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Simple Examples of One-Parameter Planar Bifurcations

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1. Introduction and Statement of Main Results

In this paper we give simple and low degree examples of one-parameter polynomial families of planar differential equations which present generic, codimension one, isolated, compact bifurcations. In contrast with some examples which appear in the usual text books each bifurcation occurs when the bifurcation parameter is zero. We study the total number of limit cycles that the examples present and we also make their phase portraits on the Poincaré sphere.

Consider a differential equation $\dot{x} = X(x)$, where X belongs to a family of smooth vector fields $\mathcal{F}$ and $x \in \mathbb{R}^2$. Fix a compact set $K \subset \mathbb{R}^2$ and a distance d between the elements of $\mathcal{F}$ restricted to K. Following [1], we will say that X gives a dynamical system of first degree of structural instability in K if it is structurally unstable in K, whereas any sufficiently d-close system inside $\mathcal{F}$ is either structurally stable in K or topologically equivalent to X. It is well known, (see also [1]), that for generic families of one-parameter differential equations passing through a differential equation of first degree of structural instability only the following compact bifurcations appear:

(1) Saddle-node.

(2) Saddle connection.

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(3) Andronov-Hopf bifurcation.

(4) Saddle loop.

(5) Saddle-node loop.

(6) Semi-stable periodic orbit.

Note that the above list does not include non isolated codimension one bifurcations. These bifurcations are studied in [15].

We will say that an one-parameter family of planar differential equations

$$\dot{x} = P(x, y, \lambda) \ , \qquad \dot{y} = Q(x, y, \lambda) \ , \qquad x, y, \lambda \in \mathbb{R},$$

is a family of polynomial systems of degree n (PSn) if P and Q are polynomials in (x, y) and $\max(\deg P, \deg Q) = n$.

It is clear that for $PS1$ the bifurcations considered do not appear. So simplest examples should appear for PSn with $n \geq 2$.

Examples of the six types of bifurcations are well known (see [1], [10] or [15]). These examples either are simple but the degree of the PSn is not the lowest possible, or they have the lowest degree but for them the value of the parameter for which the bifurcation occurs is not analytically known. This is the case for instance for the Bogdanov-Takens bifurcation, in which a saddle loop bifurcation appears (see [14]). On the other hand in [6] the authors prove that most of the generic local three-parameter bifurcations appear for $PS2$, but the systems considered are not easy to study.

In this paper we present examples of one-parameter bifurcations with the following properties: the PSn are simple, the bifurcation takes when the bifurcation parameter is zero and $n = 2$ (except in example (6) where $n = 3$). We remark that of course bifurcation (6) appears for $PS2$ but we have not been able to find any example satisfying the previous properties. The main problem to find such an example is to have a concrete $PS2$ -we mean with given coefficients- with a semi-stable limit cycle. As far as we know there are no examples of the above situation. In fact all known examples of $PS2$ exhibiting algebraic limit cycles are such that this limit cycle is hyperbolic, see [2, 7, 16]. On the other hand it is very easy (using polar coordinates) to construct an example of this bifurcation for $PS4$. This is the example which appears in most text books.

In the following result we list the examples that we will study in next sections. Phase portraits of these examples are drawn on the Poincaré sphere, see [11, 15]. In most cases we are also able to study the exact number of limit cycles that the systems present. This is the most difficult part of the paper.

We think that the examples that we give can be useful from a pedagogical point of view.

THEOREM. (Simple one-parameter bifurcations) *The following six examples present for $\lambda = 0$ the generic codimension one compact isolated bifurcation indicated. Their phase portraits in the Poincaré sphere are showed in Figure 1, where a phase portrait marked with a star means that for this system we do not know its number of limit cycles.*

(1) *Saddle-node,*

$$\begin{cases} \dot{x} = x^2 + \lambda\,, \\ \dot{y} = -y\,, \qquad -\infty < \lambda < \infty\,. \end{cases}$$

(2) *Saddle connection,*

$$\begin{cases} \dot{x} = x^2 - 1 - 2\lambda xy\,, \\ \dot{y} = -2xy + \lambda(1 - x^2)\,, \qquad -\infty < \lambda < \infty\,. \end{cases}$$

(3) *Andronov-Hopf bifurcation,*

$$\begin{cases} \dot{x} = -y(x - y + 1) - (x^2 + y^2 - \lambda)\,, \\ \dot{y} = x(x - y + 1)\,, \qquad -\infty < \lambda < 1/2\,. \end{cases}$$

(4) *Saddle loop,*

$$\begin{cases} \dot{x} = 3x + 5y + 2x^2 - \lambda(5x + 3y)(1 + x)\,, \\ \dot{y} = \lambda(3x + 5y + 2x^2) + (5x + 3y)(1 + x)\,, \end{cases}$$
$$\lambda^- \lesssim \lambda < 25 - 4\sqrt{39},\ \lambda^- \lesssim 0\,.$$

