{R}$ are mentioned in Section 22, mainly in accordance with [56].

The most significant change during the passing to the general $\pi\colon Y \to X$ was that of

$$\text{semispray} \quad \rightsquigarrow \quad \text{higher-order connection} = \text{semispray distribution.}$$

Just the *higher-order (semispray) connections* and their vertical prolongations are studied in Section 23, which can be confronted e.g. with [16], [17], [19], [59] and [6], [13], [15], [41], [51], [52], [54] (for symmetries).

Sections 24 and 25 are devoted to the connections *associated* to semisprays. Here, the adopted approach meets the results on *dynamical connections* and related structures on $\mathbb{R} \times TM$ [18], [19], [9], $\mathbb{R} \times T^kM$ [17], $J^1(Y \to \mathbb{R})$ [46], T^kM [16], [12], etc. Moreover, some additional ideas are presented; namely, all *natural $\iota_{1,k}$-admissible deformations* on $\pi_{k+1,k}$ are discussed in accordance with [20], generalizing the results of [61] and [63].

The examples of Section 26, illustrating both methods of characteristics and fields of paths for O.D.E. of the first and second-order, can be compared with [57], [27].

In view of general situation, Section 27 is much more briefly drafted. Despite of this, it gives the possibility for a transparent description both of autonomous and non-autonomous concepts related with *semisprays of various types* – cf. [59] for $J^k\pi$, [16] for T^kM and [24] for the most general situation.

Throughout the work, all manifolds are smooth ($= C^\infty$), finite-dimensional, Hausdorff, second-countable and connected, which by definition means also paracompactness and thus the presence of partitions of unity. All mappings are smooth, as well, and the summation convention is used as far as possible. The notation follows completely that of the monography [56].

2. Jet prolongations of sections and morphisms

Let $\pi\colon Y \to X$ be an arbitrary fibered manifold, and let k be a natural number. Two local section of π on $U \subset X$, $\gamma_1, \gamma_2 \in \mathcal{S}_U(\pi)$, are said to be k-*equivalent* at $x \in U \subset X$ if $\gamma_1(x) = \gamma_2(x)$ and if there are fibered coordinates

(x^i, y^σ) around $\gamma_1(x) = \gamma_2(x)$ such that

$$\frac{\partial^k \gamma_1^\sigma}{\partial x^{j_1} \dots \partial x^{j_\ell}}|_x = \frac{\partial^k \gamma_2^\sigma}{\partial x^{j_1} \dots \partial x^{j_\ell}}|_x$$

for each $\sigma = 1, \dots, m$, $\ell = 1, \dots k$ and any sequence $1 \le j_1 \le \dots \le j_\ell \le n$. The particular choice of a coordinate system does not matter and the equivalence class containing a section γ is called the *k-jet* of γ at x and it is denoted by $j^k_x \gamma$.

The *k-jet manifold of* π is the set

$$J^k\pi = \{j^k_x\gamma; x \in X, \gamma \in \mathcal{S}_{\mathrm{loc}}(\pi), x \in \mathrm{Dom}(\gamma)\}$$

of all k-jets at $x \in X$. It carries a natural structure of a differentiable manifold, which can be viewed as the total space of $(k+1)$ different projections

$$\begin{aligned} \pi_k &\colon J^k\pi \to X, \quad \pi_k(j^k_x\gamma) = x, \\ \pi_{k,\ell} &\colon J^k\pi \to J^\ell\pi, \quad \pi_{k,\ell}(j^k_x\gamma) = j^\ell_x\gamma, \end{aligned} \tag{2.1}$$

for $\ell = 0, \dots, k-1$, where $j^0_x\gamma := \gamma(x)$ and accordingly $J^0\pi := Y$ (it should be mentioned that in what follows miscellaneous projections appear – in this respect, $\pi_{k,k} := \mathrm{id}_{J^k\pi}$ and $\pi_0 := \pi$). Any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on Y induces a fibered chart (V_k, ψ_k), $\psi_k = (x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1 \dots j_k})$, on $J^k\pi$ by $V_k = \pi^{-1}_{k,0}(V)$ and

$$y^\sigma_{j_1 \dots j_\ell}(j^k_x\gamma) = \frac{\partial^\ell \gamma^\sigma}{\partial x^{j_1} \dots \partial x^{j_\ell}}|_x.$$

Let $\gamma \in \mathcal{S}_U(\pi)$. Then $j^k\gamma \in \mathcal{S}_U(\pi_k)$, defined by $j^k\gamma(x) = j^k_x\gamma$ for $x \in U$, is called the *k-jet prolongation of the section* γ. Clearly $\pi_{k,\ell} \circ j^k\gamma = j^\ell\gamma$ for $k > \ell$.

We shall have frequent occasion to study main features of the above projections (2.1). Namely, all the triples $(J^k\pi, \pi_{k,\ell}, J^\ell\pi)$ and $(J^k\pi, \pi_k, X)$ are fibered manifolds. More precisely, if π is a bundle so is π_k, whereas the others are bundles even for π being a fibered manifold. Moreover, $\pi_{k,k-1}$ is an *affine* bundle for an arbitrary $k \ge 1$, whose associated vector bundle is

$$\begin{aligned} &\Big(\pi^*_{k-1,0}\left(V_\pi Y\right) \otimes \pi^*_{k-1}\left(S^k T^* X\right), \\ &\qquad \pi^*_{k-1,0}\left(\tau_Y|_{V_\pi Y}\right) \otimes S^k \tau^*_{J^{k-1}\pi}|_{\pi^*_{k-1}\left(S^k T^* X\right)}, J^{k-1}\pi\Big). \end{aligned} \tag{2.2}$$

Let (Φ, φ) be a fibered morphism between (Y, π, X) and (Y', π', X') such that φ is a diffeomorphism. The *k-th jet prolongation of* (Φ, φ) is a map

$$J^k(\Phi, \varphi)\colon J^k\pi \to J^k\pi',$$

defined by

$$J^k(\Phi, \varphi)(j^k_x\gamma) = j^k_{\varphi(x)}\, {}_{\Phi}\gamma_{\varphi} \tag{2.3}$$

for any $j^k_x\gamma \in J^k\pi$, where ${}_{\Phi}\gamma_{\varphi} := \Phi \circ \gamma \circ \varphi^{-1} \in \mathcal{S}_{\varphi(U)}(\pi')$. In particular,

$$J^k(\Phi, \mathrm{id}_X)(j^k_x\gamma) = j^k_x(\Phi \circ \gamma). \tag{2.4}$$

It is clear that $(J^k(\Phi, \varphi), \varphi)$ or $(J^k(\Phi, \varphi), J^\ell(\Phi, \varphi))$ are fibered morphisms between π_k and π'_k or $\pi_{k,\ell}$ and $\pi'_{k,\ell}$, respectively.

3. Total derivatives

A distinguished role is played by a short exact sequence

$$0 \longrightarrow V_{\pi_k}J^k\pi \longrightarrow TJ^k\pi \longrightarrow \pi_k^*(TX) \longrightarrow 0 \tag{3.1}$$

of vector bundles over $J^k\pi$, resulting to the exact sequence

$$0 \longrightarrow \mathcal{X}^v_X(J^k\pi) \longrightarrow \mathcal{X}(J^k\pi) \longrightarrow \mathcal{X}(\pi_k) \longrightarrow 0$$

of modules of sections. The sequence (3.1) does not canonically split in general; nevertheless, there is a splitting when pulled-back to $TJ^{k+1}\pi$. In fact,

$$\begin{aligned}(\pi^*_{k+1,k}(TJ^k\pi), \pi^*_{k+1,k}(\tau_{J^k\pi}), J^{k+1}\pi) &\cong \\ &\cong (\pi^*_{k+1,k}(V_{\pi_k}J^k\pi) \oplus H_{\pi_{k+1,k}}, \pi^*_{k+1,k}(\tau_{J^k\pi}), J^{k+1}\pi),\end{aligned} \tag{3.2}$$

with

$$H_{\pi_{k+1,k}} := \bigcup_x Tj^k\gamma(T_xX)$$

being the subbundle of the *k-th holonomic tangent vectors*; the *k-th holonomic lift* of $\xi \in T_xX$ by $\gamma \in \mathcal{S}_U(\pi)$, $x \in U$, is defined as the pair $(j^{k+1}_x\gamma, Tj^k\gamma(\xi)) \in H_{\pi_{k+1,k}}$. Following (3.2), there is a decomposition

$$\mathcal{X}(\pi_{k+1,k}) = \mathcal{X}^v(\pi_{k+1,k}) \oplus \mathcal{X}^h(\pi_{k+1,k}) \tag{3.3}$$

of the module of vector fields along $\pi_{k+1,k}$ to the vertical and horizontal submodules

$$\mathcal{X}^v(\pi_{k+1,k}) = \mathcal{S}(\pi^*_{k+1,k}(\tau_{J^k\pi})|_{V_{\pi_k}J^k\pi}),$$
$$\mathcal{X}^h(\pi_{k+1,k}) = \mathcal{S}(\pi^*_{k+1,k}(\tau_{J^k\pi})|_{H_{\pi_{k+1,k}}}).$$

The k-th holonomic lift of $\frac{\partial}{\partial x^i}$ is the (local) vector field

$$D_i^{k+1,k} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k} y^\sigma_{j_1\dots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} \in \mathcal{X}^h(\pi_{k+1,k}),$$

called the *i-th total (formal) derivative*, $i = 1, \dots, n$. Each total derivative can be characterized as a derivation working on functions, i.e. a mapping $\mathcal{F}(J^k\pi) \to \mathcal{F}(J^{k+1}\pi)$; in particular

$$y^\sigma_{j_1\dots j_k i} = D_i^{k+1,k}(y^\sigma_{j_1\dots j_k}).$$

As expected, the total derivatives (and consequently the corresponding holonomic lifts) are related by the following commutative diagram

$$\begin{array}{ccc} J^{k+1}\pi & \xrightarrow{D_i^{k+1,k}} & TJ^k\pi \\ {\scriptstyle \pi_{k+1,\ell+1}}\downarrow & & \downarrow{\scriptstyle T\pi_{k,\ell}} \\ J^{\ell+1}\pi & \xrightarrow{D_i^{\ell+1,\ell}} & TJ^\ell\pi \end{array}$$

for $k > \ell > 0$.

Various compositions lead to a possibility of defining various *higher-order derivations* of functions; for example

$$D_{j_1\dots j_k} := D^{k,k-1}_{j_k} D^{k-1,k-2}_{j_{k-1}} \dots D^{1,0}_{j_1}(f) \tag{3.4}$$

for $f \in \mathcal{F}(Y)$ is an element of $\mathcal{F}(J^k\pi)$ (i.e. D_i here denotes just $D_i^{1,0}$).

In this respect, (2.4) locally reads

$$(x^i, y^\sigma, y^\sigma_i, \dots, y^\sigma_{j_1\dots j_k}) \mapsto \big(x^i, \Phi^\sigma, D_i(\Phi^\sigma), \dots, D_{j_1\dots j_k}(\Phi^\sigma)\big)$$

for $y'^\sigma \circ \Phi = \Phi^\sigma(x^j, y^\lambda)$.

4. Prolongations of vector fields

First, a deep relation between the functors J^k and V should be recalled. Namely, there is a canonical izomorphism

$$\nu_k \colon J^k\left(\pi \circ \tau_Y|_{V_\pi Y}\right) \to V_{\pi_k} J^k \pi \tag{4.1}$$

over X: considering the maps $\gamma\colon U \times \mathbb{R} \to Y$ such that $\pi \circ \gamma(x,t) = x$, for $x \in X$, $t \in \mathbb{R}$, and denoting $\sigma_{x_0}(t) = j^k_{x_0}\gamma(x,t)$ the curve lying entirely within the fiber $(J^k\pi)_{x_0}$, we can define a π_k-vertical vector $\dot{\sigma}_{x_0}(t)$. The izomorphism identifies this vector with the k-jet $j^k_{x_0}(\dot{\gamma}(x,t))$. In coordinates, (4.1) is represented by a rearrangement

$$\begin{aligned}(x^i, y^\sigma, \dot{y}^\sigma, y^\sigma_j, \dot{y}^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, \dot{y}^\sigma_{j_1\dots j_k}) &\stackrel{\nu_k}{\longmapsto} \\ &\stackrel{\nu_k}{\longmapsto} (x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, \dot{y}^\sigma, \dot{y}^\sigma_j, \dots, \dot{y}^\sigma_{j_1\dots j_k}).\end{aligned}$$

For any π-vertical vector field $\zeta \in \mathcal{X}^v_X(Y)$ can be then its *k-th jet prolongation* $\mathcal{J}^k\zeta \in \mathcal{X}^v_X(J^k\pi)$ defined by

$$\mathcal{J}^k\zeta = \nu_k \circ J^k(\zeta, \mathrm{id}_X)\colon J^k\pi \to V_{\pi_k}J^k\pi, \tag{4.2}$$

i.e. locally by

$$\mathcal{J}^k\zeta = \zeta^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{\ell=1}^{k} D_{j_1\dots j_\ell}(\zeta^\sigma)\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}. \tag{4.3}$$

In terms of commutative diagrams:

$$\begin{array}{ccccccc}
X & \xleftarrow{\pi\circ\tau_Y} & V_\pi Y & \xleftarrow{(\pi\circ\tau_Y)_{k,0}} & J^k(\pi\circ\tau_Y) & \xrightarrow{\nu_k} & V_{\pi_k}J^k\pi \\
{\scriptstyle \mathrm{id}_X}\uparrow & & \uparrow{\scriptstyle \zeta} & & \uparrow{\scriptstyle J^k(\zeta,\mathrm{id}_X)} & & \uparrow{\scriptstyle \mathcal{J}^k\zeta} \\
X & \xleftarrow{\pi} & Y & \xleftarrow{\pi_{k,0}} & J^k\pi & = & J^k\pi\,.
\end{array}$$

The flow $\{\alpha^{\mathcal{J}^k\zeta}_t\}$ is the prolongation

$$\alpha^{\mathcal{J}^k\zeta}_t = J^k(\alpha^\zeta_t, \mathrm{id}_X), \tag{4.4}$$

which gives a way of defining the prolongation for π-projectable vector fields.

Let $\zeta \in \mathcal{X}_X(Y)$ has the projection $\zeta_0 \in \mathcal{X}(X)$. The *k-th jet prolongation* of ζ is a vector field $\mathcal{J}^k\zeta \in \mathcal{X}_Y(J^k\pi)$ whose flow is the prolongation

$$\alpha^{\mathcal{J}^k\zeta}_t = J^k(\alpha^\zeta_t, \alpha^{\zeta_0}_t), \tag{4.5}$$

which means

$$\mathcal{J}^k\zeta(j^k_x\gamma) = \left\{\frac{d}{dt}J^k(\alpha^\zeta_t, \alpha^{\zeta_0}_t)(j^k_x\gamma)\right\}_{t=0}.$$

For a construction of the k-th prolongation of an *arbitrary* $\zeta \in \mathcal{X}(Y)$ one needs a relationship between $J^k(\pi \circ \tau_Y)$ and $TJ^k\pi$. There is a surjective bundle morphism

$$\mu_k \colon J^k(\pi \circ \tau_Y) \to TJ^k\pi,$$

defined by

$$\mu_k(j^k_x\xi) = \nu_k(j^k_x(\xi - T\gamma \circ T\pi \circ \xi)) + Tj^k\gamma(T\pi \circ \xi(x))$$

for $\gamma = \tau_Y \circ \xi$. The *$k$-th jet prolongation* of $\zeta \in \mathcal{X}(Y)$ is then

$$\mathcal{J}^k\zeta(j^k_x\gamma) = \mu_k(j^k_x(\zeta \circ \gamma)). \tag{4.6}$$

The local expression is

$$\mathcal{J}^k\zeta = \zeta^i\frac{\partial}{\partial x^i} + \zeta^\sigma\frac{\partial}{\partial y^\sigma} + \sum_{\ell=1}^{k}\zeta^\sigma_{j_1\ldots j_\ell}\frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}}, \tag{4.7}$$

where the components $\zeta^\sigma_{j_1\ldots j_\ell}$ are defined by

$$\zeta^\sigma_{j_1\ldots j_\ell} = D_{j_1\ldots j_\ell}(\zeta^\sigma) - D_{j_1\ldots j_\ell}(\zeta^i)y^\sigma_i = D_{j_1\ldots j_\ell}(\zeta^\sigma - y^\sigma_i\zeta^i) + \zeta^i y^\sigma_{j_1\ldots j_\ell i}. \tag{4.8}$$

5. The contact structure and Cartan distribution

Dually to (3.2), there is a decomposition

$$\begin{aligned}(\pi^*_{k+1,k}(T^*J^k\pi), \pi^*_{k+1,k}(\tau^*_{J^k\pi}), J^{k+1}\pi) \cong \\ \cong (\pi^*_{k+1,k}(\pi^*_k(T^*X)) \oplus C^*_{\pi_{k+1,k}}, \pi^*_{k+1,k}(\tau^*_{J^k\pi}), J^{k+1}\pi),\end{aligned} \tag{5.1}$$

where

$$C^*_{\pi_{k+1,k}} := \bigcup_x \pi^*_{k+1,k}(\ker(j^k\gamma)^*)$$

is the subbundle of the elements $(j^{k+1}_x\gamma, \eta) \in \pi^*_{k+1,k}(T^*J^k\pi) \subset T^*J^{k+1}\pi$ satisfying $(j^k\gamma)^*(\eta) = 0$. According to (5.1), there is a decomposition

$$\Omega^1(\pi_{k+1,k}) = \Omega^1(\pi_{k+1}) \oplus \Omega^1_c(\pi_{k+1,k}) \tag{5.2}$$

of the module of $\pi_{k+1,k}$-horizontal 1-forms on $J^{k+1}\pi$ (in other words, 1-forms along $\pi_{k+1,k}$) to the submodule

$$\Omega^1(\pi_{k+1}) = \mathcal{S}(\pi^*_{k+1}(\tau^*_X))$$

of π_{k+1}-horizontal forms together with the submodule

$$\Omega^1_c(\pi_{k+1,k}) = \mathcal{S}(\pi^*_{k+1,k}(\tau^*_{J^k\pi})|_{C^*_{\pi_{k+1,k}}})$$

the sections of which are the *contact* 1*-forms* on $J^{k+1}\pi$. A contact 1-form on $J^{k+1}\pi$ is written in coordinates as

$$\eta = \sum_{\ell=0}^{k} \eta^{j_1\ldots j_\ell}_\sigma \left(dy^\sigma_{j_1\ldots j_\ell} - y^\sigma_{j_1\ldots j_\ell i}\, dx^i\right) \tag{5.3}$$

with the canonical generators

$$\omega^\sigma_{j_1\ldots j_\ell} = dy^\sigma_{j_1\ldots j_\ell} - y^\sigma_{j_1\ldots j_\ell i}\, dx^i.$$

The role of contact forms in the higher-order situation is analogous to the first-order one: if $\psi \in \mathcal{S}_U(\pi_{k+1})$, then $\psi = j^{k+1}\gamma$ for some $\gamma \in \mathcal{S}_U(\pi)$ if, and only if,

$$\psi^*(\omega^\sigma_{j_1\ldots j_\ell}) = 0 \tag{5.4}$$

for $\sigma = 1,\ldots,m$, $1 \le j_1 \le \cdots \le j_\ell \le n$ and $\ell = 0,\ldots,k$. On the other hand, η is contact if, and only if, $(j^{k+1}\gamma)^*\eta = 0$ for all $\gamma \in \mathcal{S}_{\text{loc}}(\pi)$.

The decompositions (3.2) and (5.1) are canonically related by the *contact structure* on π_{k+1}, which consists of two complementary vector-valued 1-forms

$$h, v \in \mathcal{X}(\pi_{k+1,k}) \otimes \Omega^1(\pi_{k+1,k}).$$

In coordinates,

$$h = D^{k+1,k}_i \otimes dx^i \in \mathcal{X}^h(\pi_{k+1,k}) \otimes \Omega^1(\pi_{k+1}),$$
$$v = \sum_{\ell=0}^{k} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}} \otimes \omega^\sigma_{j_1\ldots j_\ell} \in \mathcal{X}^v(\pi_{k+1,k}) \otimes \Omega^1_c(\pi_{k+1,k}).$$

The *Cartan distribution* on $J^{k+1}\pi$ is then defined by

$$C_{\pi_{k+1,k}} = \ker[v \circ (\tau_{J^{k+1}\pi}, T\pi_{k+1,k})] \subset TJ^{k+1}\pi, \tag{5.5}$$

it is annihilated by $C^*_{\pi_{k+1,k}}$ and consequently the image of a section of π_{k+1} is an integral submanifold of $C_{\pi_{k+1,k}}$ if, and only if, it is the $(k+1)$-th prolongation of a section of π. For each $j^{k+1}_x\gamma$ one has

$$C_{\pi_{k+1,k}}|_{j^{k+1}_x\gamma} = Tj^{k+1}\gamma(T_xX) \oplus V_{\pi_{k+1,k}}J^{k+1}\pi|_{j^{k+1}_x\gamma}.$$

6. Repeated jets

Since $\pi_k\colon J^k\pi \to X$ is a fibered manifold, one can consider the so-called *repeated jets*. By $J^r\pi_k$ we denote the r-jet manifold of π_k, i.e.

$$J^r\pi_k = \{j^r_x\psi; x \in X, \psi \in \mathcal{S}_{\mathrm{loc}}(\pi_k), x \in \mathrm{Dom}(\psi)\}.$$

The induced coordinates on $J^r\pi_k$ will be denoted by

$$\begin{aligned} & x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, \\ & y^\sigma_{;i}, \dots, y^\sigma_{;i_1\dots i_r}, \\ & y^\sigma_{j;i}, \dots, y^\sigma_{j;i_1\dots i_r}, \\ & \vdots \\ & y^\sigma_{j_1\dots j_k;i}, \dots, y^\sigma_{j_1\dots j_k;i_1\dots i_r}. \end{aligned} \tag{6.1}$$

A distinguished subset in $J^r\pi_k$ is formed by the prolongations of those sections which are themselves the prolongations; it means of $\psi \in \mathcal{S}(\pi_k)$ such that $\psi = j^k\gamma$, where $\gamma \in \mathcal{S}(\pi)$. It can be shown that the map

$$\iota_{r,k}\colon J^{k+r}\pi \to J^r\pi_k,$$

defined canonically for each $r, k \geq 0$ by

$$\iota_{r,k}(j^{k+r}_x\gamma) = j^r_x(j^k\gamma)$$

is an embedding, hence $J^{k+r}\pi$ will be frequently identified with its image

$$\iota_{r,k}(J^{k+r}\pi) \subset J^r\pi_k.$$

Let us restrict attention here to the particular case $r = 1$ and $k \geq 1$, which will prove to be of importance in the study of the so-called *semiholonomic jets*.

Consider thus the space $J^1\pi_k$ together with a pair of projections to $J^1\pi_{k-1}$. First, the projection $(\pi_k)_{1,0}\colon J^1\pi_k \to J^k\pi$ may be composed with $\iota_{1,k-1}$ to get the projection

$$\iota_{1,k-1} \circ (\pi_k)_{1,0}\colon J^1\pi_k \to J^1\pi_{k-1}, \tag{6.2}$$

which in coordinates reads

$$\begin{aligned} & (x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{;i}, y^\sigma_{j;i}, \dots, y^\sigma_{j_1\dots j_k;i}) \mapsto \\ & \qquad \mapsto (x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_{k-1}}, y^\sigma_i, y^\sigma_{ji}, \dots, y^\sigma_{j_1\dots j_{k-1}i}). \end{aligned} \tag{6.3}$$

Secondly, the projection

$$\pi_{k,k-1}\colon J^k\pi \to J^{k-1}\pi$$

may be prolonged as a fibered morphism between π_k and π_{k-1} over X:

$$J^1(\pi_{k,k-1},\mathrm{id}_X)\colon J^1\pi_k \to J^1\pi_{k-1}. \tag{6.4}$$

Locally,

$$\begin{aligned}(x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_k}, y^\sigma_{;i}, y^\sigma_{j;i}, \dots, y^\sigma_{j_1\dots j_k;i}) \mapsto \\ \mapsto (x^i, y^\sigma, y^\sigma_j, \dots, y^\sigma_{j_1\dots j_{k-1}}, y^\sigma_{;i}, y^\sigma_{j;i}, \dots, y^\sigma_{j_1\dots j_{k-1};i}).\end{aligned} \tag{6.5}$$

The point is that for a given $j^1_x\psi \in J^1\pi_k$, both projections lie within the same fiber with respect to $(\pi_k)_{1,0}$, which enables to define the space $\widehat{J}^{k+1}\pi$ of *semiholonomic* $(k+1)$*-jets* as the subset in $J^1\pi_k$ on which both projections (6.2) and (6.4) coincide, i.e.

$$\widehat{J}^{k+1}\pi = \left\{ j^1_x\psi \in J^1\pi_k; J^1(\pi_{k,k-1},\mathrm{id}_X)(j^1_x\psi) = \iota_{1,k-1}\circ(\pi_k)_{1,0}(j^1_x\psi)\right\}.$$

Due to (6.3) and (6.5), the local equations for semiholonomic $(k+1)$-jets are

$$y^\sigma_{;i} = y^\sigma_i, y^\sigma_{j;i} = y^\sigma_{ji}, \dots, y^\sigma_{j_1\dots j_{k-1};i} = y^\sigma_{j_1\dots j_{k-1}i}. \tag{6.6}$$

It can be shown that $\widehat{J}^{k+1}\pi$ is a submanifold of $J^1\pi_k$ which can be defined as the kernel of the *k-jet Spencer operator*

$$\mathrm{Sp}_k\colon J^1\pi_k \to V_{\pi_{k-1}}J^{k-1}\pi \otimes \pi^*_{k-1}(T^*X),$$

defined by the requirement on $\mathrm{Sp}_k(j^1_x\psi)$ to be just the element (of the total space of the vector bundle associated to $(\pi_{k-1})_{1,0}$) such that

$$J^1(\pi_{k,k-1},\mathrm{id}_X)(j^1_x\psi) + \mathrm{Sp}_k(j^1_x\psi) = \iota_{1,k-1}\circ(\pi_k)_{1,0}(j^1_x\psi)$$

with respect to the affine structure.

Assembling the various facts we can see that there are the inclusions

$$J^{k+1}\pi \subset \widehat{J}^{k+1}\pi \subset J^1\pi_k,$$

where

$$\widehat{\pi}_{k+1,k} := (\pi_k)_{1,0}\colon J^1\pi_k \supset \widehat{J}^{k+1}\pi \to J^k\pi$$

is an affine subbundle of $(\pi_k)_{1,0}$ with the associated vector bundle (over $J^k\pi$) whose total space is

$$\pi^*_{k,0}(V_\pi Y) \otimes \pi^*_k \left(S^kT^*X \otimes T^*X\right) \cong V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k(T^*X) \subset V_{\pi_k} J^k\pi \otimes \pi^*_k(T^*X) \tag{6.7}$$

This express the fact that while on $J^{k+1}\pi$ all derivative coordinates are totally symmetric, those on $\widehat{J}^{k+1}\pi$ are totally symmetric except for the highest-order ones. In this respect, the elements of $\iota_{1,k}(J^{k+1}\pi) \subset J^1\pi_k$ are called $(k+1)$-*holonomic jets*.

In what follows, the subspace $T^*M \otimes S^kT^*M \subset \otimes^{k+1}T^*M$ will become of importance. Let us denote by s, a and A the *symmetrization, antisymmetrization* and *asymmetrization* linear projectors $s\colon \otimes^k T^*M \to S^kT^*M$, $a\colon \otimes^k T^*M \to \Lambda^kT^*M$, $A = \mathrm{id} - s\colon \otimes^k T^*M \to A^kT^*M$. In particular, for $k = 2$ one gets $A \equiv a$, i.e. $\otimes^2T^*M = S^2T^*M \oplus \Lambda^2T^*M$, and in general

$$\otimes^kT^*M = S^kT^*M \oplus A^kT^*M$$

with $\Lambda^kT^*M \subset A^kT^*M = \ker s(T^*M)$. Denoting $\lozenge^2_{k-1}T^*M = A(T^*M \otimes S^kT^*M)$, one has

$$T^*M \otimes S^kT^*M = S^{k+1}T^*M \oplus \lozenge^2_{k-1}T^*M. \tag{6.8}$$

Moreover, $\lozenge^2_{k-1}T^*M$ can be identified with its image in $\Lambda^2T^*M \otimes S^{k-1}T^*M$ by the vector bundle morphism

$$\delta\colon T^*M \otimes S^kT^*M \to \Lambda^2T^*M \otimes S^{k-1}T^*M$$

defined by $\delta(\omega \otimes u) = (-1)^k \omega \wedge \delta_k u$, where $\delta_k\colon S^kT^*M \to T^*M \otimes S^{k-1}T^*M$ is the composition

$$S^kT^*M \hookrightarrow \otimes^kT^*M = T^*M \otimes \otimes^{k-1}T^*M \overset{\mathrm{id}\times s}{\longrightarrow} T^*M \otimes S^{k-1}T^*M.$$

Due to (6.7) and (2.2), one gets a canonical splitting of the affine bundle $\widehat{\pi}_{k+1,k}$, expressed in terms of the total spaces by

$$\widehat{J}^{k+1}\pi \cong J^{k+1}\pi \times_{J^k\pi} \pi^*_{k,0}(V_\pi Y \otimes \pi^*(\lozenge^2_{k-1}T^*X)), \tag{6.9}$$

which gives rise to natural projections

$$\begin{aligned} s_k &\colon \widehat{J}^{k+1}\pi \to J^{k+1}\pi, \\ r_k &\colon \widehat{J}^{k+1} \to \pi^*_{k,0}(V_\pi Y \otimes \pi^*(\lozenge^2_{k-1}T^*X)), \end{aligned} \tag{6.10}$$

expressing the totally symmetric or asymmetric part of every highest-order derivative coordinate $y^\sigma_{j_1\ldots j_k;\,i}$, respectively. In particular, for $k = 1$ (6.9) reads

$$\widehat{J}^2\pi \cong J^2\pi \times_{J^1\pi} \pi^*_{1,0}(V_\pi Y \otimes \pi^*(\Lambda^2 T^*X)). \tag{6.11}$$

Notice finally the relationship between higher-order and repeated prolongations of a fibered morphism $\Phi\colon Y \to Y'$ over X, which become interesting for prolongations of connections:

$$J^r\left(J^k(\Phi, \mathrm{id}_X), \mathrm{id}_X\right)|_{J^{k+r}\pi} \equiv J^{k+r}(\Phi, \mathrm{id}_X). \tag{6.12}$$

7. Connections on $\pi_k : J^k\pi \to X$

Let us briefly put a list of concepts representing a (global) connection Σ on π_k:

- a section
$$\Sigma\colon J^k\pi \to J^1\pi_k$$
of $(\pi_k)_{1,0}$, which in coordinates reads as
- the system
$$y^\sigma_{;\,i} = \Sigma^\sigma_{;\,i}, \ldots, y^\sigma_{j_1\ldots j_k;\,i} = \Sigma^\sigma_{j_1\ldots j_k;\,i}$$
with
- the family of $\Sigma^\sigma_{;\,i}, \ldots, \Sigma^\sigma_{j_1\ldots j_k;\,i} \in \mathcal{F}(J^k\pi)$, being transformed like the coordinates $y^\sigma_{;\,i}, \ldots, y^\sigma_{j_1\ldots j_k;\,i}$;
- the *horizontal form*
$$h_\Sigma\colon J^k\pi \to TJ^k\pi \otimes \pi^*_k(T^*X), \quad h_\Sigma = D_{\Sigma i} \otimes dx^i,$$
of Σ, where
- $$D_{\Sigma i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k} \Sigma^\sigma_{j_1\ldots j_\ell;\,i} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}}$$
are the generators of
- the n-dimensional π_k-*horizontal distribution* $H_\Sigma = \mathrm{Im} h_\Sigma$, representing

- a splitting of the exact sequence

$$0 \longrightarrow V_{\pi_k} J^k\pi \longrightarrow TJ^k\pi \longrightarrow \pi_k^*(TX) \longrightarrow 0$$

and thus

- the direct sum decomposition $TJ^k\pi = H_\Sigma \oplus V_{\pi_k} J^k\pi$.

