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Complexification Norms and Estimates for Polynomials

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We denote the algebraic complexification of a real vector space X by $X_{\mathbb{C}}$. Addition in this space is defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

and multiplication by complex numbers as

$$(a + ib)(x, y) = (ax - by, bx + ay).$$

The complex vector space $X_{\mathbb{C}}$ can also be defined in terms of the real tensor product $X \otimes \mathbb{R}^2$. A typical element of this space may be represented by $x \otimes e_1 + y \otimes e_2$, where e_1, e_2 are orthogonal unit vectors in $\mathbb{R}^2$. This can be viewed as a complex vector space by defining $(a + ib)[x \otimes e_1 + y \otimes e_2] = (ax - by) \otimes e_1 + (ay + bx) \otimes e_2$. In both complexification processes it is unambiguous to write a typical element of the complex space $X_{\mathbb{C}}$ as $z = x + iy$, where $x, y \in X$.

The complexification of normed spaces is more interesting as it does not present us with a single satisfactory approach. The natural candidate for the norm on the complex space is

$$\|z\|_{\mathrm{nat}} = \sqrt{\|x\|_X^2 + \|y\|_X^2}.$$

This clearly imitates the modulus for the complex numbers. It does not however respect the complex homogeneity of $X_{\mathbb{C}}$. The following three properties are easily seen to be derived from properties exhibited by the modulus on complex numbers. Let $(X, \|\,\|_X)$ be a real normed space, $X_{\mathbb{C}}$ be the algebraic complexification of X and $\gamma_{\mathbb{C}}$ be a complex norm on $X_{\mathbb{C}}$.

$$\gamma_{\mathbb{C}}(x) = \|x\|_X, \quad \forall x \in X \tag{1}$$

$$\gamma_{\mathbb{C}}(x) \le \gamma_{\mathbb{C}}(x + iy), \quad \gamma_{\mathbb{C}}(y) \le \gamma_{\mathbb{C}}(x + iy), \quad \forall x, y \in X \tag{2}$$

$$\gamma_{\mathbb{C}}(x + iy) = \gamma_{\mathbb{C}}(x - iy), \quad \forall x, y \in X \tag{3}$$

We say that a norm that satisfies (1) and (2) is *desirable*. A norm that satisfies (1) and (3) is said to be *reasonable* [6]. Muñoz, Sarantopolous and Tonge show that a reasonable norm is desirable. We do not know if the reverse is true, however we suspect that it is not. If the complex norm on the complexified space is desirable then completeness and separability are inherited from the real space, when applicable. In addition we get that desirable norms are equivalent. In fact, if α and β are desirable norms then

$$\frac{1}{2}\alpha \leq \beta \leq 2\,\alpha.$$

We have found that the smallest desirable norm has many different representations. The following version was first introduced by A.E. Taylor [5],

$$\gamma_\Phi(z) = \sup_{\phi \in B_{X^*}} \sqrt{\phi(x)^2 + \phi(y)^2},$$

where $z = x + i\,y \in X_{\mathbb{C}}$.

We can embed $(X_{\mathbb{C}},\, \gamma_{p,q})$, $1 \leq p, q \leq \infty$ isometricly in the Lebesgue–Bochner space $L_q(T;\, X \oplus_p X)$, where

$$\gamma_{p,q}(z) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\|x\cos\theta - y\sin\theta\|_X^p + \|x\sin\theta + y\cos\theta\|_X^p \right)^{\frac{q}{p}} d\theta \right)^{\frac{1}{q}},$$

$$\gamma_{p,\infty}(z) = \sup_\theta \left(\|x\cos\theta - y\sin\theta\|_X^p + \|x\sin\theta + y\cos\theta\|_X^p \right)^{\frac{1}{p}}.$$

For $p = \infty$ these reduce to the following

$$\gamma_{\infty,q}(z) = \left(\frac{1}{2\pi} \int_0^{2\pi} \max\left\{ \|x\cos\theta - y\sin\theta\|_X,\ \|x\sin\theta + y\cos\theta\|_X \right\}^q d\theta \right)^{\frac{1}{q}},$$

$$\gamma_{\infty,\infty}(z) = \sup_\theta \|x\cos\theta - y\sin\theta\|_X.$$

Not all of these norms are desirable, indeed some do not even satisfy property (1) above. The norm $\gamma_{\infty,\infty}$, on the other hand, is not only desirable but is the smallest desirable norm.