(5) *Saddle-node loop,*

$$\begin{cases} \dot{x} = y(x - 1 - \lambda) + (x^2 + y^2 - 1)\,, \\ \dot{y} = -x(x - 1 - \lambda)\,, \qquad -1 < \lambda < \infty\,. \end{cases}$$

(6) *Semi-stable periodic orbit,*

$$\begin{cases} \dot{x} = -x + (4 + \lambda)y + ((1 + \lambda)x + (1 - \lambda)y)(x^2 + y^2 - 4xy)\,, \\ \dot{y} = -\lambda x + (4\lambda - 1)y + ((\lambda - 1)x + (\lambda + 1)y)(x^2 + y^2 - 4xy)\,, \end{cases}$$
$$-\sqrt{3}/2 < \lambda < 1/2\,.$$

In each example we give the range of values that we consider to ensure that phase portraits in the Poincaré sphere do not change. In Figure 1 the range of variation of λ is always (λ^-, λ^+) where these values are the values that appear in this statement.

As we will see, several of the examples are semicomplete families of rotated vector fields (SCFRVF), see [5, 13]. It is well known that these families give a suitable way to vary the parameter to obtain generic bifurcations.

In Section 2 we will study the number of limit cycles that the PSn considered can have. Section 3 deals with their phase portraits.

2. Study of the Number of Limit Cycles

It is well known that if a $PS2$ with an invariant ellipse has a limit cycle then it is the ellipse itself, see [3] or [17, pp. 256–258]. In the next proposition we give a simple, different and self contained proof of the above fact for some $PS2$ which includes Examples 3 and 5. The proof presented is based on the approach of [8].

Proposition 1. *System*

$$\begin{cases} \dot{x} = y(ax+by+c) + d(x^2+y^2-e)\,, \\ \dot{y} = -x(ax+by+c)\,, \end{cases} \tag{7}$$

has no periodic orbits different from $x^2+y^2-e=0$ if $ad \neq 0$. Furthermore when this curve is a periodic orbit it is a hyperbolic limit cycle.

Proof. Note that the periodic orbits of system (7) (that we denote by $(\dot{x}, \dot{y}) = X(x,y)$) can not cut the straight line $ax+by+c=0$ since $a\dot{x} + b\dot{y}|_{ax+by+c=0} = ad(x^2+y^2-e)$ and $x^2+y^2-e=0$ is an invariant curve for (7). So we can take the Dulac function $B(x,y) = (ax+by+c)^{-1}(x^2+y^2-e)^{-1}$. Then

$$\mathrm{div}(BX) = \frac{-ad}{(ax+by+c)^2}\,,$$

and so does not change sign. Hence, from Bendixson-Dulac Criterion system (7) has no periodic orbits in the simply connected components of $K = \mathbb{R}^2 - \{(x,y)\,|\,x^2+y^2-e=0\} \cup \{(x,y)\,|\,ax+by+c=0\}$. To finish the proof it is enough to show that, for $e \geq 0$, it is impossible that (7) has a periodic orbit Γ surrounding $x^2+y^2-e=0$. Assume first that $e>0$ and that Γ exists.

Put Γ_ε for $\{(x,y)\,|\,x^2+y^2-(e+\varepsilon)=0\}$ and consider the region D_ε with boundaries Γ, $\Gamma_{-\varepsilon}$ counterclockwise and Γ_ε clockwise. By Stokes Theorem

$$\iint_{D\varepsilon} \mathrm{div}BX\,dxdy = \int_\Gamma (BX)_1\,dy-(BX)_2\,dx + \int_{-\Gamma_\varepsilon+\Gamma_{-\varepsilon}} (BX)_1\,dy-(BX)_2\,dx\,.$$

Note that the left hand side of the above equality is not zero even if we take limit when ε goes to zero. On the other hand the first term of the right hand side of the equality is always zero because Γ is a solution of (7). So if we show that $\lim_{\varepsilon\to 0}\int_{-\Gamma_\varepsilon+\Gamma_{-\varepsilon}}(BX)_1\,dy-(BX)_2\,dx=0$ we will have a contradiction and the proof will follow for $e>0$. To prove this first we compute

$$\begin{aligned}\int_{\Gamma_\varepsilon}(BX)_1\,dy-(BX)_2\,dx &= \int_0^{2\pi}\Big(x(BX)_1+y(BX)_2\Big)\,d\theta\\ &= \int_0^{2\pi}\left[x\left\{\frac{y}{x^2+y^2-e}+\frac{d}{ax+by+c}\right\}+y\left\{\frac{-x}{x^2+y^2-e}\right\}\right]d\theta\\ &= \int_0^{2\pi}\frac{dx}{ax+by+c}d\theta = \int_0^{2\pi}\frac{d\cos\theta}{a\cos\theta+b\sin\theta+\frac{c}{\sqrt{e+\varepsilon}}}d\theta\,.\end{aligned}$$

Hence $\lim_{\varepsilon\to 0}\int_{-\Gamma_\varepsilon+\Gamma_{-\varepsilon}}(BX)_1\,dy-(BX)_2\,dx=0$.