Recall also the *curvature* of Σ, which is

$$R_\Sigma = \frac{1}{2}[h_\Sigma, h_\Sigma] \colon J^k\pi \to V_{\pi_k} J^k\pi \otimes \pi_k^*(\Lambda^2 T^* X).$$

locally expressed by

$$R_\Sigma = \sum_{\ell=0}^{k} D_{\Sigma i}\left(\Sigma^\sigma_{j_1\ldots j_\ell;\,p}\right) \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}} \otimes dx^i \wedge dx^p. \tag{7.1}$$

Due to $\widehat{J}^{k+1}\pi \subset J^1\pi_k$, a *semiholonomic connection* on π_k (for $k \geq 1$) is:

- a section
$$\widehat{\Gamma}^{(k+1)} \colon J^k\pi \to \widehat{J}^{k+1}\pi$$
of $\widehat{\pi}_{k+1,k}$, which in coordinates reads as
- the system
$$y^\sigma_{;i} = y^\sigma_i, \ldots, y^\sigma_{j_1\ldots j_{k-1};\,i} = y^\sigma_{j_1\ldots j_{k-1} i}, y^\sigma_{j_1\ldots j_k;\,i} = \widehat{\Gamma}^\sigma_{j_1\ldots j_k;\,i} \tag{7.2}$$
with
- the family of (generally nonsymmetric in the highest derivatives) $\widehat{\Gamma}^\sigma_{j_1\ldots j_k;\,i} \in \mathcal{F}(J^k\pi)$, being transformed like $y^\sigma_{j_1\ldots j_k;\,i}$;
- the *horizontal form*
$$h_{\widehat{\Gamma}^{(k+1)}} \colon J^k\pi \to TJ^k\pi \otimes \pi_k^*(T^*X), \quad h_{\widehat{\Gamma}^{(k+1)}} = D_{\widehat{\Gamma}^{(k+1)}i} \otimes dx^i.$$
where
- $$D_{\widehat{\Gamma}^{(k+1)}i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\ldots j_\ell i} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}} + \widehat{\Gamma}^\sigma_{j_1\ldots j_k;\,i} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} \tag{7.3}$$
are the generators of

- the n-dimensional π_k-*horizontal distribution* $H_{\widehat{\Gamma}^{(k+1)}} = \mathrm{Im} h_{\widehat{\Gamma}^{(k+1)}}$, creating again
- the direct sum decomposition $TJ^k\pi = H_{\widehat{\Gamma}^{(k+1)}} \oplus V_{\pi_k} J^k\pi$.

The most important difference between a semiholonomic connection $\widehat{\Gamma}^{(k+1)}$ and a general connection Σ on π_k is that evidently

$$H_{\widehat{\Gamma}^{(k+1)}} \subset C_{\pi_{k,k-1}}$$

which become interesting when dealing with such connections as representants of equations. In this respect, by (7.1-3) we have (set $\pi_{k,k-2} \equiv \pi_k$ for $k=1$)

$$R_{\widehat{\Gamma}^{(k+1)}} \colon J^k\pi \to V_{\pi_{k,k-2}} J^k\pi \otimes \pi_k^*(\Lambda^2 T^* X)$$

with local expression

$$\begin{aligned} R_{\widehat{\Gamma}^{(k+1)}} = {} & \widehat{\Gamma}^\sigma_{j_1\dots j_{k-1}p;i} \frac{\partial}{\partial y^\sigma_{j_1\dots j_{k-1}}} \otimes dx^i \wedge dx^p \\ & + D_{\widehat{\Gamma}^{(k+1)}i} \left(\widehat{\Gamma}^\sigma_{j_1\dots j_k;p} \right) \frac{\partial}{\partial y^\sigma_{j_1\dots j_k}} \otimes dx^i \wedge dx^p. \end{aligned} \tag{7.4}$$

8. A $(k+1)$-connection on a fibered manifold

Of all types of connections on π_k, the so-called $(k+1)$-*holonomic connections* on π_k or more briefly $(k+1)$-*connections* on π are intrinsically related to the theory of higher-order equations. Due to $J^{k+1}\pi \hookrightarrow \widehat{J}^{k+1}\pi \subset J^1\pi_k$, a $(k+1)$-*connection* on π (for $k \geq 1$) is a section (both global and local versions can appear)

$$\Gamma^{(k+1)} \colon J^k\pi \to J^{k+1}\pi$$

of $\pi_{k+1,k}$, with local expression

$$y^\sigma_{j_1\dots j_{k+1}} = \Gamma^\sigma_{j_1\dots j_{k+1}}, \tag{8.1}$$

where the (totally symmetric) functions $\Gamma^\sigma_{j_1\dots j_{k+1}} \in \mathcal{F}(J^k\pi)$ are the *components* of $\Gamma^{(k+1)}$, being transformed like $y^\sigma_{j_1\dots j_{k+1}}$. The *horizontal form* of $\Gamma^{(k+1)}$ is the π_k-projectable (onto identity) vector-valued 1-form

$$h_{\Gamma^{(k+1)}} \colon J^k\pi \to TJ^k\pi \otimes \pi_k^*(T^*X)$$

defined in virtue of the contact structure on π_{k+1} by

$$h_{\Gamma^{(k+1)}}(\zeta) = \mathrm{pr}_2 \left[h \left(\Gamma^{(k+1)}(j^k_x\gamma), \zeta \right) \right] \tag{8.2}$$

for each $\zeta \in T_{j^k_x\gamma} J^k\pi$. Locally,

$$h_{\Gamma^{(k+1)}} = D_{\Gamma^{(k+1)}i} \otimes dx^i, \tag{8.3}$$

where

$$\begin{aligned} D_{\Gamma^{(k+1)}i} &= D^{k+1,k}_i \circ \Gamma^{(k+1)} \\ &= \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\ldots j_\ell i} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}} + \Gamma^\sigma_{j_1\ldots j_k i} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} \end{aligned} \tag{8.4}$$

is the *i-th absolute derivative* with respect to $\Gamma^{(k+1)}$. The *vertical form*

$$v_{\Gamma^{(k+1)}} = I_{TJ^k\pi} - h_{\Gamma^{(k+1)}} : J^k\pi \to V_{\pi_k} J^k\pi \otimes T^* J^k\pi,$$

is defined by

$$v_{\Gamma^{(k+1)}}(\zeta) = \mathrm{pr}_2 \left[v \left(\Gamma^{(k+1)}(j^k_x\gamma), \zeta \right) \right]$$

and it is locally expressed by

$$v_{\Gamma^{(k+1)}} = \sum_{\ell=0}^{k-1} \frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}} \otimes \omega^\sigma_{j_1\ldots j_\ell} + \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} \otimes \omega^{\Gamma^{(k+1)}\sigma}_{j_1\ldots j_k},$$

where

$$\omega^{\Gamma^{(k+1)}\sigma}_{j_1\ldots j_k} = dy^\sigma_{j_1\ldots j_k} - \Gamma^\sigma_{j_1\ldots j_k i}\, dx^i = \omega^\sigma_{j_1\ldots j_k} \circ \Gamma^{(k+1)}.$$

The direct sum decomposition generated by $h_{\Gamma^{(k+1)}}$ and $v_{\Gamma^{(k+1)}}$ is

$$TJ^k\pi = H_{\Gamma^{(k+1)}} \oplus V_{\pi_k} J^k\pi$$

with $H_{\Gamma^{(k+1)}} = \mathrm{Im} h_{\Gamma^{(k+1)}} = \ker v_{\Gamma^{(k+1)}}$ spanned equivalently by the vector fields $D_{\Gamma^{(k+1)}i}$ or by the forms $\omega^\sigma_{j_1\ldots j_\ell}$, $\omega^{\Gamma^{(k+1)}\sigma}_{j_1\ldots j_k}$, which means that again

$$H_{\Gamma^{(k+1)}} \subset C_{\pi_{k,k-1}}.$$

The corresponding *horizontal lift* $\mathcal{X}(\pi_k) \to \mathcal{X}(J^k\pi)$ defined by

$$\pi^*_k(TX) \overset{\Gamma^{(k+1)}\times \mathrm{id}}{\longrightarrow} \pi^*_{k+1}(TX) \overset{h}{\to} \pi^*_{k+1,k}(TJ^k\pi) \overset{\mathrm{pr}_2}{\longrightarrow} TJ^k\pi$$

locally reads

$$\xi^i \frac{\partial}{\partial x^i}|_{\pi_k(j^k_x\gamma)} \longmapsto \xi^i D_{\Gamma^{(k+1)}i}|_{j^k_x\gamma}.$$

The *affine translation* $\nabla_{\Gamma^{(k+1)}}$ generated by $\Gamma^{(k+1)}$ is defined by (see also (6.7))

$$\begin{aligned} J^{k+1}\pi \overset{\mathrm{id}\times\pi_{k+1,k}}{\longrightarrow} J^{k+1}\pi \times_{J^k\pi} J^k\pi \overset{\mathrm{id}\times\Gamma^{(k+1)}}{\longrightarrow} J^{k+1}\pi \times_{J^k\pi} J^{k+1}\pi \to \\ \pi^*_{k,0}(V_\pi Y) \otimes \pi^*_k(S^{k+1}T^*X) \subset \pi^*_{k,0}(V_\pi Y) \otimes \Big(\pi^*_k(S^kT^*X \otimes T^*X)\Big) \\ \cong V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k(T^*X), \end{aligned}$$

and locally

$$\nabla_{\Gamma^{(k+1)}} = \left(y^\sigma_{j_1\ldots j_k i} - \Gamma^\sigma_{j_1\ldots j_k i}\right) \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} \otimes dx^i.$$

The *curvature* of $\Gamma^{(k+1)}$ can be defined by

$$R_{\Gamma^{(k+1)}} = \frac{1}{2}[h_{\Gamma^{(k+1)}}, h_{\Gamma^{(k+1)}}]$$

(or equivalently by means of the *formal curvature map* R as we will present in Sec. 13). In view of (7.1) or (7.4) one gets

$$R_{\Gamma^{(k+1)}} = D_{\Gamma^{(k+1)}i}\left(\Gamma^\sigma_{j_1\ldots j_k p}\right) \frac{\partial}{\partial y^\sigma_{j_1\ldots j_k}} \otimes dx^i \wedge dx^p. \tag{8.5}$$

Finally, two facts should be noticed: first, (cf. Sec. 6) that there is a canonical splitting of each semiholonomic connection $\widehat{\Gamma}^{(k+1)}$ on π_k; namely,

$$\widehat{\Gamma}^{(k+1)} = \widehat{\Gamma}^{(k+1)}_s \times_{J^k\pi} \widehat{\Gamma}^{(k+1)}_r$$

with

$$\begin{aligned} \widehat{\Gamma}^{(k+1)}_s &= s_k \circ \widehat{\Gamma}^{(k+1)} \colon J^k\pi \to J^{k+1}\pi. \\ \widehat{\Gamma}^{(k+1)}_r &= r_k \circ \widehat{\Gamma}^{(k+1)} \colon J^k\pi \to \pi^*_{k,0}\left(V_\pi Y \otimes \pi^*(\Diamond^2_{k-1}T^*X)\right), \end{aligned} \tag{8.6}$$

Clearly, $\widehat{\Gamma}^{(k+1)}_s$ is holonomic, and $\widehat{\Gamma}^{(k+1)}$ is holonomic if, and only if, $\widehat{\Gamma}^{(k+1)}_r$ vanishes. This can be confronted with the very end of Sec. 7. Actually, one has

$$\widehat{\Gamma}^{(k+1)}_r \colon J^k\pi \to \pi^*_{k,k-1}(V_{\pi_{k-1,k-2}} J^{k-1}\pi) \otimes \pi^*_k(\Lambda^2 T^*X).$$

and it is just the canonical projection of $R_{\widehat{\Gamma}^{(k+1)}}$. In other words (see also (7.4) and (8.5)), $\widehat{\Gamma}^{(k+1)}$ is flat if, and only if, it is both holonomic and integrable.

Secondly, the differences of connections on π_k are the *soldering forms* on π_k, and according to Sec. 7, the deformations of semiholonomic connections on π_k are the sections of

$$J^k\pi \to V_{\pi_{k,k-1}} J^k\pi \otimes \pi_k^*(T^*X),$$

while those of $(k+1)$-connections on π are the sections of

$$J^k\pi \to \pi_{k,0}^*(V_\pi Y) \otimes \pi_k^*(S^{k+1}T^*X) \subset V_{\pi_{k,k-1}} J^k\pi \otimes \pi_k^*(T^*X).$$

9. Higher-order equations represented by connections

By a *k-th order differential equation* on a fibered manifold π is meant a fibered submanifold $\mathcal{E}^{(k)}$ of $\pi_k\colon J^k\pi \to X$ such that

$$\pi_{k,k-1}^{-1} \circ \pi_{k,k-1}\left(\mathcal{E}^{(k)}\right) \neq \mathcal{E}^{(k)}.$$

A *solution* of $\mathcal{E}^{(k)}$ is a section $\gamma \in \mathcal{S}_U(\pi)$ such that $j^k\gamma \subset \mathcal{E}^{(k)}$.

Also the higher-order equations are frequently defined by fibered morphisms. Thus if $\Phi\colon J^k\pi \to Y'$ is a fibered morphism of constant rank between π_k and π' over X, the corresponding *differential operator* is the mapping $\mathcal{D}_\Phi\colon \mathcal{S}_{\mathrm{loc}}(\pi) \to \mathcal{S}_{\mathrm{loc}}(\pi')$ defined by $\mathcal{D}_\Phi(\gamma)(x) = (\Phi \circ j^k\gamma)(x)$, and for any $\psi \in \mathcal{S}_U(\pi')$ satisfying $\psi(U) \subset \mathrm{Im}\Phi$, the k-th order differential equation determined by Φ and ψ is

$$\mathcal{E}^{(k)}_{\Phi,\psi} = \ker_\psi \Phi = \{j^k_x\gamma; \Phi(j^k_x\gamma) = \psi(x)\} \subset J^k\pi. \tag{9.1}$$

Accordingly, a solution of $\mathcal{E}^{(k)}_{\Phi,\psi}$ is $\gamma \in \mathcal{S}_V(\pi)$ such that

$$\mathcal{D}_\Phi(\gamma) = \psi|_V. \tag{9.2}$$

which in coordinates means a system of P.D.E.

$$\Phi^\sigma\left(x^i, \gamma^\lambda(x^i), \frac{\partial\gamma^\lambda}{\partial x^j}(x^i), \ldots, \frac{\partial^k\gamma^\lambda}{\partial x^{j_1}\ldots\partial x^{j_k}}(x^i)\right) = \psi^\sigma(x^i), \tag{9.3}$$

where $\sigma = 1, \ldots, \dim\pi'$.

The *Cartan distribution* of the k-th order equation $\mathcal{E}^{(k)} \subset J^k\pi$ is the intersection

$$C^{\mathcal{E}^{(k)}} = C_{\pi_{k,k-1}} \cap T\mathcal{E}^{(k)}, \tag{9.4}$$

carrying the most important information on the equation.

The *equation of order* $(k+1)$ *represented by a* $(k+1)$*-connection* $\Gamma^{(k+1)}$ *on* π is the submanifold

$$\mathcal{E}^{\Gamma^{(k+1)}} = \Gamma^{(k+1)}(J^k\pi) \subset J^{k+1}\pi, \tag{9.5}$$

realizing (generally nonlinear) system of P.D.E. *in normal form*, i.e. explicitly solved with respect to the highest derivatives:

$$\frac{\partial^{k+1}\gamma^\sigma}{\partial x^{j_1}\dots\partial x^{j_{k+1}}} = \Gamma^\sigma_{j_1\dots j_{k+1}}\left(x^i,\gamma^\lambda,\dots,\frac{\partial^k\gamma^\lambda}{\partial x^{j_1}\dots\partial x^{j_k}}\right). \tag{9.6}$$

A section $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ is called the *integral section* (*path*) of $\Gamma^{(k+1)}$ if it is the solution of $\mathcal{E}^{\Gamma^{(k+1)}}$; i.e. if

$$j^{k+1}\gamma = \Gamma^{(k+1)} \circ j^k\gamma. \tag{9.7}$$

Evidently, using (9.1)

$$\mathcal{E}^{\Gamma^{(k+1)}} \equiv \mathcal{E}^{(k+1)}_{\nabla_{\Gamma^{(k+1)}},0}, \tag{9.8}$$

which corresponds to the characterization of integral sections as those $\gamma \in \mathcal{S}_{\mathrm{loc}}(\pi)$ whose *covariant derivative*

$$\nabla_{\Gamma^{(k+1)}}(\gamma) := \nabla_{\Gamma^{(k+1)}} \circ j^{k+1}\gamma \tag{9.9}$$

vanishes. On the other hand, a $(k+1)$-connection $\Gamma^{(k+1)}$ represents a *Pfaffian system*

$$\left.\begin{aligned} \omega^\sigma &= 0 \\ &\vdots \\ \omega^\sigma_{j_1\dots j_{k-1}} &= 0 \\ \omega^{\Gamma^{(k+1)}\sigma}_{j_1\dots j_k} &= 0 \end{aligned}\right\} \equiv \left\{\begin{aligned} dy^\sigma &= y^\sigma_i\,dx^i \\ &\vdots \\ dy^\sigma_{j_1\dots j_{k-1}} &= y^\sigma_{j_1\dots j_{k-1}i}\,dx^i \\ dy^\sigma_{j_1\dots j_k} &= \Gamma^\sigma_{j_1\dots j_k i}\,dx^i, \end{aligned}\right. \tag{9.10}$$

hence $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^k\gamma(U)$ is an integral manifold of $H_{\Gamma^{(k+1)}}$, i.e. for each $x \in U$ it holds

$$T_x j^k\gamma(T_xU) \subset H_{\Gamma^{(k+1)}}(j^k_x\gamma). \tag{9.11}$$

In terms of $h_{\Gamma^{(k+1)}}$, (9.11) reads

$$h_{\Gamma^{(k+1)}}|_{j^k\gamma} \equiv Tj^k\gamma \circ T\pi_k \colon T_{j^k_x\gamma}J^k\pi \to T_{j^k_x\gamma}J^k\pi. \tag{9.12}$$

The integrability conditions list is not much surprising. A $(k+1)$-connection $\Gamma^{(k+1)}$ on π is *integrable* if, and only if, one of the following equivalent conditions holds:

- For an arbitrary $y \in Y$, there is a unique integral section of $\Gamma^{(k+1)}$ passing through it.
- The horizontal distribution $H_{\Gamma^{(k+1)}}$ is completely integrable.
- $[D_{\Gamma^{(k+1)}i}, D_{\Gamma^{(k+1)}p}] = 0$ for all i, p.
- The connection $\Gamma^{(k+1)}$ is *flat*, i.e. $R_{\Gamma^{(k+1)}} = 0$.
- $J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)} \in J^{k+2}\pi$.
- The components of $\Gamma^{(k+1)}$ satisfy

$$D_{\Gamma^{(k+1)}i}\left(\Gamma^\sigma_{j_1\dots j_k p}\right) = D_{\Gamma^{(k+1)}p}\left(\Gamma^\sigma_{j_1\dots j_k i}\right) \tag{9.13}$$

  for arbitrary $i, p = 1, \dots, n$.

Denote by

$$C^{\Gamma^{(k+1)}} := C_{\pi_{k+1,k}} \cap T\Gamma^{(k+1)}(J^k\pi) \tag{9.14}$$

the Cartan distribution of the equation represented by $\Gamma^{(k+1)}$. Clearly, it is a regular n-dimensional distribution on the submanifold $\Gamma^{(k+1)}(J^k\pi) \subset J^{k+1}\pi$, annihilated by the forms $\omega^\sigma_{j_1\dots j_\ell}$ ($\ell = 0, \dots, k-1$) together with $dy^\sigma_{j_1\dots j_k} - \Gamma^\sigma_{j_1\dots j_k i}\,dx^i$ and $dy^\sigma_{j_1\dots j_{k+1}} - d\Gamma^\sigma_{j_1\dots j_{k+1}}$, or equivalently spanned by the vector fields

$$\begin{aligned} T\Gamma^{(k+1)}\left(D_{\Gamma^{(k+1)}i}\right) = \frac{\partial}{\partial x^i} &+ \sum_{\ell=0}^{k-1} y^\sigma_{j_1\dots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} \\ &+ \Gamma^\sigma_{j_1\dots j_k i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_k}} + D_{\Gamma^{(k+1)}p}(\Gamma^\sigma_{j_1\dots j_k i})\frac{\partial}{\partial y^\sigma_{j_1\dots j_k p}}. \end{aligned} \tag{9.15}$$

LEMMA 9.1. *A $(k+1)$-connection $\Gamma^{(k+1)}$ on π is integrable if, and only if, the distribution $C^{\Gamma^{(k+1)}}$ is completely integrable, and a section γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^{k+1}\gamma$ is the integral mapping of $C^{\Gamma^{(k+1)}}$.*

Proof. The first part of the assertion is an immediate consequence of the $\Gamma^{(k+1)}$-compatibility of the distributions $H_{\Gamma^{(k+1)}}$ and $C^{\Gamma^{(k+1)}}$:

$$[T\Gamma^{(k+1)}(D_{\Gamma^{(k+1)}i}), T\Gamma^{(k+1)}(D_{\Gamma^{(k+1)}p})] = T\Gamma^{(k+1)}\left[D_{\Gamma^{(k+1)}i}, D_{\Gamma^{(k+1)}p}\right] \ ;$$

the rest follows (9.7), (9.11) and (9.14). ∎

10. Prolongations and fields of paths

Let $\mathcal{E}^{(k)} \subset J^k\pi$ be a k-th order equation on π. The *r-th prolongation* of $\mathcal{E}^{(k)}$ is the subset

$$\mathcal{E}^{(k)(r)} = J^r\mathcal{E}^{(k)} \cap J^{k+r}\pi$$

with $J^r\mathcal{E}^{(k)} \subset J^r\pi_k$. For the equation $\mathcal{E}^{(k)}_{\Phi,\psi}$ defined by (9.1),

$$\mathcal{E}^{(k)(r)}_{\Phi,\psi} = \left\{ j^{k+r}_x\gamma ; J^r(\Phi, \mathrm{id}_X) \circ \iota_{r,k}(j^{k+r}_x\gamma) = j^r_x\psi \right\} \subset J^{k+r}\pi \tag{10.1}$$

is again a differential equation, now of the $(k+r)$-th order. It carries all the information on the initial $\mathcal{E}^{(k)}$, together with 'higher-order consequences' of it. In fact, $\mathcal{E}^{(k)(r)}_{\Phi,\psi}$ represents the family of P.D.E. obtained by differentiating the original equations $0, 1, \ldots, r$ times with respect to the independent variables.

As usually, we will scan the situation for connections. For a given k-connection, the r-th prolongation of $\mathcal{E}^{\Gamma^{(k)}}$ may be obtained following the idea of (10.1); nevertheless, the compatibility of such equations with the underlying structures allows us to present more suitable description.

Definition 10.1. Let $k \geq 0$ and $\Gamma^{(k+1)}\colon J^k\pi \to J^{k+1}\pi$ be an integrable $(k+1)$-connection on π. The *r-th prolongation* of the equation $\mathcal{E}^{\Gamma^{(k+1)}} \subset J^{k+1}\pi$ represented by $\Gamma^{(k+1)}$ is defined to be the submanifold

$$\mathcal{E}^{\Gamma^{(k+1)}(r)} = \mathrm{Im}\Gamma^{(k+1)(r)} \subset J^{k+r+1}\pi, \tag{10.2}$$

where $\Gamma^{(k+1)(r)}$ is the last term of the sequence of sections

$$\left(\Gamma^{(k+1)(0)}, \Gamma^{(k+1)(1)}, \ldots, \Gamma^{(k+1)(r)}\right)$$

recurrently defined for each $\ell = 1, \ldots, r$ by

$$\Gamma^{(k+1)(\ell)} := J^1(\Gamma^{(k+1)(\ell-1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}\colon J^k\pi \to J^{k+\ell+1}\pi \tag{10.3}$$

with $\Gamma^{(k+1)(0)} := \Gamma^{(k+1)}$.

The definition should be explained in more details. Note first that $\Gamma^{(k+1)(2)}$ defined by (10.3) can be rewritten to (ι's are omitted for the brevity sake)

$$\begin{aligned}\Gamma^{(k+1)(2)} &= J^1(J^1(\Gamma^{(k+1)},\mathrm{id}_X)\circ\Gamma^{(k+1)},\mathrm{id}_X)\circ\Gamma^{(k+1)}\\ &= J^1(J^1(\Gamma^{(k+1)},\mathrm{id}_X),\mathrm{id}_X)\circ J^1(\Gamma^{(k+1)},\mathrm{id}_X)\circ\Gamma^{(k+1)}\\ &= J^2(\Gamma^{(k+1)},\mathrm{id}_X)\circ J^1(\Gamma^{(k+1)},\mathrm{id}_X)\circ\Gamma^{(k+1)},\end{aligned}\tag{10.4}$$

where we have used the integrability of $\Gamma^{(k+1)}$, the inclusion $J^{k+2}\pi\hookrightarrow J^2\pi_k$ and (6.12). Now it is clear that the target space of $\Gamma^{(k+1)(2)}$ must be $J^{k+3}\pi$, and repeating the procedure one gets another sequence defining $\Gamma^{(k+1)(r)}$:

$$J^k\pi \xrightarrow{\Gamma^{(k+1)}} J^{k+1}\pi \xrightarrow{J^1(\Gamma^{(k+1)},\mathrm{id}_X)} J^{k+2}\pi \to \cdots J^{k+r}\pi \xrightarrow{J^r(\Gamma^{(k+1)},\mathrm{id}_X)} J^{k+r+1}\pi.$$

Evidently, *the equation $\mathcal{E}^{\Gamma^{(k+1)}(r)}$ consists of $(k+r+1)$-jets of integral sections of $\Gamma^{(k+1)}$*; in fact

$$\begin{aligned}j^{k+r+1}\gamma &= j^r(j^{k+1}\gamma) = j^r(\Gamma^{(k+1)}\circ j^k\gamma) = J^r(\Gamma^{(k+1)},\mathrm{id}_x)\circ j^{k+r}\gamma\\ &= J^r(\Gamma^{(k+1)},\mathrm{id}_X)\circ J^{r-1}(\Gamma^{(k+1)},\mathrm{id}_X)\circ j^{k+r-1}\gamma\\ &= \cdots = \Gamma^{(k+1)(r)}\circ j^k\gamma,\end{aligned}\tag{10.5}$$

which in coordinates means the system

$$\begin{aligned}y^\sigma_{j_1\dots j_{k+1}} &= \Gamma^\sigma_{j_1\dots j_{k+1}}\\ y^\sigma_{j_1\dots j_{k+1}i} &= D_i(\Gamma^\sigma_{j_1\dots j_{k+1}})\\ &\ \ \vdots\\ y^\sigma_{j_1\dots j_{k+1}i_1\dots i_r} &= D_{i_1\dots i_r}(\Gamma^\sigma_{j_1\dots j_{k+1}}),\end{aligned}\tag{10.6}$$

where according to (3.4) $D_i := D_i^{k+1,k}$ and

$$D_{i_1\dots i_r} := D_{i_r}^{k+r,k+r-1}\dots D_{i_1}^{k+1,k},$$

and the functions on the right side of (10.6) are canonically lifted to $J^{k+r+1}\pi$.

Definition 10.2. Let $k,r\geq 0$. By the *r-th order Cartan distribution* $C^{\Gamma^{(k+1)}(r)}$ of an integrable $(k+1)$-connection $\Gamma^{(k+1)}$ on π is meant the Cartan distribution of the r-th prolongation $\mathcal{E}^{\Gamma^{(k+1)}(r)}$, i.e.

$$C^{\Gamma^{(k+1)}(r)} := C_{\pi_{k+r+1,k+r}}\cap T\Gamma^{(k+1)(r)}(J^k\pi).\tag{10.7}$$

By definition, $C^{\Gamma^{(k+1)}(0)} = C^{\Gamma^{(k+1)}}$ (see (9.14)), and $C^{\Gamma^{(k+1)}(r)}$ is a regular n-dimensional distribution on $\Gamma^{(k+1)(r)}(J^k\pi) \subset J^{k+r+1}\pi$ annihilated by the forms $\omega^\sigma, \dots, \omega^\sigma_{j_1\dots j_{k+r}}$ restricted to $\Gamma^{(k+1)(r)}(J^k\pi)$ together with

$$dy^\sigma_{j_1\dots j_{k+1}i_1\dots i_r} - d\left(D_{i_1\dots i_r}(\Gamma^\sigma_{j_1\dots j_{k+1}})\right) \circ \Gamma^{(k+1)(r-1)}, \tag{10.8}$$

or equivalently spanned by the vector fields

$$\begin{aligned} T\Gamma^{(k+1)(r)}\left(D_{\Gamma^{(k+1)}i}\right) &= \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\dots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} + \Gamma^\sigma_{j_1\dots j_k i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_k}} \\ &+ \left(D_p(\Gamma^\sigma_{j_1\dots j_k i}) \circ \Gamma^{(k+1)}\right)\frac{\partial}{\partial y^\sigma_{j_1\dots j_k p}} + \dots \\ &+ \left(D_{p_1\dots p_{r+1}}(\Gamma^\sigma_{j_1\dots j_k i}) \circ \Gamma^{(k+1)(r)}\right)\frac{\partial}{\partial y^\sigma_{j_1\dots j_k p_1\dots p_{r+1}}}. \end{aligned} \tag{10.9}$$

Let $r \geq 1$. Then by (10.6) and (10.9) we get an expected relation

$$T\Gamma^{(k+1)(r-1)} \circ D_{\Gamma^{(k+1)}i} = D_i^{k+r+1,k+r} \circ \Gamma^{(k+1)(r)}. \tag{10.10}$$

The following assertion can be proved in the same way as its 'zeroth-order' version Lemma 9.1.

LEMMA 10.1. *Let $k \geq 0, r \geq 1$. A $(k+1)$-connection $\Gamma^{(k+1)}$ on π is integrable if, and only if, its r-th order Cartan distribution $C^{\Gamma^{(k+1)}(r)}$ is completely integrable, and a section γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $j^{k+r+1}\gamma$ is the integral mapping of $C^{\Gamma^{(k+1)}(r)}$.*

DEFINITION 10.3. Let $k \geq 0, r \geq 1$. A $(k+1)$-connection $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ will be called a *field of paths* of a $(k+r+1)$-connection $\Gamma^{(k+r+1)}\colon J^{k+r}\pi \to J^{k+r+1}\pi$ if on V holds

$$\Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} = \Gamma^{(k+1)(r)}. \tag{10.11}$$

By definition, each field of paths is integrable, and (10.11) means just

$$H_{\Gamma^{(k+r+1)}}|_{\Gamma^{(k+1)(r-1)}(V)} \equiv C^{\Gamma^{(k+1)}(r-1)}, \tag{10.12}$$

since by (10.10)

$$D_{\Gamma^{(k+r+1)}i} \circ \Gamma^{(k+1)(r-1)} = T\Gamma^{(k+1)(r-1)} \circ D_{\Gamma^{(k+1)}i}. \tag{10.13}$$

Equivalently, if γ is an integral section of $\Gamma^{(k+1)}$, then by (10.5)

$$\Gamma^{(k+r+1)} \circ j^{k+r}\gamma = \Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)} \circ j^k\gamma = \Gamma^{(k+1)(r)} \circ j^k\gamma = j^{k+r+1}\gamma,$$

which means that if γ is an integral section of a field of paths $\Gamma^{(k+1)}$ of $\Gamma^{(k+r+1)}$, then it is the integral section (a path) of $\Gamma^{(k+r+1)}$. In other words, $H_{\Gamma^{(k+1)}}$ defines a foliation of V such that each leaf of this foliation is an integral section of $\Gamma^{(k+r+1)}$. The local expression of (10.11–13) reads

$$\Gamma^\sigma_{j_1\dots j_{k+1}i_1\dots i_r} \circ \Gamma^{(k+1)(r-1)} = D_{i_1\dots i_r}(\Gamma^\sigma_{j_1\dots j_{k+1}}) \circ \Gamma^{(k+1)(r-1)}. \tag{10.14}$$

Globally speaking, each field of paths represents a (local) order-reduction of the given equation. In this respect, the problem of finding the integral sections of a given integrable higher-order connection can be transferred to the problem of looking for and then solving of its fields of paths; the transitivity of the relation 'to be a field of paths of a higher-order connection' is evident. In this respect, the *method of fields of paths* will be discussed in Sec. 17.

Let Γ be an integrable connection on a fibered manifold $\pi\colon Y \to X$. A p-form $\eta \in \Omega^p(Y)$ is called *invariant* with respect to Γ if it is invariant with respect to the corresponding horizontal distribution H_Γ. Here we can present the following result.