We previously mentioned that the complexification of a vector space can be achieved by means of tensor products. It therefore is natural to look at reasonable crossnorms as a source of desirable norms. We have found that all desirable norms on $X_{\mathbb{C}}$ give rise to reasonable crossnorms on $X \otimes \mathbb{R}^2$. The reverse is not so straightforward. We need the metric mapping property to shaw that a reasonable crossnorm on $X \otimes \mathbb{R}^2$ is a complex norm on $X_{\mathbb{C}}$. Properties (1) and (2) then follow immediately. This gives us that the projective norm ir and the injective norm π give rise to the largest and smallest desirable norms respectively.

The complexification of a linear functional $\phi : X \to \mathbb{R}$ is denoted by $\phi_{\mathbb{C}} : X_{\mathbb{C}} \to \mathbb{C}$ and is given by

$$\phi_{\mathbb{C}}(z) = \phi(x) + i\phi(y),$$

where $z = x+iy \in X_{\mathbb{C}}$. The importance of the desirability properties becomes more apparent at this stage.

THEOREM. *Let $(X_{\mathbb{C}}, \gamma)$ be a complexification of the real normed space X. Then γ is a desirable norm on $X_{\mathbb{C}}$ if and only if $\|\phi_{\mathbb{C}}\|_\gamma = \|\phi\|$ for every $\phi \in X^*$.*

The complexification of the dual of a real normed space can be identified with the dual of the complexification of the space by means of the mapping $J_X : (X^*)_{\mathbb{C}} \to (X_{\mathbb{C}})^*$ which is given by

$$J_X\phi(z) = (\phi_1(x) - \phi_2(y)) + i(\phi_1(y) + \phi_2(x)),$$

where $z = x + iy \in X_{\mathbb{C}}$ and $\phi = \phi_1 + \phi_2 \in (X^*)_{\mathbb{C}}$. If α is a desirable norm on $X_{\mathbb{C}}$ and if we define $\alpha^*(\phi) = \sup_{\alpha(z)\leq 1} |J_X\phi(z)|$ to be the norm on $(X^*)_{\mathbb{C}}$ then this mapping is an isomorphism. This is a desirable norm and we find that $X_{\mathbb{C}}$ is reflexive if X is. It is clear that $\pi^* = \varepsilon$ and $\varepsilon^* = \pi$, however the general situation is wide open. Therefore we pose the following.

QUESTION 1. What is α^* for various α?

The complexification of linear operators is quite similar to the above. For $L \in \mathcal{L}(X;Y)$ we define $L_{\mathbb{C}}(x + iy) = L(x) + iL(y)$. We do not have that the norm is preserved here. It can grow by a factor of up to 2. This factor will depend on the complex norms placed on the domain and range spaces. Therefore we define

$$\|L_{\mathbb{C}}\|_{\alpha,\beta} = \sup_{\alpha(z)\leq 1} \beta(L_{\mathbb{C}}(z)),$$

and

$$\rho((X_{\mathbb{C}}, \alpha), (Y_{\mathbb{C}}, \beta)) = \inf\{K : \|L_{\mathbb{C}}\|_{\alpha,\beta} \leq K\|L\|_{\alpha,\beta},\ L \in \mathcal{L}(X;Y)\}.$$

Thus if

$$\rho(\alpha,\beta) = \sup\{\rho((X_{\mathbb{C}},\alpha),\, (Y_{\mathbb{C}},\beta)) : X, Y \text{ real normed spaces}\}.$$

we get that $1 \leq \rho(\alpha,\beta) \leq 2$. Much work remains to be done on evaluating these bounds [1], [2], [4]. It is clear that $\rho(\varepsilon,\pi) = 1$ and $\rho(\pi,\varepsilon) = 2$, but the general question is open.