The case $e=0$ follows in a similar way taking as D_ε the region with boundaries Γ counterclockwise and Γ_ε clockwise.

Assume now that $\Gamma_0=\{(x(t),y(t)),\ t\in[0,T]\}$ is a periodic orbit with period T. We will prove that it is a hyperbolic limit cycle. We compute

$$\begin{aligned}\int_0^T \mathrm{div}X(x(t),y(t))\,dt &= \int_0^T\Big(ay(t)+(2d-b)x(t)\Big)\,dt\\ &= \int_0^T\frac{ax'(t)+(b-2d)y'(t)}{ax(t)+by(t)+c}dt = \int_{\Gamma_e}\frac{a\,dx+(b-2d)dy}{ax+by+c}\\ &= sgn(-c)\iint_D\frac{2ad}{(ax+by+c)^2}dxdy\neq 0.\end{aligned}$$

Note that we have used Stokes Theorem in last equality and that D denotes the region surrounded by Γ_0. We remark also that Γ_0, for $e>0$, is a periodic orbit if and only if $\Gamma_0\cap\{(x,y):\ ax+by+c=0\}=\emptyset$. ∎

Next example and proof are based on ideas of [9].

Proposition 2. *System*

$$\begin{cases} x = -x + (4+\lambda)y + ((1+\lambda)x + (1-\lambda)y)(x^2+y^2-4xy)\,, \\ \dot{y} = -\lambda x + (4\lambda - 1)y + ((\lambda-1)x + (\lambda+1)y)(x^2+y^2-4xy)\,, \end{cases} \tag{8}$$

has two hyperbolic limit cycles for $0 < \lambda < 1/2$ *and exactly one semistable limit cycle if* $\lambda = 0$. *For* $\lambda \lesssim 0$ *it has no limit cycles in a neighborhood of* $x^2 + y^2 = 1$.

Proof. If we write system (8) in polar coordinates we have

$$\begin{cases} \dot{r} = a(\theta)r + f(\theta)r^3\,, \\ \dot{\theta} = b(\theta) + g(\theta)r^2\,, \end{cases}$$

where $a(\theta) = -1 + 4\sin\theta\cos\theta + 4\lambda\sin^2\theta$, $b(\theta) = -\lambda + 4\lambda\sin\theta\cos\theta - 4\sin^2\theta$, $f(\theta) = (\lambda+1)(1-4\sin\theta\cos\theta)$, $g(\theta) = (\lambda-1)(1-4\sin\theta\cos\theta)$.

Putting $\rho = r^2$ we obtain

$$\begin{cases} \dot{\rho} = 2a(\theta)\rho + 2f(\theta)\rho^2\,, \\ \dot{\theta} = b(\theta) + g(\theta)\rho\,. \end{cases}$$

Note that periodic orbits of (8) can not cut the set $K = \{(\rho,\theta)\,|\,\dot{\theta} = 0\}$ because on K, $\dot{\rho} = 2\rho\left[\frac{a(\theta)g(\theta) - f(\theta)b(\theta)}{g(\theta)}\right] = 2\rho\frac{\lambda^2+1}{\lambda-1}(1 - 4\sin\theta\cos\theta + 4\sin^2\theta)$ does not change sign.

So periodic orbits of (8) are 2π periodic solutions of

$$\frac{d\rho}{d\theta} = S(\rho,\theta) = 2\frac{a(\theta)\rho + f(\theta)\rho^2}{b(\theta) + g(\theta)\rho}\,. \tag{9}$$

For this differential equation we define the Poincaré return map h, on $\theta = \pi/2$ as follows $h(x) = \rho(5\pi/2; x)$ where $\rho(\theta; x)$ is the solution of (9) such that ρ is x on $\theta = \pi/2$. Using results of [12] we have that a expression for $h'''(x)$, when it is defined, is

$$h'''(x) = h'(x)\left[\frac{3}{2}\left(\frac{h''(x)}{h'(x)}\right)^2 + \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} \frac{\partial^3 S}{\partial \rho^3}(\rho(\theta;x),\theta)\exp\left\{2\int_{\frac{\pi}{2}}^{\theta}\frac{\partial S}{\partial\rho}(\rho(\phi;x),\phi)d\phi\right\}d\theta\right]\,.$$