PROPOSITION 10.1. *Let $\Gamma^{(k+1)}\colon V \subset J^k\pi \to J^{k+1}\pi$ be a field of paths of a given $(k+r+1)$-connection $\Gamma^{(k+r+1)}$. Let η be a p-form defined on $\pi^{-1}_{k+r,k}(V) \subset J^{k+r}\pi$. If η is invariant with respect to $\Gamma^{(k+r+1)}$, then $(\Gamma^{(k+1)(r-1)})^*\eta$ is invariant with respect to $\Gamma^{(k+1)}$. Conversely, if a p-form ρ on V is invariant with respect to $\Gamma^{(k+1)}$, then $\pi^*_{k+r,k}\rho$ is on $\Gamma^{(k+1)(r-1)}(V)$ invariant with respect to $\Gamma^{(k+r+1)}$.*

Proof. The assertion is an immediate consequence of the relationships between both horizontal distributions, together with the well-known properties of the Lie derivative. Thus from $\Gamma^{(k+1)(r-1)}$-compatibility (see (10.13)) we have

$$\mathcal{L}_\xi\left((\Gamma^{(k+1)(r-1)})^*\eta\right) = (\Gamma^{(k+1)(r-1)})^*(\mathcal{L}_\zeta\eta) = 0,$$

and analogously from $\pi_{k+r,k}$-compatibility (which is trivial)

$$\mathcal{L}_\zeta(\pi^*_{k+r,k}\rho) = \pi^*_{k+r,k}(\mathcal{L}_\xi\rho) = 0,$$

where $\xi \in H_{\Gamma^{(k+1)}}$ and $\zeta = T\Gamma^{(k+1)(r-1)}\xi \in C^{\Gamma^{(k+1)}(r-1)}$. ∎

11. Symmetries and vertical prolongations

This section represents a direct higher-order generalizations of the ideas studied in [64]. The canonical ι-inclusions are involved implicitly.

Lemma 11.1. *Let $\Gamma^{(k+1)}$ be an integrable $(k+1)$-connection on π. Then for any $\Gamma^{(k+1)}$-horizontal vector field $\zeta^{(k)}$ on $J^k\pi$ holds*

$$\mathcal{J}^r\zeta^{(k)}\circ\Gamma^{(k+1)(r-1)}=T\Gamma^{(k+1)(r-1)}\circ\zeta^{(k)}. \tag{11.1}$$

Proof. Notice first that for an arbitrary π, the mapping $\mathcal{J}^r\colon\mathcal{X}(Y)\to\mathcal{X}(J^r\pi)$ assigning the r-th jet prolongation $\mathcal{J}^r\zeta$ to each ζ is linear over $\mathcal{F}(Y)$ when restricted to the subbundle of vectors horizontal with respect to a connection Γ on π – see (4.7), (4.8). Substituting π_k for π and using the corresponding inclusions, the same holds for higher-order connections on π, hence it is sufficient to verify the assertion for the generators $D_{\Gamma^{(k+1)}i}$ of $H_{\Gamma^{(k+1)}}$. To do this, we use (4.6) properly modified to the situation studied:

$$\begin{aligned}\mathcal{J}^r\zeta^{(k)}(j^{k+r}_x\gamma)&=\mu^{(k)}_r(j^r_x(\zeta^{(k)}\circ j^k\gamma))\\&=\nu^{(k)}_r(j^r_x(\zeta^{(k)}\circ j^k\gamma-Tj^k\gamma\circ T\pi_k\circ\zeta^{(k)}\circ j^k\gamma))\\&\quad+Tj^{k+r}\gamma(T\pi_k\circ\zeta^{(k)}\circ j^k\gamma(x)),\end{aligned}$$

with $\mu^{(k)}_r$ and $\nu^{(k)}_r$ defined for π_k analogously as for π in Sec. 4. Suppose now $\zeta^{(k)}=D_{\Gamma^{(k+1)}i}$ and $j^{k+r}\gamma=\Gamma^{(k+1)(r-1)}\circ j^k\gamma$. We can evidently suppose γ to be the integral section of $\Gamma^{(k+1)}$ and by (9.12) the first term of the above expression vanishes, while the second one is

$$T\Gamma^{(k+1)(r-1)}\circ Tj^k\gamma\circ T\pi_k(\zeta^{(k)}(j^k_x\gamma))=T\Gamma^{(k+1)(r-1)}\zeta^{(k)}(j^k_x\gamma).$$

∎

As a consequence, (10.10) becomes

$$\mathcal{J}^rD_{\Gamma^{(k+1)}i}\circ\Gamma^{(k+1)(r-1)}=D^{k+r+1,k+r}_i\circ\Gamma^{(k+1)(r)}. \tag{11.2}$$

Summing up the above ideas, we get the assertion on decompositions of tangent spaces to the equations studied.

Proposition 11.1. *Let $\Gamma^{(k+1)}$ be an integrable $(k+1)$-connection on π, and let $\zeta^{(k)}\in\mathcal{X}(J^k\pi)$. Then there is a direct sum decomposition*

$$\begin{gathered}\mathcal{J}^r\zeta^{(k)}\circ\Gamma^{(k+1)(r-1)}=\mathcal{J}^r(h_{\Gamma^{(k+1)}}\circ\zeta^{(k)})\circ\Gamma^{(k+1)(r-1)}\\+V\Gamma^{(k+1)(r-1)}\circ v_{\Gamma^{(k+1)}}\circ\zeta^{(k)}+(\mathcal{J}^r\zeta^{(k)}\circ\Gamma^{(k+1)(r-1)})^{\pi_{k+r,k}},\end{gathered}$$

where

$$\mathcal{J}^r(h_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)(r-1)} \in C^{\Gamma^{(k+1)(r-1)}}$$
$$V\Gamma^{(k+1)(r-1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)} \in V\Gamma^{(k+1)(r-1)}(V_{\pi_k} J^k\pi)$$
$$(\mathcal{J}^r\zeta^{(k)} \circ \Gamma^{(k+1)(r-1)})^{\pi_{k+r,k}} \in V_{\pi_{k+r,k}} J^{k+r}\pi.$$

In particular,

$$\mathcal{J}^r(v_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)(r-1)} = V\Gamma^{(k+1)(r-1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)} + (\mathcal{J}^r\zeta^{(k)} \circ \Gamma^{(k+1)(r-1)})^{\pi_{k+r,k}}. \tag{11.3}$$

The proposition represents a contribution to the *internal* geometry of equations under consideration, and as such it can be viewed as an internal version of results presented in terms of the so-called *characterizable connections* which will be studied in Sec. 15. The bridge between these points of view is created by fields of paths. For instance, for $\Gamma^{(k+1)}$ being a field of paths of $\Gamma^{(k+r+1)}$ it holds by (10.13) and (11.2)

$$\mathcal{J}^r D_{\Gamma^{(k+1)}i} \circ \Gamma^{(k+1)(r-1)} = D_{\Gamma^{(k+r+1)}i} \circ \Gamma^{(k+1)(r-1)}. \tag{11.4}$$

In what follows, $\Gamma^{(k+1)}$ is supposed to be an integrable $(k+1)$-connection on π. Since it is a (particular case of) a connection on π_k, the ideas of [64] can be more or less similarly repeated.

DEFINITION 11.1. A vector field $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$ will be called a *k-th order symmetry* (briefly *k-symmetry*) of $\Gamma^{(k+1)}$ if $\zeta^{(k)}$ and $\mathcal{J}^1\zeta^{(k)}$ are $\Gamma^{(k+1)}$-related, i.e.

$$\mathcal{J}^1\zeta^{(k)} \circ \Gamma^{(k+1)} = T\Gamma^{(k+1)} \circ \zeta^{(k)}. \tag{11.5}$$

The set of all k-symmetries of $\Gamma^{(k+1)}$ will be denoted by $\mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$.

PROPOSITION 11.2. *Any $\Gamma^{(k+1)}$-horizontal vector field is a k-symmetry of $\Gamma^{(k+1)}$.*

Proof. See Lemma 11.1 for $r = 0$. ∎

COROLLARY 11.1. *A vector field $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$ is a k-symmetry of $\Gamma^{(k+1)}$ if, and only if, one of the following equivalent conditions holds:*

$$\mathcal{J}^1(v_{\Gamma^{(k+1)}} \circ \zeta^{(k)}) \circ \Gamma^{(k+1)} = V\Gamma^{(k+1)} \circ v_{\Gamma^{(k+1)}} \circ \zeta^{(k)}. \tag{11.6}$$

$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\zeta^{(k)})} h_{\Gamma^{(k+1)}} = 0. \tag{11.7}$$

Proof. The relation (11.6) is a direct consequence of Propositions 11.1 and 11.2. The equivalence of (11.7) with (11.6) can be obtained analogously to [64], where now

$$\widetilde{\mathcal{L}}_{(\zeta^{(k)},\mathcal{J}^1\zeta^{(k)})}\Gamma^{(k+1)} = (\Gamma^{(k+1)}, -\mathcal{L}_{\zeta^{(k)}}h_{\Gamma^{(k+1)}}) \tag{11.8}$$

one gets by substituting $\pi_{k+1,k} \hookrightarrow (\pi_k)_{1,0}$ for $\pi_{1,0}$. ∎

Since

$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\zeta^{(k)})}h_{\Gamma^{(k+1)}} = [v_{\Gamma^{(k+1)}}(\zeta^{(k)}), D_{\Gamma^{(k+1)}i}] \otimes dx^i,$$

and

$$\begin{aligned}[v_{\Gamma^{(k+1)}}(\zeta^{(k)}), D_{\Gamma^{(k+1)}i}] &= [\zeta^{(k)} - h_{\Gamma^{(k+1)}}(\zeta^{(k)}), D_{\Gamma^{(k+1)}i}] \\ &= [\zeta^{(k)}, D_{\Gamma^{(k+1)}i}] - [\zeta^j D_{\Gamma^{(k+1)}j}, D_{\Gamma^{(k+1)}i}] \\ &= [\zeta^{(k)}, D_{\Gamma^{(k+1)}i}] + D_{\Gamma^{(k+1)}i}(\zeta^j)D_{\Gamma^{(k+1)}j},\end{aligned}$$

$\zeta^{(k)}$ is a k-symmetry of $\Gamma^{(k+1)}$ if, and only if,

$$[D_{\Gamma^{(k+1)}i}, \zeta^{(k)}] = D_{\Gamma^{(k+1)}i}(\zeta^j)D_{\Gamma^{(k+1)}j} \tag{11.9}$$

for $i = 1, \ldots, n$. In this arrangement, *the k-symmetries of $\Gamma^{(k+1)}$ are just the symmetries of the horizontal distribution $H_{\Gamma^{(k+1)}}$* (in sense of [64]).

Again, the projectability of symmetries is essential.

PROPOSITION 11.3. *A π_k-projectable vector field $\zeta^{(k)}$ on $J^k\pi$ is a k-symmetry of $\Gamma^{(k+1)}$ if, and only if, one of the following equivalent conditions holds:*

- (11.10) $$\mathcal{L}_{\zeta^{(k)}}h_{\Gamma^{(k+1)}} = 0.$$
- *the flow of $\zeta^{(k)}$ permutes the k-jets of integral sections of $\Gamma^{(k+1)}$.*

Proof. The first condition is a consequence of (11.7) and of the fact that

$$\mathcal{L}_{h_{\Gamma^{(k+1)}}(\zeta^{(k)})}h_{\Gamma^{(k+1)}}$$

vanishes if, and only if, $\zeta^{(k)} \in \mathcal{X}_X(J^k\pi)$. The second one is completely due to the results of [64], if again π_k is substituted for π. ∎

Denote finally by $\mathrm{Sym}_v^{(k)}(\Gamma^{(k+1)}) \subset \mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$ the submodule of π_k-vertical k-symmetries of $\Gamma^{(k+1)}$, by $\mathrm{Char}(H_{\Gamma^{(k+1)}})$ the ideal of *characteristic symmetries* of $H_{\Gamma^{(k+1)}}$ (e.i. those lying within $H_{\Gamma^{(k+1)}}$) and by $\mathrm{Shuf}(H_{\Gamma^{(k+1)}})$ the quotient algebra $\mathrm{Shuf}(H_{\Gamma^{(k+1)}}) = \mathrm{Sym}(H_{\Gamma^{(k+1)}})/\mathrm{Char}(H_{\Gamma^{(k+1)}})$ of the so-called *shuffling symmetries*. Recall that while the flow of a characteristic symmetry moves integral manifolds along themselves, any shuffling symmetry represents the whole class of symmetries whose flow rearranges the integral manifolds in the same way.

PROPOSITION 11.4. *It holds* $H_{\Gamma^{(k+1)}} \cong \mathrm{Char}(H_{\Gamma^{(k+1)}})$ *and*

$$\mathrm{Sym}_v^{(k)}(\Gamma^{(k+1)}) \cong \mathrm{Shuf}(\Gamma^{(k+1)}).$$

Proof. The verification is analogous to that presented in [64]; here the considered operator is given by the bellow mentioned relation (11.12). ∎

Let now $k \geq 1$. Locally, $\zeta^{(k)} \in \mathcal{X}(J^k\pi)$ is a symmetry of $\Gamma^{(k+1)}$ if

$$v_{\Gamma^{(k+1)}}(\zeta^{(k)}) = \varphi^\sigma \frac{\partial}{\partial y^\sigma} + \sum_{\ell=1}^{k} D_{\Gamma^{(k+1)}j_1\ldots j_\ell}(\varphi^\sigma)\frac{\partial}{\partial y^\sigma_{j_1\ldots j_\ell}}, \tag{11.11}$$

where

$$D_{\Gamma^{(k+1)}j_1\ldots j_\ell}(f) := D_{\Gamma^{(k+1)}j_\ell} D_{\Gamma^{(k+1)}j_{\ell-1}} \ldots D_{\Gamma^{(k+1)}j_1}(f),$$

and the equations for the m-tuple of *generating functions* $\varphi^\sigma = \zeta^\sigma - y_i^\sigma \zeta^i$ are

$$D_{\Gamma^{(k+1)}j_1\ldots j_{k+1}}(\varphi^\sigma) = \sum_{\ell=0}^{k} D_{\Gamma^{(k+1)}i_1\ldots i_\ell}(\varphi^\lambda)\frac{\partial \Gamma^\sigma_{j_1\ldots j_{k+1}}}{\partial y^\lambda_{i_1\ldots i_\ell}}. \tag{11.12}$$

The structure of higher-order jet prolongations and corresponding projections allows us to define some other types of symmetries; again the ι-inclusions are supposed to work implicitly.

DEFINITION 11.2. A vector field $\zeta^{(r)} \in \mathcal{X}(J^r\pi)$, $0 \leq r \leq k-1$, will be called the *r-symmetry* of $\Gamma^{(k+1)}$ if $\mathcal{J}^{k-r}\zeta^{(r)} \in \mathrm{Sym}^{(k)}(\Gamma^{(k+1)})$. The set of all r-symmetries of $\Gamma^{(k+1)}$ will be denoted by $\mathrm{Sym}^{(r)}(\Gamma^{(k+1)})$.

PROPOSITION 11.5. *A π_r-projectable vector field $\zeta^{(r)}$ on $J^r\pi$ is the r-symmetry of $\Gamma^{(k+1)}$ if, and only if, its flow permutes the r-jets of integral sections of $\Gamma^{(k+1)}$.*

Proof. The assertion is based on the relation between the flows of $\zeta^{(r)}$ and $\mathcal{J}^{k-r}\zeta^{(r)}$ occurring from (4.4). Let the flow $\{\alpha_t\}$ of $\zeta^{(r)}$ permutes the r-jets of integral sections of $\Gamma^{(k+1)}$, i.e. if γ is such an integral section, then $\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1}$ is again the r-jet of integral section ($\{\alpha_t^0\}$ is the flow of the projection ζ_0 of $\zeta^{(r)}$). If $j^k\gamma$ is the k-jet prolongation of an integral section, then $J^{k-r}(\alpha_t, \alpha_t^0) \circ j^k\gamma \circ (\alpha_t^0)^{-1}$ is by definition just

$$J^{k-r}(\alpha_t, \alpha_t^0) \circ j^{k-r}(j^r\gamma) \circ (\alpha_t^0)^{-1} = j^{k-r}(\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1}),$$

and thus again the k-jet of an integral section and the first part of the assertion is proved by means of Prop. 11.3.

Conversely, if $\mathcal{J}^{k-r}\zeta^{(r)}$ is a k-symmetry and if we are given an integral section γ of $\Gamma^{(k+1)}$, then

$$\begin{aligned} j^{k-r+1}(\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1}) &= j^1(j^{k-r}(\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1})) \\ &= j^1(J^{k-r}(\alpha_t, \alpha_t^0) \circ j^k\gamma \circ (\alpha_t^0)^{-1}) \\ &= \Gamma^{(k+1)} \circ J^{k-r}(\alpha_t, \alpha_t^0) \circ j^k\gamma \circ (\alpha_t^0)^{-1} \\ &= \Gamma^{(k+1)} \circ j^{k-r}(\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1}) \end{aligned}$$

and consequently $\alpha_t \circ j^r\gamma \circ (\alpha_t^0)^{-1}$ is again the r-jet of an integral section. ∎

Of course, our main concern is with vector fields on Y as generators of invariant transformations on sections; in this respect, zero-symmetries will be referred to briefly as *symmetries*. In this case, $\zeta \in \mathcal{X}(Y)$ is a symmetry of an integrable $\Gamma^{(k+1)}$ if, and only if, one of the following equivalent conditions hold:

$$\mathcal{J}^{k+1}\zeta \circ \Gamma^{(k+1)} = T\Gamma^{(k+1)} \circ \mathcal{J}^k\zeta. \tag{11.13}$$

$$\mathcal{J}^1(v_{\Gamma^{(k+1)}} \circ \mathcal{J}^k\zeta) \circ \Gamma^{(k+1)} = V\Gamma^{(k+1)} \circ v_{\Gamma^{(k+1)}} \circ \mathcal{J}^k\zeta. \tag{11.14}$$

$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\mathcal{J}^k\zeta)} h_{\Gamma^{(k+1)}} = 0. \tag{11.15}$$

$$[D_{\Gamma^{(k+1)}i}, \mathcal{J}^k\zeta] = D_{\Gamma^{(k+1)}i}(\zeta^j) D_{\Gamma^{(k+1)}j}. \tag{11.16}$$

where $D_{\Gamma^{(k+1)}i}(\zeta^j)$ denotes briefly just $D_{\Gamma^{(k+1)}i}(\pi^*_{k,0}(\zeta^j)) \equiv \pi^*_{k,1}(D^{1,0}(\zeta^j))$. If in addition $\zeta \in \mathcal{X}_X(Y)$, then it is a symmetry of $\Gamma^{(k+1)}$ if, and only if, its flow permutes the integral sections of $\Gamma^{(k+1)}$.

Remark 11.1. The symmetries of the Cartan distribution $C_{\pi_{k,k-1}}$ on $J^k\pi$ are called *contact* vector fields. By the well-known *Bäcklund's theorem*, in the case of $m = \dim \pi = 1$ and if $\zeta^{(k)}$ is contact, then it is the $(k-1)$-th prolongation of a contact vector field on $J^1\pi$. If $m > 1$, then $\zeta^{(k)}$ is the k-th prolongation of a vector field on Y. In this respect, the *external symmetry* of an equation $\mathcal{E}^{(k)} \subset J^k\pi$ is a contact vector field on $J^k\pi$ tangent to $\mathcal{E}^{(k)}$. In other words, its flow preserve both the Cartan distribution and the equation. The restriction of an external symmetry to $\mathcal{E}^{(k)}$ defines a symmetry of $C^{\mathcal{E}^{(k)}}$ and just the symmetries of the distribution $C^{\mathcal{E}^{(k)}}$ are called the *internal symmetries* of the equation $\mathcal{E}^{(k)}$. By (9.14) and Prop. 11.4, $\zeta^{(r)} \in \mathcal{X}(J^r\pi)$ is the r-symmetry of an integrable $\Gamma^{(k+1)}$ if $\mathcal{J}^{k-r+1}\zeta^{(r)}|_{\Gamma^{(k+1)}(J^k\pi)} \in \mathcal{X}(\Gamma^{(k+1)}(J^k\pi))$ is an internal symmetry of $\Gamma^{(k+1)}(J^k\pi)$. In particular, if $\zeta \in \mathcal{X}_X(Y)$ is such that

$$\mathcal{J}^{k+1}\zeta \circ \Gamma^{(k+1)} \in C^{\Gamma^{(k+1)}},$$

then its flow acts on the integral sections of $\Gamma^{(k+1)}$ trivially – moves them along themselves. On the other hand, a π-vertical symmetry can be viewed as representing the whole class of symmetries rearranging the integral sections in the same way.

As in first-order situation, the izomorphism (4.1) allows to define the vertical prolongations of higher-order connections. In fact, the mapping

$$V\Gamma^{(k+1)} \colon V_{\pi_k}J^k\pi \to V_{\pi_{k+1}}J^{k+1}\pi$$

realizes a lift of vertical vectors by

$$\sum_{\ell=0}^{k} \zeta^\sigma_{j_1\dots j_\ell} \frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}\Big|_z \longmapsto \sum_{\ell=0}^{k} \zeta^\sigma_{j_1\dots j_\ell} \frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}\Big|_{\Gamma^{(k+1)}(z)} + \sum_{\ell=0}^{k} \frac{\partial \Gamma^\sigma_{j_1\dots j_{k+1}}}{\partial y^\lambda_{i_1\dots i_\ell}} \zeta^\lambda_{i_1\dots i_\ell} \frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}}\Big|_{\Gamma^{(k+1)}(z)}. \tag{11.17}$$

Then the *vertical prolongation* $\mathcal{V}\Gamma^{(k+1)}$ of $\Gamma^{(k+1)}$ will be a $(k+1)$-connection on $(\pi \circ \tau_Y|_{V_\pi Y})$ defined by

$$V\Gamma^{(k+1)} \circ \nu_k = \nu_{k+1} \circ \mathcal{V}\Gamma^{(k+1)}, \tag{11.18}$$

which is projectable over $\Gamma^{(k+1)}$ within the 2-fibered manifold (see Sec. 13)

$$J^k(\pi \circ \tau_Y|_{V_\pi Y}) \overset{J^k(\tau_Y|_{V_\pi Y}, \mathrm{id}_x)}{\longrightarrow} J^k\pi \overset{\pi_k}{\longrightarrow} X. \tag{11.19}$$

To eliminate the (in fact, useless and confusing) formalism including the ν's, in what follows we work with a slight inaccuracy directly with the izomorphic

$$V_{\pi_k} J^k\pi \xrightarrow{\tau_{J^k\pi}} J^k\pi \xrightarrow{\pi_k} X \tag{11.20}$$

instead of (11.19), unless otherwise stated.

Then the following assertion can be easily verified by means of the results obtained in [23] (see also Sec. 13 for definition of $\Bbbk_{\Gamma^{(k+1)}}$).

PROPOSITION 11.6. *Let $\Gamma^{(k+1)}$ be an integrable $(k+1)$-connection on π and Ψ a connection on $\tau_{J^k\pi}\colon V_{\pi_k}J^k\pi \to J^k\pi$, satisfying $\Bbbk_{\Gamma^{(k+1)}} \circ \Psi = \mathcal{V}\Gamma^{(k+1)}$. If $\zeta^{(k)} \in \mathcal{X}_X(J^k\pi)$ is an integral section of Ψ, then $\zeta^{(k)} \in \mathrm{Sym}_v^{(k)}(\Gamma^{(k+1)})$.*

Let finally $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ be a field of paths of $\Gamma^{(k+r+1)}\colon J^{k+r}\pi \to J^{k+r+1}\pi$. Then one might ask on the relationship between the vertical (zeroth-order) symmetries of the above connections. First, since each integral section of $\Gamma^{(k+1)}$ is the integral section of $\Gamma^{(k+r+1)}$, *if $\zeta \in \mathcal{X}_X^v(Y)$ is a symmetry of $\Gamma^{(k+r+1)}$, then $\zeta|_{\pi_{k,0}(V)}$ is a symmetry of $\Gamma^{(k+1)}$*. To obtain the well-known result affirming that each vertical symmetry of an equation is the symmetry of its prolongation, here is the relation between the corresponding vertical prolongations.

PROPOSITION 11.7. *A $(k+1)$-connection $\Gamma^{(k+1)}$ is a field of paths of a $(k+r+1)$-connection $\Gamma^{(k+r+1)}$ if, and only if, $\mathcal{V}\Gamma^{(k+1)}$ is a field of paths of $\mathcal{V}\Gamma^{(k+r+1)}$.*

Proof. The assertion confirms the compatibility of introduced concepts with all the underlying functors and structures. Namely, if (10.11) holds, then

$$\begin{aligned}
\mathcal{V}\Gamma^{(k+r+1)} \circ (\mathcal{V}\Gamma^{(k+1)})^{(r-1)} &\cong V\Gamma^{(k+r+1)} \circ V\Gamma^{(k+1)(r-1)} \\
&= V(\Gamma^{(k+r+1)} \circ \Gamma^{(k+1)(r-1)}) = V(\Gamma^{(k+1)(r)}) \\
&= V\left(J^r(\Gamma^{(k+1)}, \mathrm{id}_X) \circ J^{r-1}(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \cdots \circ \Gamma^{(k+1)}\right) \\
&= VJ^r(\Gamma^{(k+1)}, \mathrm{id}_X) \circ VJ^{r-1}(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \cdots \circ V\Gamma^{(k+1)} \\
&\cong J^r(\mathcal{V}\Gamma^{(k+1)}, \mathrm{id}_X) \circ J^{r-1}(\mathcal{V}\Gamma^{(k+1)}, \mathrm{id}_X) \circ \cdots \circ \mathcal{V}\Gamma^{(k+1)} = (\mathcal{V}\Gamma^{(k+1)})^{(r)}.
\end{aligned}$$

The converse can be obtained analogously. ∎

COROLLARY 11.2. *If ζ is a symmetry of $\Gamma^{(k+1)}$, then it is a symmetry of $\Gamma^{(k+1)(r)}$.*

Proof. We have to prove the relation (cf. (11.13)):

$$\mathcal{J}^{k+r+1}\zeta \circ \Gamma^{(k+1)(r)} = \mathcal{V}\Gamma^{(k+r+1)} \circ \mathcal{J}^{k+r}\zeta \circ \Gamma^{(k+1)(r-1)}.$$

Let $z \in \operatorname{Dom}\Gamma^{(k+1)}$, $z = j^k_x\gamma$ with γ being the integral section of $\Gamma^{(k+1)}$. Then (again the ν's-izomorphisms are omitted)

$$\begin{aligned}
&\mathcal{V}\Gamma^{(k+r+1)} \circ \mathcal{J}^{k+r}\zeta \circ \Gamma^{(k+1)(r-1)}(z)\\
&= \mathcal{V}\Gamma^{(k+r+1)} \circ \mathcal{J}^{k+r}\zeta \circ \Gamma^{(k+1)(r-1)} \circ j^k\gamma(x)\\
&= \mathcal{V}\Gamma^{(k+r+1)} \circ \mathcal{J}^{k+r}\zeta \circ j^{k+r}\gamma(x)\\
&= \mathcal{V}\Gamma^{(k+r+1)} \circ j^{k+r}(\zeta \circ \gamma)(x) = j^{k+r+1}(\zeta \circ \gamma)(x)\\
&= \mathcal{J}^{k+r+1}\zeta \circ j^{k+r+1}\gamma(x) = \mathcal{J}^{k+r+1}\zeta \circ \Gamma^{(k+1)(r)} \circ j^k\gamma(x)\\
&= \mathcal{J}^{k+r+1}\zeta \circ \Gamma^{(k+1)(r)}(z).
\end{aligned}$$

∎

12. AN EXAMPLE

The following example can be equivalently considered both globally on the trivial bundle $\mathrm{pr}_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ and locally on a fibered manifold $\pi\colon Y \to X$ with $\dim Y = 4$ and $\dim X = 2$. The (fibered) coordinates we are dealing with are denoted by

$$x = x^1, y = x^2;\ u = y^1, v = y^2,\ u_x = y^1_1, u_y = y^1_2, v_x = y^2_1, v_y = y^2_2$$

on the 8-dimensional $J^1\pi$. Let us consider an integrable connection Γ on π, generating the first-order equations:

$$\begin{aligned}
u_x &= yv.\\
u_y &= xv.\\
v_x &= yu.\\
v_y &= xu.
\end{aligned} \tag{12.1}$$

Then the generators of H_Γ are:

$$\begin{aligned}
D_{\Gamma x} &= \frac{\partial}{\partial x} + yv\frac{\partial}{\partial u} + yu\frac{\partial}{\partial v}.\\
D_{\Gamma y} &= \frac{\partial}{\partial y} + xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v}.\\
\omega^{\Gamma u} &= du - yvdx - xvdy.\\
\omega^{\Gamma v} &= dv - yudx - xudy.
\end{aligned}$$

and the generators of C^Γ:

$$\begin{aligned}
T\Gamma(D_{\Gamma x}) =& \frac{\partial}{\partial x} + yv\frac{\partial}{\partial u} + yu\frac{\partial}{\partial v}\\
&+ y^2u\frac{\partial}{\partial u_x} + (v+xyu)\frac{\partial}{\partial u_y} + y^2v\frac{\partial}{\partial v_x} + (u+xyv)\frac{\partial}{\partial v_y},\\
T\Gamma(D_{\Gamma y}) =& \frac{\partial}{\partial y} + xv\frac{\partial}{\partial u} + xu\frac{\partial}{\partial v}\\
&+ (v+xyu)\frac{\partial}{\partial u_x} + x^2u\frac{\partial}{\partial u_y} + (u+xyv)\frac{\partial}{\partial v_x} + x^2v\frac{\partial}{\partial v_y}.
\end{aligned} \tag{12.2}$$

The first prolongation of the equation $\mathcal{E}^\Gamma \subset J^1\pi$ is the submanifold

$$\mathcal{E}^{\Gamma(1)} = J^1(\Gamma, \mathrm{id}_X) \circ \Gamma(Y) \subset J^2\pi,$$

whose local expression is

$$\begin{array}{ll}
 & u_{xx} = yv_x\\
u_x = yv & u_{xy} = v + yv_y\\
u_y = xv & u_{yy} = xv_y\\
v_x = yu & v_{xx} = yu_x\\
v_y = xu & v_{xy} = u + yu_y\\
 & v_{yy} = xu_y,
\end{array} \tag{12.3}$$

which corresponds to the generators of C^Γ (see (12.2)).