Question 2. Given α, β, what is $\rho(\alpha,\beta)$?

In a manner similar to the complexification of dual spaces we can define an isometric embedding

$$J_{X,Y} : \mathcal{L}(X;Y)_{\mathbb{C}} \longrightarrow \mathcal{L}(X_{\mathbb{C}};Y_{\mathbb{C}})$$

where

$$\lambda_{\alpha,\beta} = \sup_{\alpha(z)\leq 1} \beta(J_{X,Y}L(z))$$

is a complex norm on $\mathcal{L}(X;Y)_{\mathbb{C}}$. We see that $\lambda_{\alpha,\beta}$ is desirable if and only if α and β are desirable and $\rho(\alpha,\beta) = 1$. Thus $\lambda_{\pi,\varepsilon}$ is not desirable. This has implications for the complexification of multilinear mappings. Concrete representations for these new norms are difficult to arrive at.

Question 3. Given desirable norms α and β does there exist an explicit representation of $\lambda_{\alpha,\beta}$?

Let $A \in \mathcal{L}(^2X \times Y)$ then we can define its complexification is given by

$$A_{\mathbb{C}}(x+iy,\, u+iv) = A(x,u) - A(y,v) + i[A(x,v) + A(y,u)].$$

The linear mapping associated with $A_{\mathbb{C}}$ can be related to the complexification of the linear mapping associated with A by means of the isometric embedding J_Y. This gives us that

$$\|A_{\mathbb{C}}\| \leq \rho(\alpha,\beta^*)\|A\| \leq 2\|A\|,$$

where α and β are the norms on $X_{\mathbb{C}}$ and $Y_{\mathbb{C}}$ respectively. Vector valued bilinear mappings fare slightly worse. If $A \in \mathcal{L}(^2X \times Y; Z)$ then we use the isometric embedding $J_{Y,Z}$ to get

$$\|A_{\mathbb{C}}\| \leq \rho(\alpha,\lambda_{\beta,\gamma})\rho(\beta,\gamma)\|A\| \leq 2^2\|A\|,$$

where α, β and γ are the norms on $X_{\mathbb{C}}$, $Y_{\mathbb{C}}$ and $Z_{\mathbb{C}}$ respectively. In general the worst case is 2^n for n-homogeneous vector valued multilinear mappings. This can be reduced by a factor of 2 for the scalar valued case. In both cases the inequalities are sharp. If we put the largest desirable norm on the complexifications of each of the domain spaces then we get that the norm of the complexified mapping is equal to that of the real one. This raises two more questions.

QUESTION 4. For what spaces can 2^n be improved?

QUESTION 5. What are the best constants for other norms?

The restriction of an n-linear form $A : X \longrightarrow \mathbb{R}$ to the diagonal yields an n-homogeneous polynomial P. We can therefore complexify P by defining $P_{\mathbb{C}}(z) = A_{\mathbb{C}}(z, \dots, z)$. The polarization formula gives us that

$$\|P_{\mathbb{C}}\|_{\gamma} \leq 2^{n-1}\frac{n^n}{n!}\|P\|,$$

where γ is a desirable norm on $X_{\mathbb{C}}$. If X is a Hilbert space then this can be improved by noting that the polarization constant for such a space is one. Furthermore if we use the largest desirable norm on the complexification of a Hilbert space then we get

$$\|P_{\mathbb{C}}\|_{\pi} = \|P\|.$$

An alternative approach is to use the following integral representation of the complexification of a polynomial.

THEOREM. *If $P \in \mathcal{P}(^nX)$ then*

$$P_{\mathbb{C}}(z) = \frac{2^n}{2\pi}\int_0^{2\pi} P(x\cos\theta + y\sin\theta)e^{in\theta}\,d\theta,$$

where $z = x + iy \in X_{\mathbb{C}}$.

This clearly gives us that

$$\|P_{\mathbb{C}}\|_{\gamma} \leq 2^n\|P\|.$$

This has been improved by Muñoz, Sarantopolous and Tonge [6] to 2^{n-1} and they show that this is sharp.

The details of this paper can be found in the authors thesis [3].

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