On the other hand, we have the equations of a 2-connection $\Gamma^{(2)}$ when extracting the second part of (12.3):

$$\begin{array}{ll}
u_{xx} = yv_x & v_{xx} = yu_x\\
u_{xy} = v + yv_y & v_{xy} = u + yu_y\\
u_{yy} = xv_y & v_{yy} = xu_y.
\end{array} \tag{12.4}$$

Clearly, Γ (12.1) is a field of paths of $\Gamma^{(2)}$ (12.4). For example, the generators of $H_{\Gamma^{(2)}}$ are

$$\begin{aligned}
D_{\Gamma^{(2)}x} =& \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + v_x\frac{\partial}{\partial v} + \\
&+ yv_x\frac{\partial}{\partial u_x} + (v + yv_y)\frac{\partial}{\partial u_y} + yu_x\frac{\partial}{\partial v_x} + (u + yu_y)\frac{\partial}{\partial v_y}. \\
D_{\Gamma^{(2)}y} =& \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + v_y\frac{\partial}{\partial v} + \\
&+ (v + yv_y)\frac{\partial}{\partial u_x} + xv_y\frac{\partial}{\partial u_y} + (u + yu_y)\frac{\partial}{\partial v_x} + xu_y\frac{\partial}{\partial v_y}.
\end{aligned}$$

and the vertical prolongation $\mathcal{V}\Gamma^{(2)}$ means (under the above identifications) the lift

$$\begin{aligned}
&\left(\zeta^u\frac{\partial}{\partial u} + \zeta^v\frac{\partial}{\partial v} + \zeta^{u_x}\frac{\partial}{\partial u_x} + \zeta^{u_y}\frac{\partial}{\partial u_y} + \zeta^{v_x}\frac{\partial}{\partial v_x} + \zeta^{v_y}\frac{\partial}{\partial v_y}\right)|_z \longmapsto \\
&\quad \left(\zeta^u\frac{\partial}{\partial u} + \zeta^v\frac{\partial}{\partial v} + \zeta^{u_x}\frac{\partial}{\partial u_x} + \zeta^{u_y}\frac{\partial}{\partial u_y} + \zeta^{v_x}\frac{\partial}{\partial v_x} + \zeta^{v_y}\frac{\partial}{\partial v_y}\right)|_{\Gamma^{(2)}(z)} \\
&\qquad + \left(y\zeta^{v_x}\frac{\partial}{\partial u_{xx}} + (\zeta^v + y\zeta^{v_y})\frac{\partial}{\partial u_{xy}} + x\zeta^{v_y}\frac{\partial}{\partial u_{yy}}\right. \\
&\qquad\qquad \left. + y\zeta^{u_x}\frac{\partial}{\partial v_{xx}} + (\zeta^u + y\zeta^{u_y})\frac{\partial}{\partial v_{xy}} + x\zeta^{u_y}\frac{\partial}{\partial v_{yy}}\right)|_{\Gamma^{(2)}(z)}.
\end{aligned}$$

13. 2-FIBERED MANIFOLD $J^{k+1}\pi \overset{\pi_{k+1,k}}{\longrightarrow} J^k\pi \overset{\pi_k}{\longrightarrow} X$

A 2-*fibered manifold* is a quintuple $Z \overset{\rho}{\to}\!\longrightarrow Y \overset{\pi}{\to}\!\longrightarrow X$, where $\pi\colon Y \to X$ and $\rho\colon Z \to Y$ and thus also $\pi \circ \rho\colon Z \to X$ are fibered manifolds. For any $\gamma \in \mathcal{S}_X(\pi)$ and $\psi \in \mathcal{S}_Y(\rho)$ (we suppose sections to be global for the simplicity only, the same applies for local sections with having their domains in mind) (γ, id_X) or (ψ, id_X) is a morphism between id_X and π or between π and $\pi \circ \rho$, respectively. Then the composition $\psi \circ \gamma$ is a section of $\pi \circ \rho$, i.e. $(\psi \circ \gamma, \mathrm{id}_X)$ is a morphism between id_X and $\pi \circ \rho$, and

$$J^1(\psi \circ \gamma, \mathrm{id}_X) = J^1(\psi, \mathrm{id}_X) \circ J^1(\gamma, \mathrm{id}_X). \tag{13.1}$$

The situation can be represented diagrammatically:

$$
\begin{array}{ccccccc}
X & \xleftarrow{\pi_1} & J^1\pi & \xleftarrow{J^1(\rho,\mathrm{id}_X)} & J^1(\pi\circ\rho) & & \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_{1,0}}\big\downarrow & & \big\downarrow{\scriptstyle (\pi\circ\rho)_{1,0}} & & \\
X & \xleftarrow{\pi} & Y & \xleftarrow{\rho} & Z & \xleftarrow{\rho_{1,0}} & J^1\rho \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi\circ\rho} & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X. & &
\end{array}
$$

$$
\begin{array}{ccccccc}
X & \xrightarrow{J^1(\gamma,\mathrm{id}_X)} & J^1\pi & \xrightarrow{J^1(\psi,\mathrm{id}_X)} & J^1(\pi\circ\rho) & & \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_{1,0}}\big\downarrow & & \big\downarrow{\scriptstyle (\pi\circ\rho)_{1,0}} & & \\
X & \xrightarrow{\gamma} & Y & \xrightarrow{\psi} & Z & \xleftarrow{\rho_{1,0}} & J^1\rho \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi\circ\rho} & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X\,. & &
\end{array}
\tag{13.2}
$$

It is easy to see that $J^1(\gamma, \mathrm{id}_X) = j^1\gamma$ and $J^1(\psi\circ\gamma, \mathrm{id}_X) = j^1(\psi\circ\gamma)$, while $J^1(\psi, \mathrm{id}_X) \neq j^1\psi = J^1(\psi, \mathrm{id}_Y)$, the target space of which is $J^1\rho$. The relation between prolongations mentioned is realized by the map

$$\Bbbk\colon J^1\pi \times_Y J^1\rho \to J^1(\pi\circ\rho) \tag{13.3}$$

defined by

$$\Bbbk(j^1_x\gamma, j^1_{\gamma(x)}\psi) = j^1_x(\psi\circ\gamma).$$

In fibered coordinates (x^i, y^σ) on Y and $(x^i, y^\sigma, z^\alpha)$ on Z and the induced coordinates y^σ_i or $z^\alpha_i, z^\alpha_\lambda$ or v^σ_i, w^α_i on $J^1\pi$ or $J^1\rho$ or $J^1(\pi\circ\rho)$, respectively,

$$\Bbbk\colon \left\{\begin{array}{ll} x^i = x^i & \\ y^\sigma = y^\sigma & \quad v^\sigma_i = y^\sigma_i \\ z^\alpha = z^\alpha & \quad w^\alpha_i = z^\alpha_i + z^\alpha_\lambda y^\lambda_i. \end{array}\right.$$

Both sections and *projections* can be prolonged. Thus the prolongation of ρ is a map $J^1(\rho, \mathrm{id}_X)\colon J^1(\pi\circ\rho) \to J^1\pi$, defined by $J^1(\rho, \mathrm{id}_X)(j^1_x(\psi\circ\gamma)) = j^1_x\gamma$, and clearly $J^1(\pi, \mathrm{id}_X) \equiv \pi_1$.

Our main idea is the introduction and the study of the role of an arbitrary fibered morphism

$$\Phi\colon Z \to J^1\pi \tag{13.4}$$

between ρ and $\pi_{1,0}$ over Y, which in fibered coordinates reads

$$(x^i, y^\sigma, z^\alpha) \xrightarrow{\Phi} (x^i, y^\sigma, \Phi_i^\sigma(x^j, y^\lambda, z^\alpha)).$$

The point is that one of the most interesting particular cases of such a morphism is represented by $\Phi = \Gamma \circ \rho$ with $\Gamma \colon Y \to J^1\pi$ being a connection on π.

Let Φ (13.4) be an arbitrary morphism. Then by the composition

$$J^1\rho \xrightarrow{\rho_{1,0}\times \mathrm{id}} Z \times_Y J^1\rho \xrightarrow{\Phi\times \mathrm{id}} J^1\pi \times_Y J^1\rho \xrightarrow{\Bbbk} J^1(\pi \circ \rho)$$

one gets an affine morphism

$$\Bbbk_\Phi : J^1\rho \to J^1(\pi \circ \rho)$$

between $\rho_{1,0}$ and $(\pi \circ \rho)_{1,0}$ over Z, locally expressed by

$$\Bbbk_\Phi : \begin{cases} x^i = x^i \\ y^\sigma = y^\sigma \\ z^\alpha = z^\alpha \end{cases} \quad \begin{aligned} v_i^\sigma &= \Phi_i^\sigma \\ w_i^\alpha &= z_i^\alpha + z_\lambda^\alpha \Phi_i^\lambda. \end{aligned} \tag{13.5}$$

Due to the affine bundles structure, there is a canonically determined vector bundle morphism

$$\overline{\Bbbk}_\Phi \colon V_\rho Z \otimes \rho^*(T^*Y) \to V_{(\pi\circ\rho)} Z \otimes (\pi \circ \rho)^*(T^*X)$$

between the associated vector bundles, associated to $\Bbbk_\Phi$, which can be locally characterized by

$$\overline{\Bbbk}_\Phi : \begin{cases} x^i = x^i \\ y^\sigma = y^\sigma \\ z^\alpha = z^\alpha \end{cases} \quad \begin{aligned} \overline{v}_i^\sigma &= 0 \\ \overline{w}_i^\alpha &= \overline{z}_i^\alpha + \overline{z}_\lambda^\alpha \Phi_i^\lambda, \end{aligned} \tag{13.6}$$

where the dashed coordinates are those on the associated vector bundles.

In particular, we will write $\Bbbk_\Gamma$ and $\overline{\Bbbk}_\Gamma$ for $\Phi = \Gamma \circ \rho$. It is easy to see that there is an affine subbundle A_Φ in $J^1(\pi \circ \rho)$, canonically determined by Φ, such that $\mathrm{Im}\Bbbk_\Phi \subset A_\Phi \subset J^1(\pi \circ \rho)$. This can be defined locally by the equations

$$v_i^\sigma = \Phi_i^\sigma(x^j, y^\lambda, z^\alpha), \tag{13.7}$$

or more geometrically as $A_\Phi = \ker \mathrm{Sp}_\Phi$, where

$$\mathrm{Sp}_\Phi \colon J^1(\pi \circ \rho) \to V_\pi Y \otimes \pi^*(T^*X)$$

can be on the lines of the *Spencer operator* defined in such a way that $\mathrm{Sp}_\Phi(j^1_x\xi)$ is a vector satisfying

$$J^1(\rho, \mathrm{id}_X)(j^1_x\xi) + \mathrm{Sp}_\Phi(j^1_x\xi) = \Phi \circ (\pi \circ \rho)_{1,0}(j^1_x\xi).$$

In other words, $j^1_x\xi \in A_\Phi$ if, and only if, $\Phi \circ (\pi \circ \rho)_{1,0}(j^1_x\xi) = J^1(\rho, \mathrm{id}_X)(j^1_x\xi)$. The associated vector bundle $\overline{A}_\Phi$ to A_Φ is (whatever Φ is)

$$\overline{A}_\Phi = V_\rho Z \otimes (\pi \circ \rho)^*(T^*X) \subset V_{(\pi\circ\rho)} Z \otimes (\pi \circ \rho)^*(T^*X)$$

and it generally does not split except for ρ being an affine or vector bundle.

For a connection $\Psi \colon Z \to J^1\rho$ on ρ with the components $\Psi^\alpha_i, \Psi^\alpha_\lambda$ and a connection $\Xi \colon Z \to J^1(\pi \circ \rho)$ on $\pi \circ \rho$ with the components $\Xi^\sigma_i, \Xi^\alpha_i$ (together with Γ on π mentioned above), one gets the diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{J^1(\gamma,\mathrm{id}_X)\equiv j^1\gamma} & J^1\pi & \xrightarrow{J^1(\psi,\mathrm{id}_X)} & J^1(\pi\circ\rho) & & \\
\Big\downarrow \mathrm{id}_X & & \Gamma\Big\uparrow & & \Xi\Big\uparrow & & \\
X & \xrightarrow{\gamma} & Y & \xrightarrow{\psi} & Z & \xrightarrow{\Psi} & J^1\rho \\
\Big\downarrow \mathrm{id}_X & & \pi\Big\downarrow & & \xi\Big\uparrow & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X\,. & &
\end{array}$$

Let $\Phi \colon Z \to J^1\pi$ be a fibered morphism over Y and $\Psi \colon Z \to J^1\rho$ a connection on ρ. Then there is a connection $\Xi \colon Z \to J^1(\pi\circ\rho)$ on $\pi\circ\rho$ canonically determined by the pair Φ, Ψ, defined by $\Xi = \Bbbk_\Phi \circ \Psi$ or equivalently by the composition

$$Z \xrightarrow{\Phi\times_Y\Psi} J^1\pi \times_Y J^1\rho \xrightarrow{\Bbbk} J^1(\pi \circ \rho),$$

and denoted by $\Xi = \Bbbk(\Phi, \Psi)$.

In coordinates, the components of $\Xi = \Bbbk(\Phi, \Psi)$ are by (13.5)

$$\Xi^\sigma_i = \Phi^\sigma_i, \quad \Xi^\alpha_i = \Psi^\alpha_i + \Psi^\alpha_\lambda \Phi^\lambda_i. \tag{13.8}$$

As a corollary, one gets: let $\Gamma \colon Y \to J^1\pi$ be a connection on π and $\Psi \colon Z \to J^1\rho$ a connection on ρ. Then there is a connection $\Xi \colon Z \to J^1(\pi \circ \rho)$ on $\pi \circ \rho$

canonically determined by the pair Γ, Ψ, defined by $\Xi = \mathbb{k}_\Gamma \circ \Psi$ or equivalently by the composition

$$Z \overset{(\Gamma\circ\rho)\times_Y \Psi}{\longrightarrow} J^1\pi \times_Y J^1\rho \overset{\mathbb{k}}{\longrightarrow} J^1(\pi \circ \rho).$$

and denoted by $\Xi = \mathbb{k}(\Gamma, \Psi)$.

It is easy to see that $\mathbb{k}(\Gamma, \Psi)$ is *projectable* over Γ, which means that

$$\Gamma \circ \rho = J^1(\rho, \mathrm{id}_X) \circ \mathbb{k}(\Gamma, \Psi).$$

In coordinates, the components of $\Xi = \mathbb{k}(\Gamma, \Psi)$ are by (13.8)

$$\Xi^\sigma_i = \Gamma^\sigma_i, \quad \Xi^\alpha_i = \Psi^\alpha_i + \Psi^\alpha_\lambda \Gamma^\lambda_i. \tag{13.9}$$

On the other hand, there is a family of distinguished sections

$$\varphi \colon Z \to V_\rho Z \otimes \rho^*(T^*Y)$$

(i.e. of soldering forms on ρ or deformations of connections on ρ), determined by any morphism Φ.

A soldering form $\varphi \colon Z \to V_\rho Z \otimes \rho^*(T^*Y)$ on ρ is called a Φ-*admissible deformation* on ρ if $\varphi(z) \in \ker \overline{\mathbb{k}}_\Phi$ for all $z \in Z$.

By (13.6), the condition of the Φ-admissibility for the components φ^α_i and φ^α_λ of φ means

$$\varphi^\alpha_i + \varphi^\alpha_\lambda \Phi^\lambda_i = 0. \tag{13.10}$$

The meaning of this concept is transparent: with a fixed Φ it holds $\mathbb{k}(\Phi, \Psi_1) = \mathbb{k}(\Phi, \Psi_2)$ if, and only if, $\varphi = h_{\Psi_1} - h_{\Psi_2}$ is Φ-admissible. Equivalently, with a fixed Φ and Ξ and a given Ψ such that $\Xi = \mathbb{k}(\Phi, \Psi)$, the knowledge of the Φ-admissible deformations family means the knowledge of *all* such connections Ψ on ρ. The following lemmas were proved in [23].

Lemma 13.1. *Let $\psi \in S_{\mathrm{loc}}(\rho)$ be an integral section of a connection Ψ on ρ. Then $\gamma \in S_{\mathrm{loc}}(\pi)$ is an integral section of a connection Γ on π if, and only if, $\xi = \psi \circ \gamma \in S_{\mathrm{loc}}(\pi \circ \rho)$ is an integral section of the connection $\Xi = \mathbb{k}(\Gamma, \Psi)$.*

Lemma 13.2. *Let Γ be an integrable connection on π and $\psi \in \mathcal{S}_{\mathrm{loc}}(\rho)$. Then the following diagram commutes if, and only if, $\xi = \psi \circ \gamma$ is an integral section of $\Xi = \mathbb{k}(\Gamma, \Psi)$ for each integral section γ of Γ:*

$$\begin{array}{ccccccc}
J^1\pi & \xrightarrow{\ \mathrm{id}\ } & J^1\pi & \xrightarrow{J^1(\psi,\mathrm{id}_X)} & J^1(\pi\circ\rho) & \xleftarrow{\ \mathbb{k}_\Gamma\ } & J^1\rho \\
{\scriptstyle j^1\gamma}\Big\uparrow & & {\scriptstyle \Gamma}\Big\uparrow & & \Big\uparrow{\scriptstyle \Xi} & & \Big\uparrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow{\ \gamma\ } & Y & \xrightarrow{\ \psi\ } & Z & \xrightarrow{\ \Psi\ } & J^1\rho\,.
\end{array}$$

In this section we observe the situation described by the diagram

$$
\begin{array}{ccccccc}
X & \xleftarrow{(\pi_k)_1} & J^1\pi_k & \xleftarrow{J^1(\pi_{k+1,k},\mathrm{id}_X)} & J^1\pi_{k+1} & & \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle (\pi_k)_{1,0}}\big\downarrow & & {\scriptstyle (\pi_{k+1})_{1,0}}\big\downarrow & & \\
X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+1,k}} & J^{k+1}\pi & \xleftarrow{(\pi_{k+1,k})_{1,0}} & J^1\pi_{k+1,k} \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_k}\big\downarrow & & {\scriptstyle \pi_{k+1}}\big\downarrow & & \\
X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X\,. & &
\end{array}
\tag{13.11}
$$

The canonical map $\Bbbk$ (13.3) is now

$$\Bbbk\colon J^1\pi_k \times_{J^k\pi} J^1\pi_{k+1,k} \to J^1\pi_{k+1}. \tag{13.12}$$

It does not effect the coordinates up to $y^\sigma_{j_1\dots j_k;i}$ and its equations are

$$y^\sigma_{j_1\dots j_{k+1};i} = z^\sigma_{j_1\dots j_{k+1}i} + \sum_{\ell=0}^{k} z^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+1}\lambda} y^\lambda_{r_1\dots r_\ell;i}, \tag{13.13}$$

where by $z^\sigma_{j_1\dots j_{k+1}i}$, $z^\sigma_{j_1\dots j_{k+1}\lambda}$, $\dots$,$z^{\sigma i_1\dots i_k}_{j_1\dots j_{k+1}\lambda}$ we denote the induced derivative coordinates on $J^1\pi_{k+1,k}$. The first order of business is to mention the role of the canonical embedding

$$\iota_{1,k}\colon J^{k+1}\pi \hookrightarrow J^1\pi_k$$

which is in coordinates expressed by

$$y^\sigma_{;i} = y^\sigma_i, \dots, y^\sigma_{j_1\dots j_k;i} = y^\sigma_{j_1\dots j_k i}. \tag{13.14}$$

Comparing the above with Sec. 6, we see that $\mathrm{Sp}_{\iota_{1,k}} \equiv \mathrm{Sp}_{k+1}$ and consequently $A_{\iota_{1,k}} \equiv \widehat{J}^{k+2}\pi \subset J^1\pi_{k+1}$.

Notice that if $\Phi\colon J^{k+1}\pi \to J^1\pi_k$ is an arbitrary fibered morphism over $J^k\pi$, then since the vertical bundle associated to $(\pi_k)_{1,0}$ is $V_{\pi_k}J^k\pi \otimes \pi_k^*(T^*X)$, the difference $\Phi - \iota_{1,k}$ is a fibered morphism $J^k\pi \to V_{\pi_k}J^k\pi \otimes \pi_k^*(T^*X)$ and thus

$$\Phi_a := \iota_{1,k} + a(\Phi - \iota_{1,k}) \tag{13.15}$$

is a fibered morphism $J^{k+1}\pi \to J^1\pi_k$ over $J^k\pi$ for any $a \in \mathbb{R}$.

DEFINITION 13.1. The *formal curvature map* is the map

$$R\colon J^1\pi_{k+1,k} \to \pi^*_{k+1,k}\left(V_{\pi_k} J^k\pi \otimes \pi^*_k\left(\Lambda^2 T^*X\right)\right) \tag{13.16}$$

defined for each $j^1_{j^k_x\gamma}\chi \in J^1\pi_{k+1,k}$ by

$$R(j^1_{j^k_x\gamma}\chi) = r_{k+1} \circ J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi(j^k_x\gamma). \tag{13.17}$$

The map R is well-defined, since for each $\chi \in \mathcal{S}(\pi_{k+1,k})$ it holds

$$\iota_{1,k} \circ (\pi_{k+1})_{1,0} \circ J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi = \iota_{1,k} \circ \chi \circ (\pi_k)_{1,0} \circ \iota_{1,k} \circ \chi = \iota_{1,k} \circ \chi,$$

and

$$J^1(\pi_{k+1,k}, \mathrm{id}_X) \circ J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi = J^1(\pi_{k+1,k} \circ \chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi = \iota_{1,k} \circ \chi.$$

hence

$$J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \chi \in \widehat{J}^{k+2}\pi. \tag{13.18}$$

Then by Sec. 6,

$$\begin{aligned} R\colon J^1\pi_{k+1,k} \to \pi^*_{k+1,0}\left(V_\pi Y \otimes \pi^*\left(\lozenge^2_k T^*X\right)\right) \hookrightarrow \\ \pi^*_{k+1,0}\left(V_\pi Y\right) \otimes \pi^*_{k+1}\left(\Lambda^2 T^*X \otimes S^k T^*X\right) \cong \\ \pi^*_{k+1,k}\left(V_{\pi_{k,k-1}} J^k\pi \otimes \pi^*_k\left(\Lambda^2 T^*X\right)\right) \\ \subset \pi^*_{k+1,k}\left(V_{\pi_k} J^k\pi \otimes \pi^*_k\left(\Lambda^2 T^*X\right)\right). \end{aligned}$$

Moreover, due to the well-known splitting we have

$$\begin{aligned} \pi^*_{k+1,k}\left(V_{\pi_k} J^k\pi \otimes \pi^*_k\left(T^*X\right)\right) = \\ \iota^*_{1,k} \circ (\pi_k)^*_{1,0}\left(V_{\pi_k} J^k\pi \otimes \pi^*_k\left(T^*X\right)\right) \cong \\ \iota^*_{1,k}\left(V_{(\pi_k)_{1,0}} J^1\pi_k\right) \subset V_{\pi_{k+1}} J^{k+1}\pi, \end{aligned}$$

which means that

$$R\colon J^1\pi_{k+1,k} \to V_{\pi_{k+1}} J^{k+1}\pi \otimes \pi^*_{k+1}(T^*X). \tag{13.19}$$

Consequently, we can define (for $a, b \in \mathbb{R}$) the affine morphism

$$\mathbb{k}^{a,b}_\Phi\colon J^1\pi_{k+1,k} \to J^1\pi_{k+1}$$

between $(\pi_{k+1,k})_{1,0}$ and $(\pi_{k+1})_{1,0}$ over $J^{k+1}\pi$ by

$$\mathbb{k}_\Phi^{a,b} = \mathbb{k}_{\Phi_a} + bR. \tag{13.20}$$

It is easy to see that regarding a curvature of the connections in question, one gets that

$$R_{\Gamma^{(k+1)}} = -\mathrm{pr}_2 \circ R \circ j^1\Gamma^{(k+1)} =$$
$$-\mathrm{pr}_2 \circ r_{k+1} \circ J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)} \colon J^k\pi \to V_{\pi_{k,k-1}} J^k\pi \otimes \pi_k^*(\Lambda^2 T^*X).$$

As to be expected, the same characterization can be presented for a (first-order) connection Γ on π, i.e.

$$R_\Gamma = -\mathrm{pr}_2 \circ r_1 \circ J^1(\Gamma, \mathrm{id}_X) \circ \Gamma \colon Y \to V_\pi Y \otimes \pi^*(\Lambda^2 T^*X),$$

hence $k = 0$ will be allowed when speaking on the curvature of a $(k+1)$-connection on π.

14. Connections on $\pi_{k+1,k} : J^{k+1}\pi \to J^k\pi$

Another type of connections playing a crucial role in our discussion is represented by those on the affine bundle $\pi_{k+1,k} \colon J^{k+1}\pi \to J^k\pi$. Let us again give a summary of the notions characterizing a (global) connection Ξ on $\pi_{k+1,k}$. First of all, it is a (global) section

$$\Xi \colon J^{k+1}\pi \to J^1\pi_{k+1,k}$$

of $(\pi_{k+1,k})_{1,0}$ and thus its local equations are

$$\begin{aligned} z^\sigma_{j_1\dots j_{k+1}i} &= \Xi^\sigma_{j_1\dots j_{k+1}i} \\ z^\sigma_{j_1\dots j_{k+1}\lambda} &= \Xi^\sigma_{j_1\dots j_{k+1}\lambda} \\ &\vdots \\ z^{\sigma i_1\dots i_k}_{j_1\dots j_{k+1}\lambda} &= \Xi^{\sigma i_1\dots i_k}_{j_1\dots j_{k+1}\lambda}, \end{aligned} \tag{14.1}$$

with the components from $\mathcal{F}(J^{k+1}\pi)$. The *horizontal form* of Ξ is

$$h_\Xi \colon J^{k+1}\pi \to TJ^{k+1}\pi \otimes \pi^*_{k+1,k}(T^*J^k\pi),$$

having the local expression

$$h_\Xi = D_{\Xi i} \otimes dx^i + \sum_{\ell=0}^{k} D^{i_1\dots i_\ell}_{\Xi\lambda} \otimes dy^\lambda_{i_1\dots i_\ell} \tag{14.2}$$

with the *absolute derivatives*

$$
\begin{aligned}
D_{\Xi i} &= \frac{\partial}{\partial x^i} + \Xi^\sigma_{j_1\dots j_{k+1} i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}}\\
D_{\Xi\lambda} &= \frac{\partial}{\partial y^\lambda} + \Xi^\sigma_{j_1\dots j_{k+1}\lambda}\frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}}\\
&\vdots\\
D^{i_1\dots i_k}_{\Xi\lambda} &= \frac{\partial}{\partial y^\lambda_{i_1\dots i_k}} + \Xi^{\sigma i_1\dots i_k}_{j_1\dots j_{k+1}\lambda}\frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}}.
\end{aligned}
\tag{14.3}
$$

Therefore, the complementary *vertical form*

$$v_\Xi \colon J^{k+1}\pi \to V_{\pi_{k+1,k}}J^{k+1}\pi \otimes T^*J^{k+1}\pi$$

reads locally as

$$v_\Xi = \frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}} \otimes \omega^{\Xi\sigma}_{j_1\dots j_{k+1}} \tag{14.4}$$

with

$$\omega^{\Xi\sigma}_{j_1\dots j_{k+1}} = dy^\sigma_{j_1\dots j_{k+1}} - \sum_{\ell=0}^{k} \Xi^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda} dy^\lambda_{i_1\dots i_\ell} - \Xi^\sigma_{j_1\dots j_{k+1} i}\, dx^i.$$

The dimension of the $\pi_{k+1,k}$-*horizontal distribution* H_Ξ, generating the decomposition

$$TJ^{k+1}\pi = H_\Xi \oplus V_{\pi_{k+1,k}}J^{k+1}\pi, \tag{14.5}$$

equals to $\dim J^k\pi$ (the codimension is just the fibre dimension of $\pi_{k+1,k}$), and the corresponding *horizontal lift*

$$
\begin{aligned}
\pi^*_{k+1,k}(TJ^k\pi) \ni \left(\zeta^i \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k} \zeta^\sigma_{j_1\dots j_\ell}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}\right)|_{\pi_{k+1,k}(j^{k+1}_x\gamma)} \longmapsto\\
\left(\zeta^i D_{\Xi i} + \sum_{\ell=0}^{k} \zeta^\sigma_{j_1\dots j_\ell} D^{j_1\dots j_\ell}_{\Xi\sigma}\right)|_{j^{k+1}_x\gamma} \in TJ^{k+1}\pi
\end{aligned}
\tag{14.6}
$$

realizes a splitting of the exact sequence

$$0 \to V_{\pi_{k+1,k}}J^{k+1}\pi \to TJ^{k+1}\pi \to \pi^*_{k+1,k}(TJ^k\pi) \to 0. \tag{14.7}$$

The *affine translation* generated by Ξ is evidently

$$\nabla_\Xi \colon J^1\pi_{k+1,k} \to V_{\pi_{k+1,k}} J^{k+1}\pi \otimes \pi^*_{k+1,k}(T^*J^k\pi). \tag{14.8}$$

Since $\pi_{k+1,k}$ is an affine bundle, *affine* connections could be studied, as well. Such a connection must represent an affine bundle morphism between $\pi_{k+1,k}$ and $(\pi_{k+1,k})_1$ over $J^k\pi$, which means that the corresponding components are affine in $y^\sigma_{j_1\dots j_{k+1}}$.

The point of the importance of connections on $\pi_{k+1,k}$ in the geometry of equations studied is that *the integral sections of connections on* $\pi_{k+1,k}$ *are (local)* $(k+1)$*-connections on* π. Such a connection $\Gamma^{(k+1)} \in \mathcal{S}(\pi_{k+1,k})$ must satisfy

$$j^1\Gamma^{(k+1)} = \Xi \circ \Gamma^{(k+1)},$$

which in coordinates reads as a first-order system of P.D.E.

$$\begin{aligned}
\frac{\partial \Gamma^\sigma_{j_1\dots j_{k+1}}}{\partial x^i} &= \Xi^\sigma_{j_1\dots j_{k+1} i}(x^r, y^\nu, \dots, y^\nu_{r_1\dots r_k}, \Gamma^\nu_{r_1\dots r_{k+1}}) \\
\frac{\partial \Gamma^\sigma_{j_1\dots j_{k+1}}}{\partial y^\lambda} &= \Xi^\sigma_{j_1\dots j_{k+1}\lambda}(x^r, y^\nu, \dots, y^\nu_{r_1\dots r_k}, \Gamma^\nu_{r_1\dots r_{k+1}}) \\
&\vdots \\
\frac{\partial \Gamma^\sigma_{j_1\dots j_{k+1}}}{\partial y^\lambda_{i_1\dots i_k}} &= \Xi^{\sigma i_1\dots i_k}_{j_1\dots j_{k+1}\lambda}(x^r, y^\nu, \dots, y^\nu_{r_1\dots r_k}, \Gamma^\nu_{r_1\dots r_{k+1}}).
\end{aligned} \tag{14.9}$$

All the characterizations of the integrability of Ξ are easily derived from the general situation for a connection on π (see also Section 7).

15. Characterizable connections

In this part, the ideas of Sec. 13b find the application. First recall that

$$\Bbbk_{\iota_{1,k}} \colon J^1\pi_{k+1,k} \to \widehat{J}^{k+2}\pi$$

is by definition

$$j^1_z\Gamma^{(k+1)} \mapsto \Bbbk(\Gamma^{(k+1)}(z), j^1_z\Gamma^{(k+1)}) = J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}(z). \tag{15.1}$$

Therefore, if $\gamma \in \mathcal{S}_U(\pi)$ is an arbitrary section of the $(k+1)$-connection $\Gamma^{(k+1)}$, then

$$\begin{aligned}
\Bbbk_{\iota_{1,k}}(j^1_{j^k_x\gamma}\Gamma^{(k+1)}) &= J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}(j^k_x\gamma) \\
&= \iota_{1,k+1}(j^{k+2}_x\gamma) \in \iota_{1,k+1}(J^{k+2}\pi).
\end{aligned} \tag{15.2}$$

Secondly, for any $\Gamma^{(k+1)}$,

$$\Bbbk_{\Gamma^{(k+1)}} := \Bbbk_{\iota_{1,k}\circ\Gamma^{(k+1)}\circ\pi_{k+1,k}} : J^1\pi_{k+1,k} \to J^1\pi_{k+1}$$

reads

$$j^1_z\chi \overset{\Bbbk_{\Gamma^{(k+1)}}}{\longrightarrow} J^1(\chi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)}(z), \tag{15.3}$$

which together with (15.2) means that for an integrable $\Gamma^{(k+1)}$ holds

$$\begin{aligned} \Bbbk_{\Gamma^{(k+1)}} \circ j^1\Gamma^{(k+1)} &= \Bbbk_{\iota_{1,k}} \circ j^1\Gamma^{(k+1)} \\ &= J^1(\Gamma^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma^{(k+1)} = \Gamma^{(k+1)(1)}, \end{aligned} \tag{15.4}$$

(see (10.3)).

On the other hand, let $\Xi\colon J^{k+1}\pi \to J^1\pi_{k+1,k}$ be a connection on $\pi_{k+1,k}$, and $\Phi\colon J^{k+1}\pi \to J^1\pi_k$ be a fibered morphism over $J^k\pi$. By (13.20),

$$\Sigma^{a,b}_{\Phi,\Xi} := \Bbbk^{a,b}_{\Phi} \circ \Xi\colon J^{k+1}\pi \to J^1\pi_{k+1}$$

is a connection on π_{k+1} for an arbitrary $a, b \in \mathbb{R}$. In particular, a (local) connection $\Gamma^{(k+1)}$ can be considered representing both the morphism $\Phi = \iota_{1,k} \circ \Gamma^{(k+1)} \circ \pi_{k+1,k}$ and the section of $\pi_{k+1,k}$. Then denoting by

$$\Sigma_{\Gamma^{(k+1)},\Xi} = \Bbbk_{\Gamma^{(k+1)}} \circ \Xi,$$

the following assertion can be presented (cf. Section 13).

Proposition 15.1. *Let $\Gamma^{(k+1)}$ be an integral section of a connection Ξ on $\pi_{k+1,k}$. Then γ is an integral section of $\Gamma^{(k+1)}$ if, and only if, $\Gamma^{(k+1)} \circ j^k\gamma$ is the integral section of $\Sigma_{\Gamma^{(k+1)},\Xi}$.*

By Sec. 6, for an arbitrary connection Ξ on $\pi_{k+1,k}$ and $b \in \mathbb{R}$,

$$\widehat{\Gamma}^{(k+2)}_{\Xi,b} := \Sigma^{0,b}_{\iota_{1,k},\Xi} = \Bbbk^{0,b}_{\iota_{1,k}} \circ \Xi\colon J^{k+1}\pi \to \widehat{J}^{k+2}\pi$$

is a semiholonomic connection on π_{k+1}, which can be analogously to (8.6) decomposed to the $(k+2)$-connection

$$\Gamma^{(k+2)}_{\Xi} := s_{k+1} \circ \widehat{\Gamma}^{(k+2)}_{\Xi,b},$$

and to a certain multiple of the composition $R \circ \Xi$ of the formal curvature R with Ξ. Just the last term finds its importance in the following definition.

DEFINITION 15.1. A connection Ξ on $\pi_{k+1,k}$ will be called *characterizable*, if

$$R \circ \Xi = 0. \tag{15.5}$$

The $(k+2)$-connection

$$\Gamma^{(k+2)}_{\Xi} = \Bbbk_{\iota_{1,k}} \circ \Xi \tag{15.6}$$

will be then called the *characteristic connection* of Ξ. Accordingly, the horizontal distribution $H_{\Gamma^{(k+2)}_{\Xi}}$ will be called the *characteristic distribution* of Ξ and the maximal-dimensional integral manifolds of the characteristic distribution (i.e. $(k+1)$-jets of integral sections of $\Gamma^{(k+2)}_{\Xi}$) are the *characteristics* of Ξ.

PROPOSITION 15.2. *A $(k+2)$-connection $\Gamma^{(k+2)}$ on π is the characteristic connection of a connection Ξ on $\pi_{k+1,k}$ if, and only if, one of the following equivalent conditions holds:*

$$\Gamma^{\sigma}_{j_1\ldots j_{k+1}i} = \Xi^{\sigma}_{j_1\ldots j_{k+1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\ldots r_\ell}_{j_1\ldots j_{k+1}\lambda} y^{\lambda}_{r_1\ldots r_\ell i}; \tag{15.7}$$

$$D_{\Gamma^{(k+2)}i} = D_{\Xi i} + \sum_{\ell=0}^{k} D^{j_1\ldots j_\ell}_{\Xi\lambda} y^{\lambda}_{j_1\ldots j_\ell i}; \tag{15.8}$$

$$h_{\Xi} - h_{\Gamma^{(k+2)}} = \sum_{\ell=0}^{k} D^{j_1\ldots j_\ell}_{\Xi\lambda} \otimes \omega^{\lambda}_{j_1\ldots j_\ell}; \tag{15.9}$$

$$H_{\Gamma^{(k+2)}} = H_{\Xi} \cap C_{\pi_{k+1,k}}. \tag{15.10}$$

Proof. The first three local conditions are immediate consequences of (13.13) and (13.14) together with (8.3), (8.4) and (14.2), (14.3).

Since $H_{\Gamma^{(k+2)}} \subset C_{\pi_{k+1,k}}$ for an arbitrary $\Gamma^{(k+2)}$ and due to $\dim(H_{\Xi} \cap C_{\pi_{k+1,k}}) = n$ for an arbitrary Ξ, (15.10) holds if, and only if, $H_{\Gamma^{(k+2)}} \subset H_{\Xi}$, which is equivalent to (15.8). ∎

In keeping with Section 13, the class of characterizable connections on $\pi_{k+1,k}$ with the same characteristic $(k+2)$-connection on π is generated by the class of $\iota_{1,k}$-admissible deformations on $\pi_{k+1,k}$. More precisely, if we call any such Ξ *associated* to $\Gamma^{(k+2)}$, then for each soldering form

$$\varphi\colon J^{k+1}\pi \to V_{\pi_{k+1,k}} J^{k+1}\pi \otimes \pi^{*}_{k+1,k}(T^{*}J^{k}\pi)$$

satisfying locally

$$\varphi^{\sigma}_{j_1\dots j_{k+1}i} + \sum_{\ell=0}^{k} \varphi^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+1}\lambda} y^{\lambda}_{r_1\dots r_\ell i} = 0, \tag{15.11}$$

$h_\Xi + \varphi$ is the horizontal form of another connection on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$.

Let us finally complete the diagram (13.11) by connections:

$$\begin{array}{ccccccc}
X & \xrightarrow{j^{k+1}\gamma} & J^{k+1}\pi & \xrightarrow{J^1(\Gamma^{(k+1)},\mathrm{id}_X)} & \widehat{J}^{k+2}\pi & = = = & \widehat{J}^{k+2}\pi \\
\mathrm{id}_X\big\downarrow & & \Gamma^{(k+1)}\big\uparrow & & \Gamma^{(k+2)}\big\uparrow & & \Bbbk_{\iota_{1,k}}\big\uparrow \\
X & \xrightarrow{j^{k}\gamma} & J^{k}\pi & \xrightarrow{\Gamma^{(k+1)}} & J^{k+1}\pi & \xrightarrow{\Xi} & J^1\pi_{k+1,k} \\
\mathrm{id}_X\big\downarrow & & \pi_k\big\downarrow & & \pi_{k+1}\big\downarrow & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X\,. & &
\end{array} \tag{15.12}$$

16. The method of characteristics

We start with the statement which turns out to be intrinsically related to both indirect integration methods we will present. On the other hand, it follows the above ideas very naturally, hence the verification is trivial. Suppose $k \geq 0$.

Proposition 16.1. *Let Ξ be a characterizable connection on $\pi_{k+1,k}$, and $\Gamma^{(k+2)}_{\Xi}$ be its characteristic $(k+2)$-connection on π. Then each integral section $\Gamma^{(k+1)}$ of Ξ is a field of paths of $\Gamma^{(k+2)}_{\Xi}$.*

Proof. Let $\Gamma^{(k+1)}$ be an integral section of Ξ. Then by (15.6) and (15.4),

$$\Gamma^{(k+2)}_{\Xi} \circ \Gamma^{(k+1)} = \Bbbk_{\iota_{1,k}} \circ \Xi \circ \Gamma^{(k+1)} = \Bbbk_{\iota_{1,k}} \circ j^1\Gamma^{(k+1)} = \Gamma^{(k+1)(1)},$$

which is just (10.11) for $r = 1$. ∎

Recall that for such $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ it holds by (10.12)

$$H_{\Gamma^{(k+2)}}(z) \equiv C^{\Gamma^{(k+1)}}(z) \subset T_z\Gamma^{(k+1)}(V)$$

for each $z \in \Gamma^{(k+1)}(V)$, which can be expressed by saying that $\Gamma^{(k+1)}$ is an 'integral including manifold' of $H_{\Gamma^{(k+2)}}$.

Since each field of paths is integrable, by Sec. 10 we get:

COROLLARY 16.1. *If Ξ is characterizable and integrable, then its characteristic connection $\Gamma^{(k+2)}_{\Xi}$ is integrable, as well.*

In fact, the maximal integral manifolds of H_{Ξ} (integral sections of Ξ) are foliated by the characteristics, whose equations are by (9.6) and (15.7)

$$\begin{aligned}\frac{\partial^{k+2}\gamma^{\sigma}}{\partial x^{j_1}\dots\partial x^{j_{k+1}}\partial x^{i}} &= \Xi^{\sigma}_{j_1\dots j_{k+1}i}\left(x^r,\gamma^{\nu},\dots,\frac{\partial^{k+1}\gamma^{\nu}}{\partial x^{r_1}\dots\partial x^{r_{k+1}}}\right)\\ &+\sum_{\ell=0}^{k}\Xi^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+1}\lambda}\left(x^r,\gamma^{\nu},\dots,\frac{\partial^{k+1}\gamma^{\nu}}{\partial x^{r_1}\dots\partial x^{r_{k+1}}}\right)\frac{\partial^{\ell+1}\gamma^{\lambda}}{\partial x^{r_1}\dots\partial x^{r_\ell}\partial x^{i}}.\end{aligned}\tag{16.1}$$

In other words, under the integrability conditions, the looking for solutions of the first-order system (14.9) can be transferred to the looking for the solutions of the $(k+2)$-th order system (16.1) – the integral sections of Ξ are 'pieced together' by characteristics.

Moreover, knowing an r-dimensional integral submanifold M_r of H_{Ξ}, the characteristics can be applied when constructing an integral submanifold $M_{\geq r}$ of dimension $\geq r$ containing M_r – this task is the well-known *Cauchy initial problem*. Clearly, the case when M_r in itself is foliated by characteristics must be eliminated, in such a case $M_{\geq r}\equiv M_r$. In this respect, a point $z\in M_r$ can be called *characteristic* (with respect to Ξ) if $T_zM_r\supset H_{\Gamma^{(k+2)}_{\Xi}}(z)$, and the Cauchy problem is solvable just around the non-characteristic points of M_r. It is evident that the integrability of Ξ is not necessary for the integrability of $\Gamma^{(k+2)}_{\Xi}$. Nevertheless, the above method of characteristics can be applied, as well.

The relation between the characterizability of connections on $\pi_{k+1,k}$ and the integrability of $(k+1)$-connections on π is hidden within the following construction, which completes the ideas of Sec. 13.

DEFINITION 16.1. Let $\Gamma^{(k+1)}$ be a $(k+1)$-connection on π. The *formal mixed $\Gamma^{(k+1)}$-curvature map* is the map

$$\kappa_{\Gamma^{(k+1)}}: J^1\pi_{k+1,k}\to\pi^*_{k+1,k}(V_{\pi_{k,k-1}}J^k\pi\otimes\pi^*_k(\Lambda^2T^*X))\tag{16.2}$$

defined for each $j^1_{j^k_x\gamma}\chi\in J^1\pi_{k+1,k}$ by means of the F-N bracket as

$$\kappa_{\Gamma^{(k+1)}}(j^1_{j^k_x\gamma}\chi)=(\chi(j^k_x\gamma),[h_{\Gamma^{(k+1)}}-h_{\chi},h_{\chi}](j^k_x\gamma)).\tag{16.3}$$

The motivation of the definition is similar to that of Def. 13.1; namely, if $\widetilde{\Gamma}^{(k+1)}$ is another $(k+1)$-connection on π, then

$$\begin{aligned}\kappa(\widetilde{\Gamma}^{(k+1)},\Gamma^{(k+1)}) &:= \mathrm{pr}_2 \circ \kappa_{\Gamma^{(k+1)}} \circ j^1\widetilde{\Gamma}^{(k+1)} \\ &= [h_{\Gamma^{(k+1)}} - h_{\widetilde{\Gamma}^{(k+1)}}, h_{\widetilde{\Gamma}^{(k+1)}}]\colon J^k\pi \to V_{\pi_{k,k-1}} J^k\pi \otimes \pi_k^*(\Lambda^2 T^*X)\end{aligned} \tag{16.4}$$

is the so-called *mixed curvature* of the pair $\Gamma^{(k+1)}$ and $\widetilde{\Gamma}^{(k+1)}$. Since

$$\varphi = h_{\Gamma^{(k+1)}} - h_{\widetilde{\Gamma}^{(k+1)}}$$

is a soldering form on π_k, the mixed curvature $\kappa(\widetilde{\Gamma}^{(k+1)},\Gamma^{(k+1)})$ is nothing but the φ-torsion τ_φ of $\widetilde{\Gamma}^{(k+1)}$, which locally means

$$\begin{aligned}&\kappa(\widetilde{\Gamma}^{(k+1)},\Gamma^{(k+1)}) \\ &\quad= \left(D_{\widetilde{\Gamma}^{(k+1)}i}(\varphi^\sigma_{j_1\dots j_k p}) - \frac{\partial \widetilde{\Gamma}^\sigma_{j_1\dots j_k i}}{\partial y^\lambda_{r_1\dots r_k}} \varphi^\lambda_{r_1\dots r_k p} \right) \frac{\partial}{\partial y^\sigma_{j_1\dots j_k}} \otimes dx^i \wedge dx^p,\end{aligned} \tag{16.5}$$

where $\varphi^\sigma_{j_1\dots j_k p} = \Gamma^\sigma_{j_1\dots j_k p} - \widetilde{\Gamma}^\sigma_{j_1\dots j_k p}$. Moreover, due to [45] we have

$$\kappa(\widetilde{\Gamma}^{(k+1)},\Gamma^{(k+1)}) = R_{\Gamma^{(k+1)}} - R_{\widetilde{\Gamma}^{(k+1)}} - \frac{1}{2}[\varphi,\varphi] \tag{16.6}$$

and thus e.g. also

$$\kappa(\widetilde{\Gamma}^{(k+1)},\Gamma^{(k+1)}) - \kappa(\Gamma^{(k+1)},\widetilde{\Gamma}^{(k+1)}) = 2(R_{\Gamma^{(k+1)}} - R_{\widetilde{\Gamma}^{(k+1)}}).$$

Let us return to the definition. First, the local expression of $\kappa_{\Gamma^{(k+1)}}$ can be immediately derived when substituting $y^\sigma_{j_1\dots j_{k+1}}$ for $\widetilde{\Gamma}^\sigma_{j_1\dots j_{k+1}}$, $z^\sigma_{j_1\dots j_{k+1}i}$ for $\frac{\partial \widetilde{\Gamma}^\sigma_{j_1\dots j_{k+1}}}{\partial x^i}$, $z^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+1}\lambda}$ for $\frac{\partial \widetilde{\Gamma}^\sigma_{j_1\dots j_{k+1}}}{\partial y^\lambda_{r_1\dots r_\ell}}$ and $D_i^{k+1,k}$ for $D_{\widetilde{\Gamma}^{(k+1)}i}$ in (16.5); for example if $k=0$, then

$$\mathrm{pr}_2 \circ \kappa_\Gamma = \left(\frac{\partial \Gamma^\sigma_j}{\partial x^i} + \frac{\partial \Gamma^\sigma_j}{\partial y^\lambda} y^\lambda_i - z^\sigma_{ji} - z^\sigma_{j\lambda} y^\lambda_i - z^\sigma_{i\lambda}(\Gamma^\lambda_j - y^\lambda_j) \right) \frac{\partial}{\partial y^\sigma} \otimes dx^i \wedge dx^j.$$

Secondly, (16.6) may be now rewritten to

$$\kappa_{\Gamma^{(k+1)}} = \widehat{R}_{\Gamma^{(k+1)}} - R - \frac{1}{2}\widehat{\kappa}_{\Gamma^{(k+1)}}$$

with $\widehat{R}_{\Gamma^{(k+1)}} := R \circ j^1\Gamma^{(k+1)} \circ (\pi_{k+1,k})_1$ and $\widehat{\kappa}_{\Gamma^{(k+1)}}$ being defined analogously to $\kappa_{\Gamma^{(k+1)}}$ by

$$\widehat{\kappa}_{\Gamma^{(k+1)}}(j^1_{j^k_x\gamma}\chi) = (\chi(j^k_x\gamma), [h_{\Gamma^{(k+1)}} - h_\chi, h_{\Gamma^{(k+1)}} - h_\chi](j^k_x\gamma)).$$

Let now Ξ be a connection on $\pi_{k+1,k}$. Then

$$\kappa_{\Gamma^{(k+1)},\Xi} = \kappa_{\Gamma^{(k+1)}} \circ \Xi \colon J^{k+1}\pi \to \pi^*_{k+1,k}(V_{\pi_{k+1,k}}J^k\pi \otimes \pi^*_k(\Lambda^2T^*X))$$

represents a 'curvature-like' term generated by $\Gamma^{(k+1)}$ and Ξ, where

$$\widehat{R}_{\Gamma^{(k+1)}} \circ \Xi = R \circ j^1\Gamma^{(k+1)} \circ \pi_{k+1,k}$$

does not depend on Ξ and it vanishes if, and only if, $\Gamma^{(k+1)}$ is integrable, $R \circ \Xi$ does not depend on $\Gamma^{(k+1)}$ and it vanishes if, and only if, Ξ is characterizable, and finally $\widehat{\kappa}_{\Gamma^{(k+1)}} \circ \Xi$ integrates $\Gamma^{(k+1)}$ and Ξ together: if $\Gamma^{(k+1)}$ is an integral section of Ξ, then $\widehat{\kappa}_{\Gamma^{(k+1)}} \circ \Xi \circ \Gamma^{(k+1)} = \widehat{\kappa}_{\Gamma^{(k+1)}} \circ j^1\Gamma^{(k+1)} = 0$. In particular, if Ξ is characterizable with the integral section $\Gamma^{(k+1)}$, then $\kappa_{\Gamma^{(k+1)},\Xi} = 0$.

17. The method of fields of paths – Part I

The ideas of Sections 16 and 10 suggest the second indirect integration method, dual to the method of characteristics. Actually (by Prop. 16.1 and Coroll. 16.1), if Ξ is an integrable characterizable connection on $\pi_{k+1,k}$ associated to the $(k+2)$-connection $\Gamma^{(k+2)}$ on π, then each integral section of $\Gamma^{(k+2)}$ is locally embedded in a field of paths $\Gamma^{(k+1)}$ which is the integral section of Ξ. Accordingly, the problem of the looking for the solutions of the $(k+2)$-th order system represented by $\Gamma^{(k+2)}$ can be transferred to the looking for an integrable and characterizable connection Ξ on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$, and after this to the solving of the corresponding $(k+1)$-th order fields of paths. As already mentioned (cf. Sec. 10), if $\Gamma^{(k)}$ is a field of paths of $\Gamma^{(k+1)}$ which is the field of paths of $\Gamma^{(k+2)}$, then $\Gamma^{(k)}$ is a field of paths of $\Gamma^{(k+2)}$, and the procedure can be repeated.

Definition 17.1. Let $\Gamma^{(k+2)}$ be an integrable $(k+2)$-connection on π. A (generally local) integrable connection Ξ on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$ is called the *integral* of $\Gamma^{(k+2)}$.

Denoting here by $\Xi_{(\ell+1)}$ the integral of $\Gamma^{(\ell+2)}$, the following diagram can be presented.

$$
\begin{array}{ccccccc}
X & \xrightarrow{j^{k+1}\gamma} & J^{k+1}\pi & \xrightarrow{\Gamma^{(k+2)}} & J^{k+2}\pi & & \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_{k+1,k}}\big\downarrow & & {\scriptstyle \Gamma^{(k+2)}}\big\uparrow & & \\
X & \xrightarrow{j^{k}\gamma} & J^{k}\pi & \xrightarrow{\Gamma^{(k+1)}} & J^{k+1}\pi & \xrightarrow{\Xi_{(k+1)}} & J^1\pi_{k+1,k} \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_{k,k-1}}\big\downarrow & & {\scriptstyle \Gamma^{(k+1)}}\big\uparrow & & \\
X & \xrightarrow{j^{k-1}\gamma} & J^{k-1}\pi & \xrightarrow{\Gamma^{(k)}} & J^{k}\pi & \xrightarrow{\Xi_{(k)}} & J^1\pi_{k,k-1} \\
\vdots & & \vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots \\
X & \xrightarrow{j^{1}\gamma} & J^{1}\pi & \xrightarrow{\Gamma^{(2)}} & J^{2}\pi & \xrightarrow{\Xi_{(2)}} & J^1\pi_{2,1} \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi_{1,0}}\big\downarrow & & {\scriptstyle \Gamma^{(2)}}\big\uparrow & & \\
X & \xrightarrow{\gamma} & Y & \xrightarrow{\Gamma} & J^{1}\pi & \xrightarrow{\Xi} & J^1\pi_{1,0} \\
{\scriptstyle \mathrm{id}_X}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X\,. & &
\end{array}
$$

Natural question on the existence of integrals for a given $(k+2)$-connection may be considered both locally and globally. The former case can be answered in terms of first integrals.

Notice first that due to (15.10), each first integral of a characterizable Ξ is the first integral of its characteristic $\Gamma^{(k+2)}$. The converse is not true in general, nevertheless, the following assertion holds.

PROPOSITION 17.1. *Let $\Gamma^{(k+2)}$ be an integrable $(k+2)$-connection on π and $\{a^1, \dots, a^K\}$, where $K = \dim \pi_{k+1,k}$, be a set of independent first integrals of $\Gamma^{(k+2)}$, defined on some open subset $W \subset J^{k+1}\pi$. If the matrix*

$$A = \left(\frac{\partial a^L}{\partial y^\sigma_{j_1 \dots j_{k+1}}} \right) \tag{17.1}$$

is regular on W, then there is an integral Ξ of $\Gamma^{(k+2)}$ on W, defined by

$$H_\Xi = \operatorname{anih}\{da^1, \dots, da^K\}. \tag{17.2}$$

Proof. First it should be stressed that we suppose $W \subset \pi^{-1}_{k+1,0}(V)$, where (V, ψ) is a fibered chart on Y.

By definition, the distribution (17.2) is completely integrable. Let us denote by $(A^{\sigma}_{j_1 \ldots j_{k+1} L})$ the inverse matrix to A given by (17.1), where σ and $j_1 \ldots j_{k+1}$ label the rows and L the columns. Then the annihilators of H_{Ξ} are

$$dy^{\sigma}_{j_1 \ldots j_{k+1}} + A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial x^i} dx^i + \sum_{\ell=0}^{k} A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial y^{\lambda}_{i_1 \ldots i_\ell}} dy^{\lambda}_{i_1 \ldots i_\ell} \tag{17.3}$$

and it remains to show that the characteristic connection to Ξ, given by

$$\begin{aligned} \Xi^{\sigma}_{j_1 \ldots j_{k+1} i} &= -A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial x^i} \\ \Xi^{\sigma}_{j_1 \ldots j_{k+1} \lambda} &= -A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial y^{\lambda}} \\ &\vdots \\ \Xi^{\sigma i_1 \ldots i_k}_{j_1 \ldots j_{k+1} \lambda} &= -A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial y^{\lambda}_{i_1 \ldots i_k}}, \end{aligned} \tag{17.4}$$

is just $\Gamma^{(k+2)}$ (this obviously automatically determines also the characterizability). Since

$$\begin{aligned} &\Xi^{\sigma}_{j_1 \ldots j_{k+1} i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1 \ldots r_\ell}_{j_1 \ldots j_{k+1} \lambda} y^{\lambda}_{r_1 \ldots r_\ell i} \\ &\qquad = -A^{\sigma}_{j_1 \ldots j_{k+1} L} \left(\frac{\partial a^L}{\partial x^i} + \sum_{\ell=0}^{k} \frac{\partial a^L}{\partial y^{\lambda}_{r_1 \ldots r_\ell}} y^{\lambda}_{r_1 \ldots r_\ell i} \right) \\ &\qquad = A^{\sigma}_{j_1 \ldots j_{k+1} L} \frac{\partial a^L}{\partial y^{\lambda}_{r_1 \ldots r_{k+1}}} \Gamma^{\lambda}_{r_1 \ldots r_{k+1} i} = \Gamma^{\sigma}_{j_1 \ldots j_{k+1} i}, \end{aligned}$$

the proof is completed (see (15.7)). ∎

It should be noticed that if $\Gamma^{(k+2)}$ is integrable, then the existence of a set of independent first integrals satisfying the condition (17.1) is due to the horizontality of $H_{\Gamma^{(k+2)}}$.

The problem of global integrals is much more complicated. In fact, two questions appear in terms of the above considerations. First, whether there exist transformations 'converse' to those of Sec. 15, allowing a global assignement $\Gamma^{(k+2)} \mapsto \Xi$, and secondly, what conditions force Ξ to be the integral of $\Gamma^{(k+2)}$? As already announced, especially the first question represents an *open problem* for $\dim X > 1$ and $k \geq 1$. For $k = 0$, the following assertion can be presented, reformulating the corresponding result of [22]. It should be stressed that all concepts involved are *global*.

PROPOSITION 17.2. *Let $\Gamma^{(2)}$ be a 2-connection on π and Λ a linear connection on X. Then there is a connection*

$$\Xi_a^\Lambda = g_a^\Lambda \circ j^1\Gamma^{(2)} \tag{17.5}$$

on $\pi_{1,0}$ associated to $\Gamma^{(2)}$, being determined in virtue of a natural fibered morphism

$$g_a^\Lambda \colon J^1\pi_{2,1} \to J^1\pi_{1,0} \tag{17.6}$$

over $J^1\pi$ which is locally expressed by

$$\begin{aligned} z_{i\lambda}^\sigma &= \frac{1}{2}(z_{ik\lambda}^{\sigma k} + \delta_\lambda^\sigma \Lambda_{ki}^k) + a\delta_\lambda^\sigma(\Lambda_{ik}^k - \Lambda_{ki}^k), \\ z_{ij}^\sigma &= y_{ij}^\sigma - z_{i\lambda}^\sigma y_j^\lambda \end{aligned} \tag{17.7}$$

for an arbitrary $a \in \mathbb{R}$.

It appears that the presence of a linear connection on the base X is essential, it cannot be omitted. If $\mathcal{T}\colon TX \to V_{\tau_X}TX \otimes \Lambda^2T^*X$,

$$\mathcal{T} = \Lambda_{ij}^k \frac{\partial}{\partial x^k} \otimes dx^i \wedge dx^j,$$

is the classical *torsion* of Λ, then its *contraction* is a one-form

$$\widehat{\mathcal{T}} = \mathcal{T}_i\, dx^i$$

on X with $\mathcal{T}_i = \Lambda_{ik}^k - \Lambda_{ki}^k$. It can be shown that the linear connection Λ on X canonically generates the *soldering form of type* $\widehat{\mathcal{T}}$ on $\pi_{1,0}$, which locally reads

$$S_\Lambda = \mathcal{T}_i \frac{\partial}{\partial y_i^\sigma} \otimes (dy^\sigma - y_j^\sigma\, dx^j), \tag{17.8}$$

hence, it is trivial if, and only if, Λ is symmetric (torsion free).

As a consequence, the connection (17.5) can be written in the form

$$\Xi_a^\Lambda = \Xi_0^\Lambda + a\, S_\Lambda, \tag{17.9}$$

where the components of Ξ_0^Λ are by (17.7)

$$\begin{aligned}\Xi^\sigma_{i\lambda} &= \frac{1}{2}\left(\frac{\partial\Gamma^\sigma_{ik}}{\partial y^\lambda_k} + \delta^\sigma_\lambda\Lambda^k_{ki}\right),\\ \Xi^\sigma_{ij} &= \Gamma^\sigma_{ij} - \Xi^\sigma_{i\lambda}y^\lambda_j,\end{aligned} \tag{17.10}$$

with Γ^σ_{ij} being the components of $\Gamma^{(2)}$.

Recall finally that the family of connections on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ can be obtained by means of $\iota_{1,0}$-admissible deformations on $\pi_{1,0}$ (see (15.11)), where $\iota_{1,0} \equiv \mathrm{id}_{J^1\pi}$.

Remark 17.1. On the other hand, there is a construction of global associated connections for $k \geq 0$, but with $\dim X = 1$, established in [62]. In this situation, the role of a linear connection is played by a *volume form* on X.

18. Strong horizontal distributions

While the above ideas and results on the relationships between horizontal distributions had to do with integral methods for the corresponding connections, in what follows we deal with in some sense complementary 'strong horizontal' distributions, which will prove to be related with the symmetries of equations studied. In what follows, $k \geq 0$.

First recall that a non-vanishing (1,1)-tensor field F of constant rank on a manifold M is called an *$f(3,-1)$-structure* if $F^3 - F = 0$. Regarding a formula

$$F^3 - F = F(F-I)(F+I),$$

there is a direct sum decomposition on TM induced by any such F :

$$TM = \mathrm{Im}(F^2 - F) \oplus \mathrm{Im}(F^2 - I) \oplus \mathrm{Im}(F^2 + F) \tag{18.1}$$

with the eigenspaces corresponding to the eigenvalues -1, 0, $+1$, respectively.

Proposition 18.1. *Let Ξ be a characterizable connection on $\pi_{k+1,k}$ and $\Gamma^{(k+2)}$ be its characteristic connection. Then*

$$F_\Xi = 2h_\Xi - h_{\Gamma^{(k+2)}} - I \tag{18.2}$$

is an $f(3,-1)$-structure on $J^{k+1}\pi$ of rank $= \dim \pi_{k+1}$.

Proof. Since both h_Ξ and $h_{\Gamma(k+2)}$ are projectors and due to Prop. 15.2, which implies

$$h_\Xi \circ h_{\Gamma(k+2)} = h_{\Gamma(k+2)} \circ h_\Xi = h_{\Gamma(k+2)}. \tag{18.3}$$

it is easy to verify that

$$F_\Xi^2 = v_{\Gamma(k+2)}, \tag{18.4}$$

which immediately leads to $F_\Xi^3 - F_\Xi = 0$. The rank of F_Ξ is evident from its local expression. ∎

Next assertion should be viewed as the external version of Prop. 11.1.

COROLLARY 18.1. *Each characterizable connection Ξ on $\pi_{k+1,k}$ determines a direct sum decomposition of $TJ^{k+1}\pi$:*

$$TJ^{k+1}\pi = V_{\pi_{k+1,k}}J^{k+1}\pi \oplus H_{\Gamma(k+2)} \oplus H_{F_\Xi}, \tag{18.5}$$

given for any $\zeta^{(k+1)} \in TJ^{k+1}\pi$ by

$$\zeta^{(k+1)} = v_\Xi(\zeta^{(k+1)}) + h_{\Gamma(k+2)}(\zeta^{(k+1)}) + (h_\Xi - h_{\Gamma(k+2)})(\zeta^{(k+1)}). \tag{18.6}$$

Proof. Directly by (18.1), since

$$F_\Xi^2 - F_\Xi = 2v_\Xi, \quad F_\Xi^2 - I = -h_{\Gamma(k+2)}, \quad F_\Xi^2 + F_\Xi = 2(h_\Xi - h_{\Gamma(k+2)}). \tag{18.7}$$

∎

Evidently,

$$H_\Xi = H_{\Gamma(k+2)} \oplus H_{F_\Xi}, \tag{18.8}$$

which suggest the following definition.

DEFINITION 18.1. The eigenspace $H_{F_\Xi} = \mathrm{Im}(h_\Xi - h_{\Gamma(k+2)})$ will be called the *strong horizontal distribution* generated by Ξ.

In fact, the strong horizontality means together with (18.8) also

$$V_{\pi_{k+1}}J^{k+1}\pi = V_{\pi_{k+1,k}}J^{k+1}\pi \oplus H_{F_\Xi}. \tag{18.9}$$

It appears that there is further interesting object which can make the role of the strong horizontal distribution more transparent.

Definition 18.2. A *reduced connection* on $\pi_{k+1,k}$ is such a section

$$\Gamma_{(k+1,k)} \colon \pi^*_{k+1,k}(V_{\pi_k} J^k\pi) \to V_{\pi_{k+1}} J^{k+1}\pi$$

which is a vector bundle morphism between $\pi^*_{k+1,k}(\tau_{J^k\pi})$ and $\tau_{J^{k+1}\pi}$ over $J^{k+1}\pi$.

In other words, a reduced connection $\Gamma_{(k+1,k)}$ represents a lift

$$\mathcal{X}^v_X(\pi_{k+1,k}) \to \mathcal{X}^v_X(J^{k+1}\pi),$$

or equivalently a splitting of the exact sequence

$$0 \to V_{\pi_{k+1,k}} J^{k+1}\pi \to V_{\pi_{k+1}} J^{k+1}\pi \to \pi^*_{k+1,k}(V_{\pi_k} J^k\pi) \to 0. \tag{18.10}$$

In coordinates,

$$\begin{aligned} &\left(j^{k+1}_x\gamma, \sum_{\ell=0}^{k} \zeta^\sigma_{j_1\dots j_\ell} \frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}\Big|_{j^k_x\gamma} \right) \overset{\Gamma_{(k+1,k)}}{\longmapsto} \\ &\qquad \sum_{\ell=0}^{k} \zeta^\lambda_{i_1\dots i_\ell} \left(\frac{\partial}{\partial y^\lambda_{i_1\dots i_\ell}}\Big|_{j^{k+1}_x\gamma} + \Gamma^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda} \frac{\partial}{\partial y^\sigma_{j_1\dots j_{k+1}}}\Big|_{j^{k+1}_x\gamma} \right), \end{aligned} \tag{18.11}$$

where the functions $\Gamma^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda} \in \mathcal{F}(J^{k+1}\pi)$ are the *components* of $\Gamma_{(k+1,k)}$.

In particular, $\Gamma_{(k+1,k)}$ generates a lift

$$\mathcal{X}^v_X(J^k\pi) \to \mathcal{X}^v_X(J^{k+1}\pi)$$

by the composition

$$\begin{aligned} &J^{k+1}\pi \xrightarrow{\mathrm{id}\times\pi_{k+1,k}} J^{k+1}\pi \times J^k\pi \xrightarrow{\mathrm{id}\times\zeta^{(k)}} \\ &\qquad\qquad J^{k+1}\pi \times V_{\pi_k} J^k\pi \xrightarrow{\Gamma_{(k+1,k)}} V_{\pi_{k+1}} J^{k+1}\pi. \end{aligned} \tag{18.12}$$

This lift we will denote by $\Gamma_{(k+1,k)}(\zeta^{(k)})$. Accordingly, each reduced connection $\Gamma_{(k+1,k)}$ on $\pi_{k+1,k}$ can be identified with the decomposition

$$V_{\pi_{k+1}} J^{k+1}\pi = V_{\pi_{k+1,k}} J^{k+1}\pi \oplus H_{\Gamma_{(k+1,k)}}, \tag{18.13}$$

where the generators of the $(\dim \pi_{k+1})$-dimensional distribution

$$H_{\Gamma_{(k+1,k)}} := \mathrm{Im}\Gamma_{(k+1,k)},$$

are evident from (18.11).

DEFINITION 18.3. A vector field $\zeta^{(r)} \in \mathcal{X}^v_X(J^r\pi)$, $0 \leq r \leq k$, will be called the *r-integral section* of a reduced connection $\Gamma_{(k+1,k)}$ on $\pi_{k+1,k}$ if

$$\Gamma_{(k+1,k)}(\mathcal{J}^{k-r}\zeta^{(r)}) = \mathcal{J}^{k-r+1}\zeta^{(r)}. \tag{18.14}$$

On the other hand, reduced connections on $\pi_{k+1,k}$ work within the 2-fibered manifold (11.20). More precisely, if we are given a reduced connection $\Gamma_{(k+1,k)}$ on $\pi_{k+1,k}$ and a $(k+1)$-connection $\Gamma^{(k+1)}$ on π, then the composition

$$\begin{aligned} V_{\pi_k}J^k\pi \overset{\tau_{J^k\pi}\times\,\mathrm{id}}{\longrightarrow} J^k\pi \times V_{\pi_k}J^k\pi \overset{\Gamma^{(k+1)}\times\,\mathrm{id}}{\longrightarrow} \\ J^{k+1}\pi \times V_{\pi_k}J^k\pi \overset{\Gamma_{(k+1,k)}}{\longrightarrow} V_{\pi_{k+1}}J^{k+1}\pi, \end{aligned} \tag{18.15}$$

locally expressed by

$$\begin{aligned} &\sum_{\ell=0}^{k} \zeta^{\sigma}_{j_1\dots j_\ell}\frac{\partial}{\partial y^{\sigma}_{j_1\dots j_\ell}}|_z \longmapsto \\ &\sum_{\ell=0}^{k} \zeta^{\lambda}_{i_1\dots i_\ell}\left(\frac{\partial}{\partial y^{\lambda}_{i_1\dots i_\ell}}|_{\Gamma^{(k+1)}(z)} + \Gamma^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda}\frac{\partial}{\partial y^{\sigma}_{j_1\dots j_{k+1}}}|_{\Gamma^{(k+1)}(z)}\right), \end{aligned} \tag{18.16}$$

defines canonically (cf. (11.17) and (11.18)) the $(k+1)$-connection on $\pi \circ \tau_Y : V_\pi Y \to X$. In what follows, both (18.16) and this connection we will denote (with a slight inaccuracy) by $\Gamma_{(k+1,k)} \circ \Gamma^{(k+1)}$.

The point is that an arbitrary characterizable connection Ξ on $\pi_{k+1,k}$ naturally generates a reduced connection $\Gamma^{\Xi}_{(k+1,k)}$ on $\pi_{k+1,k}$ by the restriction of the horizontal lift (14.6) to $\pi^*_{k+1,k}(V_{\pi_k}J^k\pi)$, i.e.

$$\Gamma^{\Xi}_{(k+1,k)} = h_{\Xi}|_{\pi^*_{k+1,k}(V_{\pi_k}J^k\pi)}, \tag{18.17}$$

which in coordinates reads

$$\Gamma^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda} = \Xi^{\sigma i_1\dots i_\ell}_{j_1\dots j_{k+1}\lambda}. \tag{18.18}$$

The evident fact that

$$H_{\Gamma^{\Xi}_{(k+1,k)}} \equiv H_{F_\Xi} = H_\Xi \cap V_{\pi_{k+1}}J^{k+1}\pi$$

results in the reformulation of Corollary 18.1.

Proposition 18.2. *Each characterizable connection* Ξ *on* $\pi_{k+1,k}$ *splits into the direct sum of its characteristic connection* $\Gamma_{\Xi}^{(k+2)}$ *defined by* (15.6) *and the reduced connection* $\Gamma_{(k+1,k)}^{\Xi}$ *defined by* (18.17).

Clearly, $\Gamma_{\Xi}^{(k+2)}$ corresponds just to the lift $h_{\Xi}|_{H_{\pi_{k+1,k}}}$, i.e. $h_{\Gamma_{\Xi}^{(k+2)}} \equiv h_{\Xi} \circ h$.

As already announced, $\Gamma_{(k+1,k)}^{\Xi}$ (or equivalently the strong horizontal distribution) is closely related to the symmetries; more precisely, to the symmetries of the integral sections of Ξ (and thus by Sec. 11 also of $\Gamma_{\Xi}^{(k+2)}$).

Proposition 18.3. *Let* Ξ *be a characterizable connection on* $\pi_{k+1,k}$ *and* $\Gamma_{(k+1,k)}^{\Xi}$ *the associated reduced connection. If* $\Gamma^{(k+1)} \in \mathcal{S}_V(\pi_{k+1,k})$ *is an integral section of* Ξ, *then*

$$\Gamma_{(k+1,k)} \circ \Gamma^{(k+1)} \equiv V\Gamma^{(k+1)} \tag{18.19}$$

on $\tau_{J^k\pi}^{-1}(V)$. *Consequently, if* $\zeta^{(r)} \in \mathcal{X}_X^v(J^r\pi)$ *is an* r*-integral section of* $\Gamma_{(k+1,k)}^{\Xi}$, *then* $\zeta^{(r)}|_{\pi_{k,r}(V)}$ *is the* r*-symmetry of* $\Gamma^{(k+1)}$.

Proof. Directly by (11.17), (18.16), (18.18) and (14.9). ∎

Roughly speaking, the strong horizontal distribution generated by Ξ consists of the prolongations of the symmetries. This must be consistent with the results of Sec. 11 (see Prop. 11.6). In fact, if Ψ is a linear connection on $\tau_{J^k\pi}$, then the composition

$$J^{k+1}\pi \times V_{\pi_k} J^k\pi \xrightarrow{\mathrm{id}\times\Psi} J^{k+1}\pi \times J^1\tau_{J^k\pi} \xrightarrow{\Bbbk} V_{\pi_{k+1}} J^{k+1}\pi$$

is a reduced connection $\Gamma_{(k+1,k)}$ on $\pi_{k+1,k}$ such that for an arbitrary $(k+1)$-connection $\Gamma^{(k+1)}$ it holds $\Gamma_{(k+1,k)} \circ \Gamma^{(k+1)} = \Bbbk_{\Gamma^{(k+1)}} \circ \Psi$.

Remark 18.1. In order to certify the name *reduced connection*, we should recall that there is a natural identification of the tangent space $T(J^k\pi)_x$ of the fiber $(J^k\pi)_x = \pi_k^{-1}(x)$ with the fiber $\rho_k^{-1}(x)$, where $\rho = \pi \circ \tau_Y \colon V_\pi Y \to X$; i.e.

$$\left(V_{\pi_k} J^k\pi\right)_x \cong T(J^k\pi)_x.$$

for each $x \in X$. Consequently, each reduced connection $\Gamma_{(k+1,k)}$ on $\pi_{k+1,k}$ defines for a fixed x the mapping

$$(J^{k+1}\pi)_x \times_{(J^k\pi)_x} T(J^k\pi)_x \to T(J^{k+1}\pi)_x,$$

which is the horizontal lift with respect to a uniquely defined connection on

$$\pi_{k+1,k}\colon (J^{k+1}\pi)_x \to (J^k\pi)_x$$

(clearly, we have to consider the connected components of fibers only).

19. An example

Let us consider two connections Ξ_1, Ξ_2 on $\pi_{1,0}$ with the fibration being studied in examples of Sec. 12:

$$\begin{aligned}
\Xi_1: \quad D_{\Xi x} &= \frac{\partial}{\partial x} + (v + yv_y - \frac{1}{2}xv_x)\frac{\partial}{\partial u_y} + (u + yu_y - \frac{1}{2}xu_x)\frac{\partial}{\partial v_y} \\
D_{\Xi y} &= \frac{\partial}{\partial y} + v\frac{\partial}{\partial u_x} + \frac{1}{2}xv_y\frac{\partial}{\partial u_y} + u\frac{\partial}{\partial v_x} + \frac{1}{2}xu_y\frac{\partial}{\partial v_y} \\
D_{\Xi u} &= \frac{\partial}{\partial u} + y\frac{\partial}{\partial v_x} + \frac{1}{2}x\frac{\partial}{\partial v_y} \\
D_{\Xi v} &= \frac{\partial}{\partial v} + y\frac{\partial}{\partial u_x} + \frac{1}{2}x\frac{\partial}{\partial u_y}. \\
\Xi_2: \quad D_{\Xi x} &= \frac{\partial}{\partial x} + v\frac{\partial}{\partial u_y} + u\frac{\partial}{\partial v_y} \\
D_{\Xi y} &= \frac{\partial}{\partial y} + v\frac{\partial}{\partial u_x} + u\frac{\partial}{\partial v_x} \\
D_{\Xi u} &= \frac{\partial}{\partial u} + y\frac{\partial}{\partial v_x} + x\frac{\partial}{\partial v_y} \\
D_{\Xi v} &= \frac{\partial}{\partial v} + y\frac{\partial}{\partial u_x} + x\frac{\partial}{\partial u_y}.
\end{aligned}$$

Both connections are characterizable with the characteristic 2-connection $\Gamma^{(2)}$ presented by (12.4). While Ξ_2 is integrable with integral sections which differ from (12.1) by additive constants only, Ξ_1 is not integrable. The corresponding 2-dimensional strong horizontal distributions are generated by $D_{\Xi u}$, $D_{\Xi v}$ and for Ξ_1 it means that

$$\begin{aligned}
\mathcal{J}^1\zeta_1 &= uD_{\Xi u} + vD_{\Xi v} \\
\mathcal{J}^1\zeta_2 &= e^{xy}(D_{\Xi u} + D_{\Xi v})
\end{aligned}$$

for the symmetries ζ_1, ζ_2 of Γ (12.1) expressed by

$$\zeta_1 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v},$$
$$\zeta_2 = e^{xy}\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right).$$

20. The method of fields of paths – part II.

Let us complete the ideas of Sec. 17 within the framework of the 2-fibered manifold

$$J^{k+r}\pi \xrightarrow{\pi_{k+r,k}} J^k\pi \xrightarrow{\pi_k} X.$$

where $r \geq 2$. The corresponding diagram is now

$$(20.1)\qquad \begin{array}{ccccccc} X & \xleftarrow{(\pi_k)_1} & J^1\pi_k & \xleftarrow{J^1(\pi_{k+r,k},\mathrm{id}_X)} & J^1\pi_{k+r} & & \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle (\pi_k)_{1,0}} & & \downarrow{\scriptstyle (\pi_{k+r})_{1,0}} & & \\ X & \xleftarrow{\pi_k} & J^k\pi & \xleftarrow{\pi_{k+r,k}} & J^{k+r}\pi & \xleftarrow{(\pi_{k+r,k})_{1,0}} & J^1\pi_{k+r,k} \\ \downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \pi_k} & & \downarrow{\scriptstyle \pi_{k+r}} & & \\ X & \xleftarrow{\mathrm{id}_X} & X & \xleftarrow{\mathrm{id}_X} & X\,. & & \end{array}$$

The map

$$(20.2)\qquad \Bbbk\colon J^1\pi_k \times_{J^k\pi} J^1\pi_{k+r,k} \to J^1\pi_{k+r},$$

defined for $\psi \in \mathcal{S}_{\mathrm{loc}}(\pi_k)$ and $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$, $\mathrm{Im}\psi \subset \mathrm{Dom}\varphi$, by

$$(20.3)\qquad \Bbbk(j^1_x\psi, j^1_{\psi(x)}\varphi) = j^1_x(\varphi \circ \psi).$$

locally does not effect the coordinates

$$x^i, y^\sigma, \ldots, y^\sigma_{j_1\ldots j_{k+r}}, y^\sigma_{;i}, \ldots, y^\sigma_{j_1\ldots j_k;i},$$

and

$$(20.4)\qquad \Bbbk\colon \begin{cases} y^\sigma_{j_1\ldots j_{k+1};i} = z^\sigma_{j_1\ldots j_{k+1}i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\ldots r_\ell}_{j_1\ldots j_{k+1}\lambda} y^\lambda_{r_1\ldots r_\ell;i} \\ \qquad\vdots \\ y^\sigma_{j_1\ldots j_{k+r};i} = z^\sigma_{j_1\ldots j_{k+r}i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\ldots r_\ell}_{j_1\ldots j_{k+r}\lambda} y^\lambda_{r_1\ldots r_\ell;i} \end{cases}$$

with z's being the induced coordinates on $J^1\pi_{k+r,k}$. Clearly, there is a natural candidate for a morphism between $\pi_{k+r,k}$ and $(\pi_k)_{1,0}$ over $J^k\pi$; namely, denote by

$$\Phi_0 = \iota_{1,k} \circ \pi_{k+r,k+1} \colon J^{k+r}\pi \to J^1\pi_k \tag{20.5}$$

the composition, whose coordinate expression coincides with (13.14). Then the affine morphism

$$\Bbbk_{\Phi_0} \colon J^1\pi_{k+r,k} \to J^1\pi_{k+r}$$

defines an affine subbundle

$$A_{\pi_{k+r,k}} = \ker \mathrm{Sp}_{\Phi_0} \subset J^1\pi_{k+r}$$

(see Sec. 13), i.e. $A_{\pi_{k+r,k}}$ consists of the points $z \in J^1\pi_{k+r}$ satisfying

$$\iota_{1,k} \circ \pi_{k+r,k+1} \circ (\pi_{k+r})_{1,0}(z) = J^1(\pi_{k+r,k}, \mathrm{id}_X)(z). \tag{20.6}$$

Following the terminology of Sec. 6, such elements will be called $\pi_{k+r,k}$-*semi-holonomic jets*; the local expression of (20.6) is again just (13.4). In fact,

$$\Bbbk_{\Phi_0} \colon \begin{cases} y^\sigma_{;i} = y^\sigma_i \\ \quad\vdots \\ y^\sigma_{j_1\ldots j_k;i} = y^\sigma_{j_1\ldots j_k i} \\ y^\sigma_{j_1\ldots j_{k+1};i} = z^\sigma_{j_1\ldots j_{k+1} i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\ldots r_\ell}_{j_1\ldots j_{k+1}\lambda} y^\lambda_{r_1\ldots r_\ell\, i} \\ \quad\vdots \\ y^\sigma_{j_1\ldots j_{k+r};i} = z^\sigma_{j_1\ldots j_{k+r} i} + \displaystyle\sum_{\ell=0}^{k} z^{\sigma r_1\ldots r_\ell}_{j_1\ldots j_{k+r}\lambda} y^\lambda_{r_1\ldots r_\ell\, i}. \end{cases} \tag{20.7}$$

Thus there is a canonical inclusion

$$J^{k+r+1}\pi \subset \widehat{J}^{k+r+1}\pi \subset A_{\pi_{k+r,k}}, \tag{20.8}$$

which corresponds to the associated vector bundle

$$\overline{A}_{\pi_{k+r,k}} = V_{\pi_{k+r,k}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X) \subset V_{\pi_{k+r}} J^{k+r}\pi \otimes \pi^*_{k+r}(T^*X). \tag{20.9}$$

Remark here that the study of invariant subspaces of the above nature has been presented in [24], studied by means of the methods of natural operators.

Notice now some properties of the sections of $\pi_{k+r,k}$, called *jet fields*; again, we work with global sections for the simplicity only, the same applies (under appropriate restrictions) for the local ones.

A section $\gamma \in \mathcal{S}_U(\pi)$ will be called an *integral section* (or a *path*) of a jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ if it is the solution of the equation $\mathcal{E}^\varphi = \varphi(J^k\pi) \subset J^{k+r}\pi$, i.e. if $\varphi \circ j^k\gamma = j^{k+r}\gamma$ on U. In this respect, φ will be called *integrable* if there is an integral section of φ through each point of Y. In coordinates, the equations of φ are

$$\begin{aligned} y^\sigma_{j_1\dots j_{k+1}} &= \varphi^\sigma_{j_1\dots j_{k+1}} \\ &\vdots \\ y^\sigma_{j_1\dots j_{k+r}} &= \varphi^\sigma_{j_1\dots j_{k+r}} \end{aligned} \tag{20.10}$$

with the *components* of φ being functions on $J^k\pi$.

For an arbitrary jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$, there is a distinguished associated projection; namely, by

$$\Gamma^{(k+1)}_\varphi = \pi_{k+r,k+1} \circ \varphi \tag{20.11}$$

we get a $(k+1)$-connection $\Gamma^{(k+1)}_\varphi$ on π; in coordinates,

$$\Gamma^\sigma_{j_1\dots j_{k+1}} = \varphi^\sigma_{j_1\dots j_{k+1}}. \tag{20.12}$$

PROPOSITION 20.1. *A jet field $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is integrable if, and only if, $\Gamma^{(k+1)}_\varphi$ is integrable and $\varphi = \Gamma^{(k+1)(r-1)}_\varphi$.*

Proof. By Sec. 10, if $\Gamma^{(k+1)}_\varphi$ is integrable, so is its prolongation. Conversely, if γ is an integral section of φ, then $\Gamma^{(k+1)}_\varphi \circ j^k\gamma = \pi_{k+r,k+1} \circ \varphi \circ j^k\gamma = \pi_{k+r,k+1} \circ j^{k+r}\gamma = j^{k+1}\gamma$, so that if φ is integrable, so is $\Gamma^{(k+1)}_\varphi$. The rest of the assertion is evident from (20.10) and (10.6). ∎

As for higher-order connections, there is an n-dimensional π_{k+r-1}-horizontal distribution H_φ on $J^{k+r-1}\pi$ naturally associated with φ. In fact,

$$H_\varphi = \operatorname{span}\{D_{\varphi i},\ i = 1, \dots, n\}, \tag{20.13}$$

where the generators $D_{\varphi i}$ are defined by

$$D_{\varphi i} = D_i^{k+r,k+r-1} \circ \varphi \circ \pi_{k+r-1,k},$$

i.e. locally

$$D_{\varphi i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k-1} y^\sigma_{j_1\dots j_\ell i} \frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} + \sum_{\ell=k}^{k+r-1} \varphi^\sigma_{j_1\dots j_\ell\, i} \frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}. \tag{20.14}$$

As to be expected (and as to be proved by direct calculations in coordinates), *a section $\gamma \in \mathcal{S}_U(\pi)$ is an integral section of φ if, and only if, $j^{k+r-1}\gamma(U)$ is an integral manifold of H_φ.*

Remark 20.1. Due to the horizontality, H_φ is involutive (= completely integrable) if, and only if,

$$[D_{\varphi i}, D_{\varphi p}] = 0 \tag{20.15}$$

for all i, p. It should be stressed that this condition is *not* equivalent with the integrability of φ in the above presented sense. Nevertheless, the integral section of φ could be defined to be $\psi \in \mathcal{S}_U(\pi_{k+r-1})$ such that $\psi(U)$ is an integral manifold of H_φ. Of course, now the equations must be considered on $J^1\pi_{k+r-1}$.

Adding sections and connections, the diagram (20.1) turns out to be of the form

$$\begin{array}{ccccccc}
X & \xrightarrow{j^1\psi} & J^1\pi_k & \xrightarrow{J^1(\varphi,\mathrm{id}_X)} & J^1\pi_{k+r} & = & J^1\pi_{k+r} \\
\mathrm{id}_X\downarrow & & \Sigma_{(k)}\uparrow & & \Sigma_{(k+r)}\uparrow & & \Bbbk_{\Phi_0}\uparrow \\
X & \xrightarrow{\psi} & J^k\pi & \xrightarrow{\varphi} & J^{k+r}\pi & \xrightarrow{\Xi} & J^1\pi_{k+r,k} \\
\mathrm{id}_X\downarrow & & \pi_k\downarrow & & \pi_{k+r}\downarrow & & \\
X & \xrightarrow{\mathrm{id}_X} & X & \xrightarrow{\mathrm{id}_X} & X\,, & &
\end{array} \tag{20.16}$$

where by $\Sigma_{(\ell)}$ we denote a connection on π_ℓ.

As regards $\Sigma_{(k+r)}$, it can be called $\pi_{k+r,k}$*-semiholonomic*, if

$$\Sigma_{(k+r)} : J^{k+r}\pi \to A_{\pi_{k+r,k}}, \tag{20.17}$$

which by Sec. 7 and (20.7) means just

$$\Sigma^\sigma_{;i} = y^\sigma_i, \dots, \Sigma^\sigma_{j_1\dots j_k;i} = y^\sigma_{j_1\dots j_k i} \tag{20.18}$$

and

$$D_{\Sigma i} = \frac{\partial}{\partial x^i} + \sum_{\ell=0}^{k} y^\sigma_{j_1\dots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} + \sum_{\ell=k+1}^{k+r} \Sigma^\sigma_{j_1\dots j_\ell ; i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}}. \tag{20.19}$$

In this respect, if $\varphi \in \mathcal{S}(\pi_{k+r,k})$ is a jet field, then by (20.14) and (20.19), φ can be identified with a (special type of) $\pi_{k+r-1,k-1}$-semiholonomic connection on π_{k+r-1}. This completes the ideas of Remark 20.1.

Our main concern is with connections on $\pi_{k+r,k}$. Here are the main associated concepts (cf. Sec. 14) for an arbitrary

$$\Xi \colon J^{k+r}\pi \to J^1\pi_{k+r,k} \ :$$

the local equations

$$\begin{aligned} z^\sigma_{j_1\dots j_\ell i} &= \Xi^\sigma_{j_1\dots j_\ell i} \\ z^\sigma_{j_1\dots j_\ell \lambda} &= \Xi^\sigma_{j_1\dots j_\ell \lambda} \\ &\ \vdots \\ z^{\sigma i_1\dots i_k}_{j_1\dots j_\ell \lambda} &= \Xi^{\sigma i_1\dots i_k}_{j_1\dots j_\ell \lambda} \end{aligned} \tag{20.20}$$

for $\ell = k+1,\dots,k+r$, with Ξ's from $\mathcal{F}(J^{k+r}\pi)$;

$$h_\Xi \colon J^{k+r}\pi \to TJ^{k+r}\pi \otimes \pi^*_{k+r,k}(T^*J^k\pi),$$

where (14.2) holds with

$$\begin{aligned} D_{\Xi i} &= \frac{\partial}{\partial x^i} + \sum_{\ell=k+1}^{k+r} \Xi^\sigma_{j_1\dots j_\ell i}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} \\ D_{\Xi \lambda} &= \frac{\partial}{\partial y^\lambda} + \sum_{\ell=k+1}^{k+r} \Xi^\sigma_{j_1\dots j_\ell \lambda}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} \\ &\ \vdots \\ D^{i_1\dots i_k}_{\Xi \lambda} &= \frac{\partial}{\partial y^\lambda_{i_1\dots i_k}} + \sum_{\ell=k+1}^{k+r} \Xi^{\sigma i_1\dots i_k}_{j_1\dots j_\ell \lambda}\frac{\partial}{\partial y^\sigma_{j_1\dots j_\ell}} \ ; \end{aligned} \tag{20.21}$$

the $\pi_{k+r,k}$-horizontal distribution H_Ξ spanned by (20.21), i.e., the decomposition

$$TJ^{k+r}\pi = H_\Xi \oplus V_{\pi_{k+r,k}}J^{k+r}\pi.$$

The point is that *the integral sections* (if any) *of a connection* Ξ *on* $\pi_{k+r,k}$ *are (local) jet fields* from $\mathcal{S}(\pi_{k+r,k})$ satisfying

$$j^1\varphi = \Xi \circ \varphi, \tag{20.22}$$

the local expression of which can be easily derived by (20.20).

DEFINITION 20.1. A connection Ξ on $\pi_{k+r,k}$ will be called *characterizable*, if the connection

$$\Bbbk_{\Phi_0} \circ \Xi \tag{20.23}$$

is holonomic. The connection $\Gamma_{\Xi}^{(k+r+1)} = \Bbbk_{\Phi_0} \circ \Xi$ will be called *characteristic* to Ξ.

By (20.7) and (20.18), $\Bbbk_{\Phi_0} \circ \Xi$ is $\pi_{k+r,k}$-semiholonomic for an arbitrary Ξ; it is semiholonomic if, and only if,

$$\begin{aligned} y^{\sigma}_{j_1\dots j_{k+1}i} &= \Xi^{\sigma}_{j_1\dots j_{k+1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+1}\lambda} y^{\lambda}_{r_1\dots r_\ell\, i} \\ &\vdots \\ y^{\sigma}_{j_1\dots j_{k+r-1}i} &= \Xi^{\sigma}_{j_1\dots j_{k+r-1}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+r-1}\lambda} y^{\lambda}_{r_1\dots r_\ell\, i}, \end{aligned} \tag{20.24}$$

and it is holonomic if, moreover, the functions

$$\Gamma^{\sigma}_{j_1\dots j_{k+r}i} = \Xi^{\sigma}_{j_1\dots j_{k+r}i} + \sum_{\ell=0}^{k} \Xi^{\sigma r_1\dots r_\ell}_{j_1\dots j_{k+r}\lambda} y^{\lambda}_{r_1\dots r_\ell\, i} \tag{20.25}$$

are totally symmetric.

Analogously to Prop. 15.2 we get:

PROPOSITION 20.2. *A* $(k+r+1)$*-connection* $\Gamma^{(k+r+1)}$ *on* π *is the characteristic connection of a connection* Ξ *on* $\pi_{k+r,k}$ *if, and only if, the components of* $\Gamma^{(k+r+1)}$ *and* Ξ *are related by (20.25), which equivalently means*

$$D_{\Gamma^{(k+r+1)}i} = D_{\Xi i} + \sum_{\ell=0}^{k} D^{j_1\dots j_\ell}_{\Xi\lambda} y^{\lambda}_{j_1\dots j_\ell i} \tag{20.26}$$

or

$$H_{\Gamma^{(k+r+1)}} \subset H_{\Xi}. \tag{20.27}$$

The motivation of the above constructions is the following.

PROPOSITION 20.3. *Let* Ξ *be a characterizable connection on* $\pi_{k+r,k}$, *and* $\Gamma_{\Xi}^{(k+r+1)}$ *its characteristic connection. Let* $\varphi \in \mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$ *be an integral section of* Ξ *and* $\Gamma_{\varphi}^{(k+1)}$ *the* $(k+1)$*-connection on* π, *defined by* (20.11). *Then* $\Gamma_{\varphi}^{(k+1)}$ *is a field of paths of* $\Gamma_{\Xi}^{(k+r+1)}$ *and*

$$\varphi = \Gamma_{\varphi}^{(k+1)(r-1)}. \tag{20.28}$$

Proof. It is easy to see (cf. (15.4)) that for an arbitrary jet field φ we have

$$\Bbbk_{\Phi_0} \circ j^1\varphi = \Bbbk_{\iota_{1,k}\circ\Gamma_{\varphi}^{(k+1)}} \circ j^1\varphi = J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)}. \tag{20.29}$$

Then $J^{k+r+1}\pi \ni \Gamma_{\Xi}^{(k+r+1)}\circ\varphi = \Bbbk_{\Phi_0}\circ\Xi\circ\varphi = \Bbbk_{\Phi_0}\circ j^1\varphi = J^1(\varphi,\mathrm{id}_X)\circ\iota_{1,k}\circ\Gamma_{\varphi}^{(k+1)}$. Since by definition

$$\begin{aligned} J^1(\pi_{k+r,k+1}, \mathrm{id}_X)(j_x^{k+r+1}\gamma) &= j_x^1(\pi_{k+r,k+1} \circ j^{k+r}\gamma) = j_x^1(j^{k+1}\gamma) \\ &= j_x^{k+2}\gamma = \pi_{k+r+1,k+2}(j_x^{k+r+1}\gamma), \end{aligned}$$

we have

$$\begin{aligned} \Gamma_{\varphi}^{(k+1)(1)} &= J^1(\Gamma_{\varphi}^{(k+1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} \\ &= J^1(\pi_{k+r,k+1}, \mathrm{id}_X) \circ J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} \\ &= J^1(\pi_{k+r,k+1}, \mathrm{id}_X) \circ \Gamma_{\Xi}^{(k+r+1)} \circ \varphi = \pi_{k+r,k+2} \circ \varphi, \end{aligned}$$

in particular, $\Gamma^{(k+1)}$ is integrable. Then analogously by (10.3),

$$\Gamma_{\varphi}^{(k+1)(2)} = J^1(\pi_{k+r,k+2}, \mathrm{id}_X) \circ J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} = \pi_{k+r,k+3} \circ \varphi,$$

and the procedure ends with

$$\Gamma_{\varphi}^{(k+1)(r-1)} = J^1(\pi_{k+r,k+r-1}, \mathrm{id}_X)\circ\Gamma_{\Xi}^{(k+r+1)}\circ\varphi = \pi_{k+r+1,k+r}\circ\Gamma_{\Xi}^{(k+r+1)}\circ\varphi = \varphi,$$

which proves (20.28). Finally,

$$\begin{aligned} \Gamma_{\Xi}^{(k+r+1)} \circ \Gamma_{\varphi}^{(k+1)(r-1)} &= \Gamma_{\Xi}^{(k+r+1)} \circ \varphi = J^1(\varphi, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} \\ &= J^1(\Gamma_{\varphi}^{(k+1)(r-1)}, \mathrm{id}_X) \circ \iota_{1,k} \circ \Gamma_{\varphi}^{(k+1)} = \Gamma_{\varphi}^{(k+1)(r)}. \end{aligned}$$

∎

As usually, the situation may be described diagrammatically:

$$\begin{array}{ccccc} J^{k+1}\pi & \xrightarrow{J^1(\varphi,\mathrm{id}_X)} & J^{k+r+1}\pi & = & J^{k+r+1}\pi \\ \Gamma_{\varphi}^{(k+1)}\Big\uparrow & & \Gamma_{\Xi}^{(k+r+1)}\Big\uparrow & & \Bbbk_{\Phi_0}\Big\uparrow \\ J^k\pi & \xrightarrow{\varphi} & J^{k+r}\pi & \xrightarrow{\Xi} & J^1\pi_{k+r,k}. \end{array}$$

In this arrangement, the following definition appears very naturally; again, any connection Ξ on $\pi_{k+r,k}$ whose characteristic connection is the given $\Gamma^{(k+r+1)}$ will be called *associated* to it.

Definition 20.2. Let $\Gamma^{(k+r+1)}$ be an integrable $(k+r+1)$-connection on π. A (generally local) integrable connection Ξ on $\pi_{k+r,k}$ associated to $\Gamma^{(k+r+1)}$ is called the *$\pi_{k+r,k}$-integral* of $\Gamma^{(k+r+1)}$.

In other words, a second version of the method of fields of paths was presented. In contradiction to Sec. 17, now we are not looking for fields of paths directly, but through their prolongations. It is evident that the crucial problem is again that of the existence of $\pi_{k+r,k}$-integrals. In this respect, the following assertion can be proved in the same way as Prop. 17.1.

Proposition 20.4. *Let $\Gamma^{(k+r+1)}$ be an integrable $(k+r+1)$-connection on π and $\{a^1,\dots,a^K\}$, where $K=\dim\pi_{k+r,k}$, be a set of independent first integrals of $\Gamma^{(k+r+1)}$, defined on some open $W\subset J^{k+r}\pi$. If the matrix*

$$A=\left(\frac{\partial a^L}{\partial y^{\sigma}_{j_1\dots j_\ell}}\right),$$

where $\ell=k+1,\dots,k+r$, is regular on W, then

$$H_{\Xi}=\mathrm{anih}\{da^1,\dots,da^K\}$$

defines an $\pi_{k+r,k}$-integral of $\Gamma^{(k+r+1)}$ on W.

For an application (and in fact the motivation) of the above considerations, we refer to [65], dealing with the particular case of one-dimensional base X (and thus O.D.E.) and generalizing the *Hamilton-Jacobi* method from variational analysis studied in [39].

21. The manifold $\pi\colon \mathbb{R}\times M \longrightarrow \mathbb{R}$

Suppose we are given an m-dimensional manifold M and consider the trivial bundle

$$\pi \equiv \mathrm{pr}_1\colon \mathbb{R}\times M \to \mathbb{R}. \tag{21.1}$$

A (local) section of π is then of the form

$$\gamma = (\mathrm{id}_{\mathbb{R}}, c), \tag{21.2}$$

with c being a curve in M. By $j^1_0\gamma \mapsto \dot{c}(0)$, where γ is an arbitrary section of π on some neighbourhood of zero, a canonical izomorphism

$$(J^1\pi)_0 = \pi^{-1}(0) \cong TM \tag{21.3}$$

is realized, which immediately leads to the identification

$$J^1\pi \cong \mathbb{R}\times TM, \tag{21.4}$$

where

$$(J^1\pi, \pi_{1,0}, Y) \cong (\mathbb{R}\times TM, \mathrm{id}_{\mathbb{R}}\times \tau_M, \mathbb{R}\times M). \tag{21.5}$$

Recall the notion of the *k-th order tangent bundle* T^kM to M, which can be defined recurrently. Put $T^0M = M$, $T^1M = TM$ and $\tau^{1,0}_M = \tau_M\colon TM \to M$. Then for each $k\geq 1$, $T^{k+1}M$ is the $(k+2)m$-dimensional submanifold of TT^kM on which coincide the projections

$$\tau_{T^kM}\colon TT^kM \to T^kM \tag{21.6}$$

$$T\tau^{k,k-1}_M\colon TT^kM \to TT^{k-1}M, \tag{21.7}$$

acting within the commutative diagram

$$\begin{array}{ccc} TT^kM & \xrightarrow{T\tau^{k,k-1}_M} & TT^{k-1}M \\ {\scriptstyle\tau_{T^kM}}\big\downarrow & & {\scriptstyle\tau_{T^{k-1}M}}\big\downarrow \\ T^kM & \xrightarrow{\tau^{k,k-1}_M} & T^{k-1}M, \end{array} \tag{21.8}$$

and $\tau^{k+1,k}_M$ is the restriction $\tau_{T^kM}|_{T^{k+1}M}$.

More transparently, $T^k M$ is the set of equivalence classes of curves in M with the k-th order contact, which corresponds to the identification

$$T^k M \cong (J^k \pi)_0 = \pi_k^{-1}(0) \tag{21.9}$$

and thus to

$$J^k \pi \cong \mathbb{R} \times T^k M. \tag{21.10}$$

The fact that

$$\pi_{k+1,k} \cong \mathrm{id}_{\mathbb{R}} \times \tau_M^{k+1,k} \colon \mathbb{R} \times T^{k+1} M \to \mathbb{R} \times T^k M \tag{21.11}$$

is now a *vector* bundle could become important when speaking on connections.

Denote the coordinates: if the fibered coordinates on $Y = \mathbb{R} \times M$ are (t, q^σ), then those induced on $J^k \pi = \mathbb{R} \times T^k M$ are $(t, q^\sigma, \dots, q^\sigma_{(k)})$ with

$$q^\sigma_{(k)}(j^k_x \gamma) = \frac{d^k c^\sigma}{dt^k}|_x$$

with γ given by (21.2) and $t \circ \gamma = t$, $q^\sigma \circ \gamma = q^\sigma \circ c = c^\sigma(t)$. In what follows, we suppose t to be a *global* coordinate on $\mathbb{R}$.

As regards repeated jets, substituting π_k for π in (21.4), we have

$$J^1 \pi_k \cong \mathbb{R} \times TT^k M \tag{21.12}$$

with (by (21.5))

$$(J^1 \pi_k, (\pi_k)_{1,0}, J^k \pi) \cong (\mathbb{R} \times TT^k M, \mathrm{id}_{\mathbb{R}} \times \tau_{T^k M}, \mathbb{R} \times T^k M). \tag{21.13}$$

Since vertical vectors find wide application in our exposition, it is worth mentioning here that there is another important interpretation of $J^1 \pi$; namely,

$$J^1 \pi \cong V_\pi Y, \tag{21.14}$$

i.e. $V_\pi Y \cong \mathbb{R} \times TM \subset T(\mathbb{R} \times M) = TY$. This identification can be roughly expressed by $j^1 \gamma \mapsto \dot{\gamma}$ and it locally reads $\dot{t} = 1$. Substituting π_k for π in (21.14) yields

$$J^1 \pi_k \cong V_{\pi_k} J^k \pi \cong \mathbb{R} \times TT^k M. \tag{21.15}$$

On the other hand, (21.14) together with π_1 being substituted for π in (21.10) results in the izomorphism

$$J^k(\pi \circ \tau_Y|_{V_\pi Y}) \cong J^k \pi_1 \cong \mathbb{R} \times T^k TM \tag{21.16}$$

over X with

$$(J^k(\pi \circ \tau_Y|_{V_\pi Y}), J^k(\tau_Y|_{V_\pi Y}, \mathrm{id}_{\mathbb{R}}), J^k\pi) \\ \cong (\mathbb{R} \times T^kTM, \mathrm{id}_{\mathbb{R}} \times T^k\tau_M, \mathbb{R} \times T^kM). \tag{21.17}$$

Then the izomorphism between (11.19) and (11.20), exchanging $J^k(\tau_Y|_{V_\pi Y}, \mathrm{id}_X)$ with $\tau_{J^k\pi}|_{V_{\pi_k} J^k\pi}$, can now be described by the commutative diagram

$$\begin{array}{ccc} \mathbb{R} \times T^kTM & \xrightarrow{\mathrm{id}_{\mathbb{R}} \times \kappa_M^{(k)}} & \mathbb{R} \times TT^kM \\ {\scriptstyle \mathrm{id}_{\mathbb{R}} \times T^k\tau_M} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\mathbb{R}} \times \tau_{T^kM}} \\ \mathbb{R} \times T^kM & = & \mathbb{R} \times T^kM \end{array} \tag{21.18}$$

with $\kappa_M^{(k)}$ realizing the canonical exchange

$$(T^kTM, T^k\tau_M, T^kM) \rightsquigarrow (TT^kM, \tau_{T^kM}, T^kM);$$

evidently $\mathrm{id}_{\mathbb{R}} \times \kappa_M^{(k)} \equiv \nu_k$, i.e.

$$(q^\sigma, \dot{q}^\sigma, q^\sigma_{(1)}, \dot{q}^\sigma_{(1)}, \dots, q^\sigma_{(k)}, \dot{q}^\sigma_{(k)}) \overset{\kappa_M^{(k)}}{\longmapsto} (q^\sigma, q^\sigma_{(1)}, \dots, q^\sigma_{(k)}, \dot{q}^\sigma, \dot{q}^\sigma_{(1)}, \dots, \dot{q}^\sigma_{(k)}).$$

For $k = 1$, $\kappa_M^{(1)} : TTM \to TTM$ is denoted by κ_M.

Recall finally the family of *natural vector-valued one-forms* on $\mathbb{R} \times T^kM$, which is expressed by

$$\sum_{i=1}^{k} c_i J_i^{(k)} + \sum_{i=k+1}^{2k} c_i C_{i-k}^{(k)} \otimes dt + c_{2k+1} I_{T^kM} + c_{2k+2} I_{\mathbb{R}}, \tag{21.19}$$

where $c_i \in \mathcal{F}(\mathbb{R})$, I_{T^kM} and

$$J_i^{(k)} = \sum_{j=1}^{k-i+1} j \frac{\partial}{\partial q^\sigma_{(i+j-1)}} \otimes dq^\sigma_{(j-1)} \tag{21.20}$$

(for $i = 1, \dots, k$) are the unique natural (1,1)-tensor fields on T^kM,

$$I_{\mathbb{R}} = \frac{\partial}{\partial t} \otimes dt,$$

and

$$C_i^{(k)} = \sum_{j=1}^{k-i+1} \frac{(i+j-1)\,!}{(j-1)\,!} q^\sigma_{(j)} \frac{\partial}{\partial q^\sigma_{(i+j-1)}} \tag{21.21}$$

(for $i = 1, \dots, k$) are the *absolute (generalized Liouville) vector fields* on $T^k M$. Notice that $J_i^{(k)} = (J_1^{(k)})^i$ and $C_i^{(k)} = J_{i-1}^{(k)} C_1$ for $i = 2, \dots, k$, and that some of the above forms will become of particular importance later.

22. A connection on $\pi\colon \mathbb{R} \times M \to \mathbb{R}$

In this arrangement, a (first-order) connection $\Gamma\colon Y \to J^1\pi$ on $\pi\colon \mathbb{R}\times M \to \mathbb{R}$ can be identified with a π-vertical vector field

$$v = \Gamma^\sigma(t, q^\lambda) \frac{\partial}{\partial q^\sigma} \tag{22.1}$$

on $\mathbb{R} \times M$, which is equivalently a vector field along $\mathrm{pr}_2\colon \mathbb{R} \times M \to M$, called a *time-dependent vector field* on M. The vector field

$$D_\Gamma = \frac{\partial}{\partial t} + v$$

generates the one-dimensional (and thus completely integrable) horizontal distribution H_Γ and consequently the Pfaffian system

$$dq^\sigma = \Gamma^\sigma \, dt.$$

equivalently expressed as the first-order system of ordinary differential equations in normal form

$$\frac{dc^\sigma}{dt} = \Gamma^\sigma(t, c^\lambda).$$

The integral sections of Γ are the 'graphs' of the integral curves of v, which means that $v \circ \gamma = \dot{c}$. In particular, if v does not depend on t, the connection Γ represents the submanifold

$$\Gamma(Y) = \mathbb{R} \times \mathrm{Im} v \subset \mathbb{R} \times TM$$

and its integral sections are the genuine graphs.

In keeping with the identifications of Sec. 21, here are the mappings related to the vertical prolongations of Γ:

$$V\Gamma\colon \mathbb{R} \times TM \to \mathbb{R} \times TTM$$

is a section of $\mathrm{id}_{\mathbb{R}} \times T\tau_M$, while

$$\mathcal{V}\Gamma\colon \mathbb{R}\times TM \to \mathbb{R}\times TTM$$

defined by

$$V\Gamma = (\mathrm{id}_{\mathbb{R}} \times \kappa_M)\circ \mathcal{V}\Gamma$$

is a section of $\mathrm{id}_{\mathbb{R}} \times \tau_{TM}$. In particular, for time-independent v in (22.1) we have

$$V\Gamma \equiv \mathrm{id}_{\mathbb{R}} \times Tv. \quad \mathcal{V}\Gamma \equiv \mathrm{id}_{\mathbb{R}} \times v^c.$$

where v^c is the *complete lift* of v, defined by the following commutative diagram:

$$\begin{array}{ccccc} TM & \xleftarrow{\tau_{TM}} & TTM & \xleftarrow{\kappa_M} & TTM \\ {\scriptstyle v}\uparrow & & {\scriptstyle Tv}\uparrow & & {\scriptstyle v^c}\uparrow \\ M & \xleftarrow{\tau_M} & TM & = & TM. \end{array}$$

Notice also that for a given $\zeta \in \mathcal{X}^v_{\mathbb{R}}(\mathbb{R}\times M)$, $J^1(\zeta,\mathrm{id}_{\mathbb{R}})$ is a section of $\mathrm{id}_{\mathbb{R}} \times T\tau_M$, while its prolongation $\mathcal{J}^1\zeta$ is a section of $\mathrm{id}_{\mathbb{R}} \times \tau_{TM}$. In particular, if we consider

$$\zeta \equiv \mathrm{id}_{\mathbb{R}} \times \xi \tag{22.2}$$

with $\xi \in \mathcal{X}(M)$, then

$$J^1(\zeta,\mathrm{id}_{\mathbb{R}}) \equiv \mathrm{id}_{\mathbb{R}} \times T\xi. \quad \mathcal{J}^1\zeta \equiv \mathrm{id}_{\mathbb{R}} \times \xi^c.$$

Of course, even for $v \in \mathcal{X}(M)$, we can consider time-dependent symmetries, satisfying

$$[\zeta, \frac{\partial}{\partial t} + v] = 0.$$

23. Semispray connections

Let $k \geq 1$. In accordance with Sec. 8, a $(k+1)$-connection $\Gamma^{(k+1)}$ on $\pi\colon \mathbb{R}\times M \to \mathbb{R}$ is a section

$$\Gamma^{(k+1)}\colon \mathbb{R}\times T^kM \to \mathbb{R}\times T^{k+1}M$$

of $\mathrm{id}_{\mathbb{R}} \times \tau_M^{k+1,k}$. Any $(k+1)$-connection is characterized by its horizontal form $h_{\Gamma^{(k+1)}} = D_{\Gamma^{(k+1)}} \otimes dt$, where the absolute derivative

$$D_{\Gamma^{(k+1)}} = \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} q^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} + \Gamma^\sigma_{(k+1)} \frac{\partial}{\partial q^\sigma_{(k)}} \tag{23.1}$$

is the so-called *semispray* on $\mathbb{R} \times T^k M$, defining the one-dimensional π_k-horizontal *semispray distribution* $H_{\Gamma^{(k+1)}}$. Due to the product structure and analogously to the first-order case, $\Gamma^{(k+1)}$ can be represented by the vector field

$$w^{(k+1)} = \sum_{i=0}^{k-1} q^\sigma_{(i+1)} \frac{\partial}{\partial q^\sigma_{(i)}} + \Gamma^\sigma_{(k+1)} \frac{\partial}{\partial q^\sigma_{(k)}} \tag{23.2}$$

along $\mathrm{pr}_2 \colon \mathbb{R} \times T^k M \to T^k M$, which is nothing but a *time-dependent semispray* on $T^k M$; in the autonomous situation, a semispray on $T^k M$ is a section of $\tau_M^{k+1,k}$.

The $(k+1)$-*th order* (generally nonlinear) *system of O.D.E. represented by a* $(k+1)$-*connection* $\Gamma^{(k+1)}$ on $\pi \colon \mathbb{R}\times M \to \mathbb{R}$ can be described both globally as the
$((k+1)m+1)$-dimensional submanifold

$$\Gamma^{(k+1)}(\mathbb{R} \times T^k M) \subset \mathbb{R} \times T^{k+1} M \tag{23.3}$$

of $\mathbb{R} \times T^{k+1} M$ and locally by

$$\frac{d^{k+1} c^\sigma}{dt^{k+1}} = \Gamma^\sigma_{(k+1)} \left(t, c^\lambda, \dots, \frac{d^k c^\lambda}{dt^k} \right) \; ; \tag{23.4}$$

the Pfaffian version of (23.3), (23.4) is

$$\begin{aligned} dq^\sigma &= q^\sigma_{(1)}\, dt \\ &\vdots \\ dq^\sigma_{(k-1)} &= dq^\sigma_{(k)}\, dt \\ dq^\sigma_{(k)} &= \Gamma^\sigma_{(k+1)}\, dt. \end{aligned}$$

The integral sections of $\Gamma^{(k+1)}$ are thus the ‘graphs’ of the geodesics of the semispray (23.2) in the sense that $w^{(k+1)} \circ j^{k+1}\gamma = c^{(k+1)}$.

In agreement with Sec. 21, the vertical functor V applied to a $(k+1)$-connection $\Gamma^{(k+1)}$ gives

$$V\Gamma^{(k+1)} \colon \mathbb{R} \times TT^k M \to \mathbb{R} \times TT^{k+1} M$$

as a section of $\mathrm{id}_{\mathbb{R}} \times T\tau_M^{k+1,k}$, and the vertical prolongation $\mathcal{V}\Gamma^{(k+1)}$, defined by

$$V\Gamma^{(k+1)} \circ (\mathrm{id}_{\mathbb{R}} \times \kappa_M^{(k)}) = (\mathrm{id}_{\mathbb{R}} \times \kappa_M^{(k+1)}) \circ \mathcal{V}\Gamma^{(k+1)}.$$

is a section

$$\mathcal{V}\Gamma^{(k+1)} \colon \mathbb{R} \times T^k TM \to \mathbb{R} \times T^{k+1} TM$$

of $\mathrm{id}_{\mathbb{R}} \times \tau_{TM}^{k+1,k}$. In fact, due to (21.16), $\mathcal{V}\Gamma^{(k+1)}$ is nothing but a $(k+1)$-connection on $\pi_1 \colon \mathbb{R} \times TM \to \mathbb{R}$, i.e. a section of

$$(\pi_1)_{k+1,k} \colon J^{k+1}\pi_1 \to J^k \pi_1.$$

In particular, for the autonomous situation, when $w^{(k+1)}$ given by (23.2) is an ordinary semispray on $T^k M$, it generates canonically the semispray $\mathcal{T}w^{(k+1)}$ on $T^k TM$ by the following commutative diagram:

$$\begin{array}{ccccc}
T^{k+1}M & \xleftarrow{\tau_{T^{k+1}M}} & TT^{k+1}M & \xleftarrow{\kappa_M^{(k+1)}} & T^{k+1}TM \\
{\scriptstyle w^{(k+1)}}\uparrow & & {\scriptstyle Tw^{(k+1)}}\uparrow & & \uparrow{\scriptstyle \mathcal{T}w^{(k+1)}} \\
T^k M & \xleftarrow{\tau_{T^k M}} & TT^k M & \xleftarrow{\kappa_M^{(k)}} & T^k TM.
\end{array}$$

In other words,

$$V\Gamma^{(k+1)} \equiv \mathrm{id}_{\mathbb{R}} \times Tw^{(k+1)}.$$
$$\mathcal{V}\Gamma^{(k+1)} \equiv \mathrm{id}_{\mathbb{R}} \times \mathcal{T}w^{(k+1)}.$$

Analogously to the first-order case, for a given $\zeta \in \mathcal{X}^v_{\mathbb{R}}(\mathbb{R} \times M)$, $J^k(\zeta, \mathrm{id}_{\mathbb{R}})$ is a section of $\mathrm{id}_{\mathbb{R}} \times T^k \tau_M$, while its k-th prolongation $\mathcal{J}^k \zeta$ is a section of $\mathrm{id}_{\mathbb{R}} \times \tau_{T^k M}$. In particular, for ζ defined by (22.2) we have

$$J^k(\zeta, \mathrm{id}_{\mathbb{R}}) \equiv \mathrm{id}_{\mathbb{R}} \times T^k \xi.$$
$$\mathcal{J}^k \zeta \equiv \mathrm{id}_{\mathbb{R}} \times \xi^{T^k}.$$

where ξ^{T^k} is the *k-th flow prolongation* of ξ ($\xi^{T^0} = \xi^T \equiv \xi^c$).

In view of Sec. 11, a vector field $\zeta^{(r)} \in \mathcal{X}(\mathbb{R} \times T^r M)$, $0 \le r \le k$, is the r-symmetry of $\Gamma^{(k+1)}$ if, and only if, one of the following equivalent conditions holds:

$$\mathcal{J}^{k-r+1}\zeta^{(r)} \circ \Gamma^{(k+1)} = T\Gamma^{(k+1)} \circ \mathcal{J}^{k-r}\zeta^{(r)}. \tag{23.5}$$

$$\mathcal{J}^1(v_{\Gamma^{(k+1)}} \circ \mathcal{J}^{k-r}\zeta^{(r)}) \circ \Gamma^{(k+1)} = V\Gamma^{(k+1)} \circ v_{\Gamma^{(k+1)}} \circ \mathcal{J}^{k-r}\zeta^{(r)}. \tag{23.6}$$

$$\mathcal{L}_{v_{\Gamma^{(k+1)}}(\mathcal{J}^{k-r}\zeta^{(r)})} h_{\Gamma^{(k+1)}} = 0. \tag{23.7}$$

$$[D_{\Gamma^{(k+1)}}, \mathcal{J}^{k-r}\zeta^{(r)}] = D_{\Gamma^{(k+1)}}(\zeta^0) D_{\Gamma^{(k+1)}}. \tag{23.8}$$

where $\zeta^0 = dt(\zeta^{(r)})$. The corresponding autonomous situation for the (zero-order) symmetries of a semispray $w^{(k+1)}$ on T^kM can be then described by the following commutative diagram

$$\begin{array}{ccc} T^{k+1}M & \xrightarrow{\xi^{T^{k+1}}} & TT^{k+1}M \\ {\scriptstyle w^{(k+1)}}\uparrow & & \uparrow{\scriptstyle Tw^{(k+1)}} \\ T^kM & \xrightarrow{\xi^{T^k}} & TT^kM \end{array}$$

(cf. (23.5) and (23.6)) or by

$$[w^{(k+1)}, \xi^{T^k}] = 0$$

(cf. (23.7) and (23.8)).

For example, a vector field $\zeta^{(1)} \in \mathcal{X}^v_X(\mathbb{R} \times TM)$,

$$\zeta^{(1)} = \zeta^\sigma \frac{\partial}{\partial q^\sigma} + \zeta^\sigma_{(1)} \frac{\partial}{\partial q^\sigma_{(1)}}.$$

is the (vertical) first-order symmetry of a 2-connection $\Gamma^{(2)}$ on π (or of the corresponding semispray $D_{\Gamma^{(2)}}$) if

$$\zeta^\sigma_{(1)} = D_{\Gamma^{(2)}}(\zeta^\sigma).$$
$$D^2_{\Gamma^{(2)}}(\zeta^\sigma) = \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda}\zeta^\lambda + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} D_{\Gamma^{(2)}}(\zeta^\lambda).$$

Since for a vector field $\zeta \in \mathcal{X}^v_X(\mathbb{R} \times M)$ we have

$$\mathcal{J}^1\zeta = \zeta^\sigma \frac{\partial}{\partial q^\sigma} + D(\zeta^\sigma)\frac{\partial}{\partial q^\sigma_{(1)}}$$

ζ is a (vertical) symmetry of $\Gamma^{(2)}$ if, and only if,

$$D_{\Gamma^{(2)}} \circ D(\zeta^\sigma) = \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda}\zeta^\lambda + \frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} D(\zeta^\lambda).$$

24. Connections on $\pi_{k+1,k}\colon \mathbb{R}\times T^{k+1}M \to \mathbb{R}\times T^k M$

Let $k \geq 0$. The first jet prolongation $J^1\pi_{k+1,k}$ of $\pi_{k+1,k}$ is the manifold of 1-jets of (local) $(k+1)$-connections on π, the induced coordinates on which we denote by

$$(t, q^\sigma, \dots, q^\sigma_{(k+1)}, z^\sigma_{(k+1)}, z^\sigma_{(k+1)\lambda}, \dots, z^{\sigma(k)}_{(k+1)\lambda}).$$

where

$$\begin{aligned} z^\sigma_{(k+1)}\left(j^1_{(x,y)}\Gamma^{(k+1)}\right) &= \frac{\partial \Gamma^\sigma_{(k+1)}}{\partial t}|_{(x,y)} \\ z^\sigma_{(k+1)\lambda}\left(j^1_{(x,y)}\Gamma^{(k+1)}\right) &= \frac{\partial \Gamma^\sigma_{(k+1)}}{\partial q^\lambda}|_{(x,y)} \\ &\vdots \\ z^{\sigma(k)}_{(k+1)\lambda}\left(j^1_{(x,y)}\Gamma^{(k+1)}\right) &= \frac{\partial \Gamma^\sigma_{(k+1)}}{\partial q^\sigma_{(k)}}|_{(x,y)} \end{aligned} \tag{24.1}$$

with $x \in \mathbb{R}$, $y \in T^k M$.

There is an interesting submanifold of $J^1\pi_{k+1,k}$ having to do with the relations between the autonomous and the time-dependent situations. Namely, in accordance with (21.11), there is a canonical inclusion

$$\mathbb{R}\times J^1\tau^{k+1,k}_M \hookrightarrow J^1\pi_{k+1,k}. \tag{24.2}$$

defined by

$$(x, j^1_y w^{(k+1)}) \longmapsto j^1_{(x,y)}\Gamma^{(k+1)} \tag{24.3}$$

for $\Gamma^{(k+1)}$ being defined by $w^{(k+1)}$ (see (23.1), (23.2)). The local condition for this submanifold is by (24.1) just

$$z^\sigma_{(k+1)} = 0. \tag{24.4}$$

In this respect, the morphism

$$\mathbb{k}_{\iota_{1,k}}\colon J^1\pi_{k+1,k} \to \mathbb{R}\times T^{k+2}M \tag{24.5}$$

locally reads

$$q^\sigma_{(k+2)} = z^\sigma_{(k+1)} + \sum_{i=0}^{k} z^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)}. \tag{24.6}$$

and its restriction to (24.2) generates the morphism

$$\mathbb{k}_M^{(k+1)} \colon J^1\tau_M^{k+1,k} \to T^{k+2}M \tag{24.7}$$

over $T^{k+1}M$ with local expression

$$q^\sigma_{(k+2)} = \sum_{i=0}^{k} z^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)}. \tag{24.8}$$

A connection on $\pi_{k+1,k} \colon \mathbb{R} \times T^{k+1}M \to \mathbb{R} \times T^k M$ is a section

$$\Xi \colon \mathbb{R} \times T^{k+1}M \to J^1\pi_{k+1,k} \tag{24.9}$$

with the horizontal form

$$h_\Xi = D_{\Xi 0} \otimes dt + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} \otimes dq^\lambda_{(i)}. \tag{24.10}$$

The absolute derivatives with respect to Ξ are the vector fields

$$\begin{aligned} D_{\Xi 0} &= \frac{\partial}{\partial t} + \Xi^\sigma_{(k+1)} \frac{\partial}{\partial q^\sigma_{(k+1)}} \\ D_{\Xi\lambda} &= \frac{\partial}{\partial q^\lambda} + \Xi^\sigma_{(k+1)\lambda} \frac{\partial}{\partial q^\sigma_{(k+1)}} \\ &\vdots \\ D^{(k)}_{\Xi\lambda} &= \frac{\partial}{\partial q^\lambda_{(k)}} + \Xi^{\sigma(k)}_{(k+1)\lambda} \frac{\partial}{\partial q^\sigma_{(k+1)}} \end{aligned} \tag{24.11}$$

generating the $((k+1)m+1)$-dimensional $\pi_{k+1,k}$-horizontal distribution H_Ξ and thus the Pfaffian system

$$dq^\sigma_{(k+1)} = \Xi^\sigma_{(k+1)}\, dt + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda}\, dq^\lambda_{(i)}. \tag{24.12}$$

which in coordinates means a *first-order system of P.D.E.*

$$
\begin{aligned}
\frac{\partial \Gamma^\sigma_{(k+1)}}{\partial t} &= \Xi^\sigma_{(k+1)}(t, q^\nu, \dots, q^\nu_{(k)}, \Gamma^\nu_{(k+1)}) \\
\frac{\partial \Gamma^\sigma_{(k+1)}}{\partial q^\lambda} &= \Xi^\sigma_{(k+1)\lambda}(t, q^\nu, \dots, q^\nu_{(k)}, \Gamma^\nu_{(k+1)}) \\
&\vdots \\
\frac{\partial \Gamma^\sigma_{(k+1)}}{\partial q^\lambda_{(k)}} &= \Xi^{\sigma(k)}_{(k+1)\lambda}(t, q^\nu, \dots, q^\nu_{(k)}, \Gamma^\nu_{(k+1)})
\end{aligned}
\tag{24.13}
$$

for (local) $(k+1)$-connections on π.

As usually, the integrability conditions for Ξ are expressable in terms of its absolute derivatives (24.11) by

$$[D_{\Xi 0}, D^{(i)}_{\Xi\lambda}] = 0 \tag{24.14}$$

$$[D^{(i)}_{\Xi\lambda}, D^{(j)}_{\Xi\sigma}] = 0 \tag{24.15}$$

for $\sigma, \lambda = 1, \dots, m$ and $i, j = 0, \dots k$.

Following (24.2), a connection Λ on $\tau^{k+1,k}_M$ can be considered as a connection on $\pi_{k+1,k}$ of the particular type

$$\Xi = \mathrm{id}_{\mathbb{R}} \times \Lambda \colon \mathbb{R} \times T^{k+1}M \to \mathbb{R} \times J^1\tau^{k+1,k}_M \tag{24.16}$$

with the components $\Xi^\sigma_{(k+1)} = 0$ and $\Xi^{\sigma(i)}_{(k+1)\lambda} \in \mathcal{F}(T^{k+1}M)$. The corresponding horizontal distribution is

$$h_\Xi = \frac{\partial}{\partial t} \otimes dt + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} \otimes dq^\lambda_{(i)} = \mathrm{id}_{T\mathbb{R}} + h_\Lambda. \tag{24.17}$$

and the integral sections can be identified with the semisprays on T^kM (for $k \geq 1$) or the vector fields on M (for $k = 0$). Just the case of $k = 0$ might be of particular importance due to the fact that Λ represents a (generally *non-linear*) *connection* on $\tau_M \colon TM \to M$ with integral sections being the vector fields on M whose covariant derivative with respect to Λ vanishes, i.e. those *parallel* with respect to Λ.

The deformations of connections on $\pi_{k+1,k}$ are the soldering forms on $\pi_{k+1,k}$; a local expression of any such a $\pi_{k+1,k}$-vertical endomorphism on

$\mathbb{R} \times T^{k+1}M$ is

$$\varphi = \frac{\partial}{\partial q^{\sigma}_{(k+1)}} \otimes \left(\varphi^{\sigma}_{(k+1)}\, dt + \sum_{i=0}^{k} \varphi^{\sigma(i)}_{(k+1)\lambda}\, dq^{\lambda}_{(i)} \right). \tag{24.18}$$

Nevertheless, there is a distinguished subfamily of the above soldering forms created by the *natural soldering forms* on $\pi_{k+1,k}$. Actually, by (21.19-21), any such a soldering form is expressed by

$$\varphi = f_1 J^{(k+1)}_{k+1} + f_2 C^{(k+1)}_{k+1} \otimes dt \tag{24.19}$$

for $f_1, f_2 \in \mathcal{F}(\mathbb{R})$, i.e.

$$\varphi^{\sigma}_{(k+1)} = (k+1)\,!\, f_2 q^{\sigma}_{(1)}, \quad \varphi^{\sigma}_{(k+1)\lambda} = f_1 \delta^{\sigma}_{\lambda} \tag{24.20}$$

and the rest of the components vanishes identically. As a consequence we have:

PROPOSITION 24.1. *All natural $\iota_{1,k}$-admissible deformations on $\pi_{k+1,k}$ are of the form*

$$\varphi = f S^{(k+1)}_{k+1} \tag{24.21}$$

with

$$S^{(k+1)}_{k+1} = J^{(k+1)}_{k+1} - \frac{1}{(k+1)\,!} C^{(k+1)}_{k+1} \otimes dt \tag{24.22}$$

and $f \in \mathcal{F}(\mathbb{R})$.

Proof. By (24.6) and (24.20), a natural soldering form φ is $\iota_{1,k}$-admissible if, and only if, (24.21) holds. ∎

In coordinates,

$$S^{(k+1)}_{k+1} = \frac{\partial}{\partial q^{\sigma}_{(k+1)}} \otimes (dq^{\sigma} - q^{\sigma}_{(1)}\, dt). \tag{24.23}$$

25. Associated connections

Owing to the dimension of the base, each connection Ξ on $\pi_{k+1,k} : \mathbb{R} \times T^{k+1}M \to \mathbb{R} \times T^kM$ is characterizable and a semispray connection $\Gamma^{(k+2)} : \mathbb{R}\times T^{k+1}M \to \mathbb{R}\times T^{k+2}M$ is the characteristic $(k+2)$-connection of Ξ if, and only if, for the corresponding semispray $D_{\Gamma^{(k+2)}}$ on $\mathbb{R} \times T^{k+1}M$ holds

$$D_{\Gamma^{(k+2)}} = D_{\Xi 0} + \sum_{i=0}^{k} D^{(i)}_{\Xi\lambda} q^{\lambda}_{(i+1)}. \tag{25.1}$$

which means

$$\Gamma^{\sigma}_{(k+2)} = \Xi^{\sigma}_{(k+1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} q^{\lambda}_{(i+1)} \tag{25.2}$$

(see (24.6)); the semispray $D_{\Gamma^{(k+2)}}$ expressed by (25.1) can be called *characteristic* to Ξ, as well. The diagram (15.12) now reads

$$\begin{array}{ccccccc}
\mathbb{R} \times T^{k+1}M & \xrightarrow{J^1(\Gamma^{(k+1)},\mathrm{id}_{\mathbb{R}})} & \mathbb{R} \times T^{k+2}M & = & \mathbb{R} \times T^{k+2}M & = & \mathbb{R} \times T^{k+2}M \\
\Big\uparrow \Gamma^{(k+1)} & & \Big\uparrow \Gamma^{(k+2)} & & \Bbbk\iota_{1,k} \Big\uparrow & & \mathrm{id}_{\mathbb{R}} \times \Bbbk^{(k+1)}_{M} \Big\uparrow \\
\mathbb{R} \times T^kM & \xrightarrow{\Gamma^{(k+1)}} & \mathbb{R} \times T^{k+1}M & \xrightarrow{\Xi} & J^1\pi_{k+1,k} & \hookleftarrow & \mathbb{R} \times J^1\tau^{k+1,k}_M ,
\end{array} \tag{25.3}$$

which in particular defines the *characteristic semispray* on $T^{k+1}M$ for a connection Λ on $\tau^{k+1,k}_M$ by means of (24.16) in the autonomous case.

The *equations for characteristics* are by (23.4) and (25.2)

$$\begin{aligned}
\frac{d^{k+2}c^{\sigma}}{dt^{k+2}} &= \Xi^{\sigma}_{(k+1)} \left(t, c^{\nu}, \dots, \frac{d^{k+1}c^{\nu}}{dt^{k+1}} \right) \\
&\quad + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} \left(t, c^{\nu}, \dots, \frac{d^{k+1}c^{\nu}}{dt^{k+1}} \right) \frac{d^{i+1}c^{\lambda}}{dt^{i+1}}
\end{aligned} \tag{25.4}$$

and with respect to the ideas of Sec. 16, the looking for the solutions of the first-order P.D.E. system (24.13) can be transferred to the looking for the solutions of the $(k+2)$-th order O.D.E. system (25.4).

On the other hand, the very beginning of Sec. 17 suggests the importance of the looking for connections on $\pi_{k+1,k}$ associated to the given $(k+2)$-connection on π. In this respect, the role of another natural (1,1)-tensor field from (21.19) appears; namely,

$$h_{\Xi_0} = \frac{1}{2}\left[h_{\Gamma^{(k+2)}} + I + \frac{1}{k+2}\left(kv_{\Gamma^{(k+2)}} - 2\mathcal{L}_{D_{\Gamma^{(k+2)}}} S_1^{(k+1)}\right)\right] \tag{25.5}$$

is the horizontal form of a connection Ξ_0 on $\pi_{k+1,k}$ associated to $\Gamma^{(k+2)}$, where

$$S_1^{(k+1)} = J_1^{(k+1)} - C_1^{(k+1)} \otimes dt. \tag{25.6}$$

i.e.

$$S_1^{(k+1)} = \sum_{i=1}^{k+1} i \frac{\partial}{\partial q^\sigma_{(i)}} \otimes (dq^\sigma_{(i-1)} - q^\sigma_{(i)}\, dt). \tag{25.7}$$

The components of Ξ_0 defined by (25.5) are

$$\Xi^{\sigma(i)}_{(k+1)\lambda} = \frac{i+1}{k+2}\frac{\partial \Gamma^\sigma_{(k+2)}}{\partial q^\lambda_{(i+1)}}, \quad i = 0, \dots k. \tag{25.8}$$

$$\Xi^\sigma_{(k+1)} = \Gamma^\sigma_{(k+2)} - \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)}. \tag{25.9}$$

Following Prop. 24.1, the family of all connections on $\pi_{k+1,k}$ *naturally associated* to a $(k+2)$-connection $\Gamma^{(k+2)}$ on $\pi\colon \mathbb{R}\times M \to \mathbb{R}$ is defined by

$$h_\Xi = h_{\Xi_0} + f S^{(k+1)}_{k+1} \tag{25.10}$$

with $f \in \mathcal{F}(\mathbb{R})$.

It should be noticed that (25.5) corresponds to the fact that

$$F_{\Xi_0} = \frac{1}{k+2}\left(kv_{\Gamma^{(k+2)}} - 2\mathcal{L}_{D_{\Gamma^{(k+2)}}} S_1^{(k+1)}\right) \tag{25.11}$$

is the $f(3,-1)$-structure associated with Ξ_0.

As regards the strong horizontal subbundle and the corresponding reduced connection on $\pi_{k+1,k}$, the decomposition (18.13) is

$$\mathbb{R}\times TT^{k+1}M = \left(\mathbb{R}\times V_{\tau_M^{k+1,k}} T^{k+1}M\right) \oplus H_{\Gamma_{(k+1,k)}}$$

with

$$V_{\pi_{k+1,k}} J^k\pi \cong \mathbb{R}\times V_{\tau_M^{k+1,k}} T^{k+1}M \subset \mathbb{R}\times TT^{k+1}M \cong V_{\pi_k} J^k \pi$$

and

$$H_{\Gamma_{(k+1,k)}} = H_{\Xi} \cap V_{\pi_k} J^k\pi = \operatorname{span}\{D_{\Xi^{(i)}_\lambda},\ i = 0, \dots, k,\ \lambda = 1, \dots, m\}$$

for $\Gamma_{(k+1,k)}$ generated by a connection Ξ on $\pi_{k+1,k}$ by (18.17). Now it is evident why the components $\Gamma^{\sigma(i)}_{(k+1)\lambda}$ (or $\Xi^{\sigma(i)}_{(k+1)\lambda}$) must be transformed like those of a connection on $\tau^{k+1,k}_M$ (cf. also Remark 18.1 in view of (21.9)). Evidently, in the autonomous situation (24.16) we have

$$H^{\Xi}_{\Gamma_{(k+1,k)}} \equiv H_\Lambda.$$

Remark 25.1. Following Remark 17.1, let us finally recall the result of [62]. Let $\pi\colon Y \to X$ be an arbitrary fibered manifold over one-dimensional base X endowed by a volume form $\Omega = \omega\, dt$. By [56], there is a naturally defined vector-valued one-form

$$S^{(k+1)}_\Omega = \sum_{j+i=1}^{k} \binom{j+i+1}{i} \frac{d^j\omega}{dt^j} \frac{\partial}{\partial q^\sigma_{(j+i+1)}} \otimes (dq^\sigma_{(i)} - q^\sigma_{(i+1)}\, dt)$$

on $J^{k+1}\pi$, where i, j are non-negative integers and $\frac{d^0\omega}{dt^0} \equiv \omega$. Then

$$h_{\Xi_\Omega} = \frac{1}{2}\left[h_{\Gamma^{(k+2)}} + I + \frac{1}{k+2}\left(kv_{\Gamma^{(k+2)}} - \frac{2}{\omega}\,\mathcal{L}_{D_{\Gamma^{(k+2)}}} S^{(k+1)}_\Omega\right)\right] \tag{25.12}$$

is the horizontal form of a (global) connection Ξ_Ω on $\pi_{k+1,k}$, associated to $\Gamma^{(k+2)}$. Clearly, (25.5) corresponds to $\Omega = dt$ and $S^{(k+1)}_{dt} \equiv S^{(k+1)}_1$.

On the other hand, the result can be related to that of Prop. 17.2, for $k = 0$ and $\dim X = 1$. The 'strong horizontal components' of Ξ_Ω defined by (25.12) are then

$$\Xi^\sigma_\lambda = \frac{1}{2}\left(\frac{\partial \Gamma^\sigma_{(2)}}{\partial q^\lambda_{(1)}} - \frac{d\omega}{dt}\,\frac{1}{\omega}\,\delta^\sigma_\lambda\right)$$

with the quantity $\Lambda(t) = -\frac{d\omega}{dt}\frac{1}{\omega}$ being transformed in the same way like the component of a linear connection on X. Consequently, there is a geometric interpretation of Ξ^Λ_0 in this situation; namely, it is just Ξ_Ω for an arbitrary volume form Ω on X which is the integral section (i.e. $\Lambda^* \circ \Omega = j^1\Omega$) of the dual connection Λ^* on τ^*_X.

26. Examples

Let $k = 0$. Consider the pair of equations

$$\begin{aligned} \frac{\partial z}{\partial x} &= A(x,y,z) \\ \frac{\partial z}{\partial y} &= B(x,y,z). \end{aligned} \tag{26.1}$$

called sometimes the system of *simultaneous* equations of the first order. The integrability conditions (24.14) now read

$$\frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}A = \frac{\partial A}{\partial y} + \frac{\partial A}{\partial z}B. \tag{26.2}$$

and the general solution of (26.1) can be viewed as a one-parameter system of surfaces in $\mathbb{R}^3$. The characteristic 2-connection is the equation

$$y'' = A(x,y,y') + B(x,y,y')y' \tag{26.3}$$

and it is easy to see that in case of $A = A(x,y)$, $B = B(x,y)$, (26.2) means just the condition for (26.3) to be *exact*, i.e. $A\,dx + B\,dy = dz$.

Moreover, since

$$\begin{aligned} \frac{\partial z}{\partial x} &= A(x,y,z) - f(x)z \\ \frac{\partial z}{\partial y} &= B(x,y,z) + f(x). \end{aligned} \tag{26.4}$$

is a system associated to (26.3) by means of $S_1^{(1)}$ (see (24.23)), if $f(x)$ is such a function that

$$\frac{\partial B}{\partial x} + \frac{\partial B}{\partial z}A - \frac{\partial A}{\partial y} - \frac{\partial A}{\partial z}B = \left(\frac{\partial B}{\partial z}z + \frac{\partial A}{\partial z} - B - f\right)f - f'. \tag{26.5}$$

the system (26.3) can be reduced to the first-order one by solving (26.4). In particular, for the system

$$y'' = A(x,y) + B(x,y)y'.$$

(26.5) reads

$$\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x} = fB + f^2 + f'. \tag{26.6}$$

and the corresponding first-order equation is of the form

$$y' = \int B(x,y)\,dy + f(x)y + \varphi(x).$$

where $\varphi(x)$ is determined by (26.4) together with (26.6).

For example, for the equation

$$y'' = a(x)y' + b(x)y + c(x)$$

we have $A(x,y) = b(x)y + c(x)$, $B(x,y) = a(x)$, and (26.6) now means

$$b = a' + fa + f^2 + f'.$$

As a consequence we have: for arbitrary $a, c, f \in \mathcal{F}(\mathbb{R})$, the linear equation

$$y'' = ay' + (a' + fa + f^2 + f')y + c \tag{26.7}$$

is solvable in quadratures. Namely, the one-parameter family of fields of paths of (26.7) is created by linear equations

$$y' = (a(x) + f(x))y + \varphi(x)$$

with $\varphi(x)$ satisfying

$$\varphi'(x) = c(x) - f(x)\varphi(x).$$

Another second-order equation

$$y'' = \frac{2y'^2}{y} + \left(\frac{b}{2y^2} - ay^2\right)y' + ay^2 \;\; (= f(x,y,y')) \tag{26.8}$$

with $a, b \in \mathbb{R}$, is characteristic to

$$\begin{aligned} \frac{\partial z}{\partial x} &= ay^2 \\ \frac{\partial z}{\partial y} &= \frac{2z}{y} + \frac{b}{2y^2} - ay^2. \end{aligned} \tag{26.9}$$

The last system constitutes an integrable connection since both sides of (26.2) equals to $2ay$. It is easy to see that

$$z = Cy^2 + ay^2(x-y) - \frac{b}{6y} \;\; (= g(x,y))$$

is the general solution of (26.9) and accordingly it represents the reduction of the order of (26.8) in the sense that

$$\frac{d}{dx}g(x,y)\circ g = f(x,y,g(x,y)).$$

On the other hand, the integration of the system

$$\begin{aligned}\frac{\partial z}{\partial x} &= f(x)(z+c)\\ \frac{\partial z}{\partial y} &= g(y)(z+c).\end{aligned}$$

with $f, g \in \mathcal{F}(\mathbb{R})$, $c \in \mathbb{R}$, can be transferred to the integration of the corresponding

$$y'' = (y'+c)(f(x)+g(y)y').$$

which immediately yields

$$\ln|z+c| = \int f(x)\,dx + \int g(y)\,dy + k.$$

27. Connections in $\mathbb{R}\times T^{k+r}M \to \mathbb{R}\times T^kM \to \mathbb{R}$

We have already used the evident fact of the coincidence between semiholonomic and holonomic jets, which is due to the one-dimensionality of the base. On the other hand, the ideas of Sec. 20 suggest the possibility of an effective description of the concepts, studied up to now (both in time-dependent and autonomous situations) by means of slightly less transparent constructions.

A $\pi_{k+r,k}$-*semiholonomic* connection on $\pi_{k+r}\colon \mathbb{R}\times T^{k+r}M \to \mathbb{R}$ is a section

$$\Sigma\colon \mathbb{R}\times T^{k+r}M \to A_{\pi_{k+r,k}} \subset \mathbb{R}\times TT^{k+r}M$$

of $(\pi_{k+r})_{1,0}|_{A_{\pi_{k+r,k}}}$. It generates a one-dimensional π_{k+r}-horizontal distribution H_Σ spanned by

$$D_\Sigma = \frac{\partial}{\partial t} + \sum_{i=0}^{k} q^\sigma_{(i+1)}\frac{\partial}{\partial q^\sigma_{(i)}} + \sum_{i=k+1}^{k+r} \Sigma^\sigma_{(i+1)}\frac{\partial}{\partial q^\sigma_{(i)}}. \tag{27.1}$$

which can be called the $\pi_{k+r,k}$-*semispray* on $\mathbb{R}\times T^{k+r}M$; it should be stressed that the 'classical' semispray (23.1) is just the $\pi_{k-1,k}$-semispray on $\mathbb{R}\times T^kM$.

In a standard way, both the time-dependent and independent versions of $\pi_{k+r,k}$-semisprays on $T^{k+r}M$ can be derived, as well.

As regards connections on $\pi_{k+r,k}$, there is again a distinguished subfamily of them, which is due to the existence of a submanifold

$$\mathbb{R} \times J^1\tau_M^{k+r,k} \hookrightarrow J^1\pi_{k+r,k}. \tag{27.2}$$

This is defined analogously to (24.3) with local jet fields from $\mathcal{S}_{\mathrm{loc}}(\pi_{k+r,k})$ being substituted for local $(k+1)$-connections on π. If the annihilators of the horizontal distribution of a connection Ξ on $\pi_{k+r,k}$ are

$$dq^\sigma_{(\ell)} = \Xi^\sigma_{(\ell)}\, dt + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(\ell)\lambda}\, dq^\lambda_{(i)} \tag{27.3}$$

for $\ell = k+1, \dots, k+r$, then

$$\Xi\colon \mathbb{R} \times T^{k+r}M \to \mathbb{R} \times J^1\tau_M^{k+r,k} \tag{27.4}$$

holds if, and only if, $\Xi^\sigma_{(\ell)} = 0$ for all ℓ. If, for example, Λ is a connection on $\tau_M^{k+r,k}$, then

$$\Xi = \mathrm{id}_{\mathbb{R}} \times \Lambda$$

is of the form (27.4), which can be applied when studying the interrelations between non-autonomous and autonomous formalisms.

For an arbitrary connection Ξ on $\pi_{k+r,k}$, the connection (20.23) is $\pi_{k+r,k}$-semiholonomic, i.e. its horizontal distribution is spanned by $\pi_{k+r,k}$-semisprays, and their relations could be studied. Nevertheless, our main concern is (in accordance with the motivations and results of Sec. 20) with the *characterizable connections* on $\pi_{k+r,k}$ in sense of Def. 20.1 (see also Section 25). By (20.24), $\Bbbk_{\Phi_0} \circ \Xi$ is holonomic if, and only if,

$$\begin{aligned} q^\sigma_{(k+2)} &= \Xi^\sigma_{(k+1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+1)\lambda} q^\lambda_{(i+1)} \\ &\vdots \\ q^\sigma_{(k+r)} &= \Xi^\sigma_{(k+r-1)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+r-1)\lambda} q^\lambda_{(i+1)}. \end{aligned} \tag{27.5}$$

and the *characteristic semispray connection* $\Gamma^{(k+r+1)}$ is locally defined by

$$\Gamma^{\sigma}_{(k+r+1)} = \Xi^{\sigma}_{(k+r)} + \sum_{i=0}^{k} \Xi^{\sigma(i)}_{(k+r)\lambda} q^{\lambda}_{(i+1)}. \tag{27.6}$$

This in particular defines the *characteristic semispray* on $T^{k+r}M$ for a *characterizable connection* on $\tau_M^{k+r,k}$ by

$$w_\Lambda = \sum_{i=0}^{k+r-1} q^{\sigma}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(i)}} + \sum_{i=0}^{k} \Lambda^{\sigma(i)}_{(k+r)\lambda} q^{\lambda}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(k+r)}}.$$

As regards the jet fields $\varphi \in \mathcal{S}(\pi_{k+r,k})$, these can be identified with $\pi_{k+r,k}$-semispray distributions spanned by

$$D_\varphi = \frac{\partial}{\partial t} + \sum_{i=0}^{k-1} q^{\sigma}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(i)}} + \sum_{i=k}^{k+r-1} \varphi^{\sigma}_{(i+1)} \frac{\partial}{\partial q^{\sigma}_{(i)}}. \tag{27.7}$$

and in view of Prop. 20.3, we should recall that the equations for the prolongations $\Gamma^{(k+1)(r)}$ of a semispray connection $\Gamma^{(k+1)}$ on π are

$$\begin{aligned} q^{\sigma}_{(k+1)} &= \Gamma^{\sigma}_{(k+1)} \\ q^{\sigma}_{(k+2)} &= D(\Gamma^{\sigma}_{(k+1)}) \\ &\vdots \\ q^{\sigma}_{(k+r+1)} &= D^{(r)}(\Gamma^{\sigma}_{(k+1)}). \end{aligned}$$

where $D^{(r)} = D^{k+r,k+r-1} \dots D^{k+1,k}$ $(= \frac{d^r}{dt^r})$.

Finally, here is the 'one-dimensional' version of Prop. 20.4.

PROPOSITION 27.1. *Let $\Gamma^{(k+r+1)}$ be a $(k+r+1)$-connection on π and $\{a^1, .., a^K\}$, where $K = rm$, be a set of independent first integrals of $\Gamma^{(k+r+1)}$, defined on some open $W \subset J^{k+r}\pi$. If the matrix*

$$A = \left(\frac{\partial a^L}{\partial q^{\sigma}_{(\ell)}} \right). \tag{27.8}$$

where $\ell = k+1, \dots, k+r$, is regular on W, then

$$H_\Xi = \text{anih}\{da^1, \dots, da^K\} \tag{27.9}$$

defines an $\pi_{k+r,k}$-integral of $\Gamma^{(k+r+1)}$ on W.

Proof. First it should be stressed that we suppose $W \subset \pi^{-1}_{k+r,0}(V)$, where (V, ψ) is a fibered chart on Y.

By definition, the distribution (27.9) is completely integrable. Let us denote by $(A^{\sigma}_{(\ell)L})$ the inverse matrix to A, where σ and (ℓ) label the rows and L the columns. Then the annihilators of H_{Ξ} are

$$dq^{\sigma}_{(\ell)} + A^{\sigma}_{(\ell)L}\frac{\partial a^L}{\partial t}\, dt + \sum_{i=0}^{k} A^{\sigma}_{(\ell)L}\frac{\partial a^L}{\partial q^{\lambda}_{(i)}}\, dq^{\lambda}_{(i)}$$

and it remains to show that Ξ is characterizable and its characteristic connection is just $\Gamma^{(k+r+1)}$. Since for an arbitrary ℓ it holds

$$\begin{aligned}\Xi^{\sigma}_{(\ell)} + \sum_{i=0}^{k}\Xi^{\sigma(i)}_{(\ell)\lambda}q^{\lambda}_{(i+1)} &= -A^{\sigma}_{(\ell)L}\left(\frac{\partial a^L}{\partial t} + \sum_{i=0}^{k}\frac{\partial a^L}{\partial q^{\lambda}_{(i)}}q^{\lambda}_{(i+1)}\right)\\ &= A^{\sigma}_{(\ell)L}\left(\sum_{i=k+1}^{k+r-1}\frac{\partial a^L}{\partial q^{\lambda}_{(i)}}q^{\lambda}_{(i+1)} + \frac{\partial a^L}{\partial q^{\lambda}_{(k+r)}}\Gamma^{\lambda}_{(k+r+1)}\right)\\ &= \begin{cases} q^{\sigma}_{(\ell+1)}, & \text{for } \ell = k+1,\dots,k+r-1\\ \Gamma^{\sigma}_{(k+r+1)}, & \text{for } \ell = k+r,\end{cases}\end{aligned}$$

the proof is completed (see (27.5) and (27.6)). ∎

References

[1] Alekseevskiĭ, D.V., Vinogradov, A.M., Lychagin, V.V., "Fundamental Ideas and Concepts of Differential Geometry", Itogi nauki i tekhniki, Sovremennye problemy matematiki, 28, VINITI, 1988, Moscow (in russian).

[2] Anderson, I.M., Kamran, N., Olver, P.J., Internal, External and Generalized Symmetries, preprint.

[3] Anastasiei, M., Kawaguchi, H., A geometrical theory of the time dependent Lagrangians I. Non-linear connections *Tensor, N.S.* **48** (1989), 273–282. II. M-connections *Tensor, N.S.* **48** (1989), 283–293. III. Applications *Tensor, N.S.* **49** (1990), 296–304.

[4] Arnol′d, V.I., "Additional Parts of the Theory of Ordinary Differential Equations", Nauka, 1978, Moscow (in russian).

[5] Bliznikas, V.I., Vosylius, R. V., Non-holonomic connections, *Liet. Matem. Rink..* **27** (1) (1987), 15–27. (in russian).

[6] Cariñena, J.F., Martínez, E., Symmetry theory and Lagrangian inverse problem for time-dependent second-order differential equations *J. Phys. A: Math. Gen.,* **22** (1989), 2659–2665.

[7] Cartan, H., "Calcul Différentiel Formes Différentielles", Mir, 1971, Moscow, (transl. to russian).

[8] Crampin, M., Martínez, E. Sarlet, W., Linear connections for systems of second-order ordinary differential equations *Ann. Inst. Henri Poincaré,* **65** (1996), 223 – 249.

[9] Crampin, M., Prince, G. E., Thompson, G., A geometrical version of the Helmholtz conditions in time dependent Lagrangian dynamics *J. Phys. A. Math. Gen.,* **17** (1984), 1437 – 1447.

[10] Crampin, M., Sarlet, W., Cantrijn, F., Higher-order differential equations and higher-order lagrangian mechanics *Math. Proc. Cambridge Philos. Soc.,* **99** (1986), 565 – 587.

[11] de Andres, L.C., de León, M., Rodrigues, P.R., Connections on tangent bundles of higher order *Demonstratio Math.,* **22** (3) (1989), 607 – 632.

[12] de Andres, L.C., de León, M., Rodrigues, P.R., Connections on tangent bundles of higher order associated to regular Lagrangians *Geom. Dedicata,* **39** (1991), 12 – 18.

[13] de León, M., Marrero, J.C., Time-dependent linear Lagrangians: the inverse problem, symmetries and constants of motion, in "International Symposium on Hamiltonian Systems and Celestian Mechanics", Guanajnato, Mexico, 1991, Advanced Series in Nonlinear Dynamics, 4, World Scientific, 55 – 83, Singapore, 1993.

[14] de León, M., Martín de Diego, D., Symmetries and constants of the motion for higher order Lagrangian systems *J. Math. Phys.,* **36** (8) (1995), 4138 – 4161.

[15] de León, M., Martín de Diego, D., Symmetries and constants of the motion for higher order Lagrangian systems. II: the non-autonomous case *Extracta Mathematicae,* **9** (2) (1994), 111 – 134.

[16] de León, M., Rodrigues, P.R., "Generalized Classical Mechanics and Field Theory", North-Holland Mathematical Studies 112, North-Holland, 1985, Amsterdam.

[17] de León, M., Rodrigues, P.R., Higher order almost tangent geometry and non-autonomous Lagrangian dynamics *Supp. Rend. Circolo Mat. Palermo,* **16** (1987), 157 – 171.

[18] de León, M., Rodrigues, P.R., Dynamical connections and non-autonomous Lagrangian systems *Ann. Fac. Sci. Toulouse,* **IX** (1988), 171 – 181.

[19] de León, M., Rodrigues, P.R., "Methods of Differential Geometry in Analytical Mechanics", North-Holland Mathematics Studies, 158, North-Holland, 1989, Amsterdam.

[20] Doupovec, M., Kolář, I., Natural affinors on time-dependent Weil bundles *Arch Math. (Brno),* **27** (1991), 205 – 209.

[21] Doupovec, M., Vondra, A., On certain natural transformations between connections "Proc. Conf. Diff. Geom. and Its Appl.", Opava, 1992, Silesian University, Opava, 1993, 273 – 279.

[22] Doupovec, M., Vondra, A., Some natural operations between connections on fibered manifolds *Rendiconti del Circolo Matematico di Palermo,* Serie II - Suppl. **39** (1996), 73 – 84.

[23] Doupovec, M., Vondra, A., Natural relations between connections in 2-fibred manifolds, in "New Developments in Differential Geometry", edited by L. Tammasi and J. Szenthe, Kluwer Academic Publishers, Dordrecht, 1996, 113 – 130.

[24] Doupovec, M., Vondra, A., Invariant subspaces in higher-order jet prolongations of a fibered manifold, preprint Brno, 1997.

[25] Duzhin, S.V., Lychagin, V.V., Symmetries of distributions and quadrature of ordinary differential equations *Acta Appl. Math.,* **24** (1991), 29 – 57.

[26] Goldschmidt, H., Integrability criteria for systems of non-linear differential equations *J. Differential Geom.,* **1** (1969), 269 – 307.

[27] Kamke, E., "Differentialgleichungen", Leipzig, 1959.

[28] Kolář, I., Connections in 2-fibred manifolds *Arch. Math. (Brno),* **17** (1981), 23 – 30.

[29] Kolář, I., Some natural operations with connections *J. Nat. Acad. Math.,* **5** (1987), 127 – 141.

[30] Kolář, I., Modugno, M., On the algebraic structure on the jet prolongation of fibred manifolds *Czechoslovak Math. J.,* **40** (115) (1990), 601 – 611.

[31] Kolář, I., Modugno, M., Torsions of connections on some natural bundles *Diff. Geom. Appl.,* **2** (1992), 1 – 16.

[32] Kolář, I., Michor, P.W., Slovák, J., "Natural Operations in Differential Geometry", Springer-Verlag, 1993.

[33] Kolář, I., Slovák, J., Prolongations of vector fields to jet bundles, in "Proc. Winter School on Geometry and Physics", Srní, 1989, Suppl. Rendiconti Circolo Mat. Palermo, Serie II 21, 1989, 103 – 111.

[34] Krupka, D., Some geometrical aspects of variational problems on fibred manifolds *Folia Fac. Sci. Natur. Univ. Purk. Brun. Phys.,* **XIV** (1973), 1 – 65.

[35] Krupka, D., On the higher-order Hamilton theory in fibered spaces, in "Geometrical Methods in Physics", Proc. Conf. Diff. Geom. and Its Appl., Nové Město na Moravě, 1983, J. E. Purkyně Univ., Brno, 1984, 167 – 183.

[36] Krupka, D., Geometry of Lagrangian structures 1 *Arch. Math. (Brno),* **22** (1986), 159 – 174. Geometry of Lagrangian structures 2 *Arch. Math. (Brno),* **22** (1986), 211 – 228.

[37] Krupka,D., Janyška, J., "Lectures on differential invariants", Folia Fac. Sci. Natur. Univ. Purk. Brun. Phys., Brno, 1990.

[38] Krupková, O., Lepagean 2 – forms in higher order Hamiltonian mechanics. I. Regularity *Arch. Math. (Brno),* **22** (1986), 97 – 120. Lepagean 2 – forms in higher order Hamiltonian mechanics. II. Inverse problem *Arch. Math. (Brno),* **23** (1987), 155 – 170.

[39] Krupková, O., "Variational analysis on fibered manifolds over one-dimensional bases", Ph.D. Thesis, Dept. of Mathematics, Silesian Univ. at Opava, Czechoslovakia, 1992, 1 – 67.

[40] Krupková, O., Variational metrics on $R \times TM$ and the geometry of nonconservative mechanics *Math. Slovaca,* **44** (3) (1994) 315 – 335.

[41] Krupková, O., Symmetries and first integrals of time-dependent higher-order constrained systems *J. Geom. Phys.,* **18** (1996), 38 – 58.

[42] Krupková, O., Vondra, A., On some integration methods for connections on fibered manifolds, in "Proc. Conf. Diff. Geom. and Its Appl.", Opava, 1992, Silesian University, Opava, 1993, 89 – 101.

[43] Lychagin, V., "Lectures on Geometry of Differential Equations", Università La Sapienza, Roma, 1992.

[44] Mangiarotti, L., Modugno, M., Fibred Spaces, Jet Spaces and Connections for Field Theories, in "Proceedings of International Meeting Geometry and Physics", Florence, 1982 Pitagora Editrice, Bologna, 1983, 135 – 165.

[45] Krupková, O., Connections and differential calculus on fibred manifolds. Aplications to field theory, *Istituto di Matematica Applicata "G. Sansone", Firense,* (1989), 1 – 147.

[46] Massa, E., Pagani, E., Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics *Ann. Inst. Henri Poincaré: Phys. Theor.,* **61** (1994), 17 – 62.

[47] Modugno, M., Jet involution and prolongation of connections *Časopis Pěst. Mat.,* **114** (4) (1989), 356 – 365.

[48] Olver, P.J., "Aplications of Lie Groups to Differential Equations", Springer-Verlag, 1986.

[49] Pommaret, J.F., "Systems of Partial Differential Equations and Lie Pseudogroups", Mir, Moscow, 1983, (transl. to russian).

[50] Postnikov, M.M., "Lectures on Geometry IV. Differencial Geometry", Nauka, Moscow, 1988, (in russian).

[51] Krupková, O., Toward a classification of dynamical symmetries in classical mechanics *Bull. Austral. Math. Soc.,* **27** (1983), 53-71.

[52] Krupková, O., A complete classification of dynamical symmetries in classical mechanics *Bull. Austral. Math. Soc.,* **32** (1985), 299 – 308.

[53] Rashevskiĭ, P.K., "Geometrical Theory of Partial Differential Equations", OGIZ, Moscow, 1947, (in russian).

[54] Sarlet, W., Symmetries and alternative Lagrangians in higher-order mechanics *Physics Lett. A,* **108** (1) (1985), 14 – 18.

[55] Sarlet, W., Vandecasteele, A., Cantrijn, F., Martínez, E., Derivations of forms along a map: the framework for time-dependent second-order equations *Diff. Geom. Appl.,* **5** (1995), 171 – 203.

[56] Saunders, D.J., "The Geometry of Jet Bundles", Cambridge University Press London Mathematical Society Lecture Note Series, 142, Cambridge, 1989.

[57] Stepanov, V.V., "Course on Differential Equations", Moscow, 1950 (in russian).

[58] Vinogradov, A.M., Krasil′shchik, I.S., Lychagin, V.V., "Introduction to the Geometry of Nonlinear Differential Equations", Nauka, Moscow, 1986, (in russian).

[59] Vondra, A., Semisprays, connections and regular equations in higher order mechanics, in "Proc. Conf. Diff. Geom. and Its Appl.", Brno 1989 Word Scientific, Singapore, 1990, 276 – 287.

[60] Vondra, A., On some connections related to the geometry of regular higher – order dynamics *Sborník VA, řada B* **2** (1992), 7 – 18.

[61] Vondra, A., Sprays and homogeneous connections on $R \times TM$ *Arch. Math. (Brno),* **28** (1992), 163 – 173.

[62] Vondra, A., Natural dynamical connections *Czechoslovak Math. J.,* **41** (1991), 724 – 730.

[63] Vondra, A., Geometry of second-order connections and ordinary differential equations *Mathematica Bohemica,* **120** (2) (1995), 145 – 167.

[64] Vondra, A., Symmetries of connections on fibered manifolds *Arch. Math. (Brno),* **30** (2) (1994), 97 – 115.

[65] Vondra, A., Prolongations and fields of paths for higher-order O.D.E. represented by connections on a fibered manifold *Extracta Mathematicae,* **11** (1) (1996), 229 – 242.

[66] von Westenholz, C., " Differential Forms in Mathematical Physics", North-Holland, Amsterdam, 1981, (revised edition).

[67] Vosylius, R.V., The contravariant theory of the differential prolongation on a space with a connection in "Itogi nauki i tekhniki, Problemy geometrii (14)", Viniti, Moscow, 1983, 101 – 176. (in russian)