Note that the fixed points of $h(x)$ correspond with periodic orbits of (9). In our case

$$\begin{aligned}\frac{\partial^3 S}{\partial \rho^3}(\rho,\theta) &= 12\frac{b(\theta)g(\theta)(a(\theta)g(\theta)-b(\theta)f(\theta))}{(b(\theta)+g(\theta)\rho)^4}\\ &= \frac{(-\lambda+4\lambda\sin\theta\cos\theta-4\sin^2\theta)(1-4\sin\theta\cos\theta+4\sin^2\theta)}{(b(\theta)+\rho g(\theta))^2}\\ &\quad\cdot\frac{12(\lambda-1)(\lambda^2+1)(1-4\sin\theta\cos\theta)^2}{(b(\theta)+\rho g(\theta))^2}.\end{aligned}$$

This last expression is greater or equal than zero if $0 \leq \lambda < 1/2$. Assume that $\lambda > 0$. Then $h(0) = 0$ and since $h'''(x) > 0$, using Rolle's Theorem we have that $h(x) = x$ has at most two positive solutions (taking into account their multiplicities). Consider now the case $\lambda = 0$. In this case h is not defined in a neighborhood of zero, but it is clear that there is a x_0 on $\theta = \pi/2$ such that $h(x_0) > x_0$. Studying the infinity in the Poincaré compactification (see next section) there is y_0, also on $\theta = \pi/2$, big enough and such that $h(y_0) > y_0$. Furthermore all periodic orbits of (9) cut $\theta = \pi/2$ between $\rho = x_0$ and $\rho = y_0$. So again, since $h'''(x) > 0$, using Rolle's Theorem, h(x) has at most two positive solutions (taking into account their multiplicities).

In order to finish the proof it is enough to note that (9) is a SCFRVF with respect to λ. Then since for $\lambda = 0$, $x^2 + y^2 - 1 = 0$ is a semistable limit cycle (this assertion follows by tedious calculations) the study for $\lambda \geq 0$ is complet. That near $x^2 + y^2 - 1 = 0$ there are no limit cycles for $\lambda < 0$ follows for the non intersection property of SCFRVF. In fact there are no limit cycles for $\lambda \lesssim 0$ in the region covered by the limit cycles of system (8) varying λ from 0 to 1/2. ∎

We think that system (8) has no limit cycles for $\lambda < 0$ but we have not found any proof.

3. Study of the Examples

(1) System (1) can be integrated and it is clear that its phase portrait is like (1) of Figure 1. Furthermore this is the usual example of saddle-node bifurcation presented in most text books.

(2) System (2) is a SCFRVF with respect to the parameter λ. This system was already considered in [1, p. 212]. For $\lambda = 0$ it can be integrated, while

for $\lambda \neq 0$ its phase portrait in the Poincaré sphere follows without major difficulties. Its phase portraits are shown in (2) of Figure 1.

(3) System (3) has $x^2 + y^2 - \lambda = 0$ as an invariant curve. Using this fact it is not difficult to prove that there is an Andronov-Hopf bifurcation when λ crosses 0, because this invariant curve is precisely the limit cycle that appears for $\lambda > 0$. From Proposition 1, it follows that this limit cycle is hyperbolic and unique. We also could have used results of [4], because (changing t by $-t$) (3) is a bounded $PS2$ with a unique finite singularity and it is proved in that paper that these systems have at most one limit cycle and that when it exists it is hyperbolic. Hence (3) of Figure 1 follows.

(4) System (4) (denoted by $(\dot{x}, \dot{y}) = X(x, y)$) is a SCFRVF. Furthermore for $\lambda = 0$, $y^2/2 - x^2/2 - x^3/3 = 0$ is an invariant algebraic curve for it. This curve gives the loop connection. This connection is a hyperbolic repeller graph because the divergence of $X(x, y)$ at $(0, 0)$ is positive, see [15]. Therefore, varying λ, a saddle loop bifurcation appears. In fact, in a small enough neighbourhood of the loop, the system has no (resp. one) limit cycles if $\lambda \lesssim 0$ (resp. $\lambda \gtrsim 0$).

We have not been able to control the global number of limit cycles for system (4) for $\lambda \neq 0$. We present here the proof that for $\lambda = 0$ it has no limit cycles. Take the Dulac function $B(x, y) = (y^2/2 - x^2/2 - x^3/3)^{-7/6}$. Then $\text{div}(BX) = -(y^2/2 - x^2/2 - x^3/3)^{-7/6}$ and hence by the Bendixson-Dulac Criterion the proof follows. Phase portraits of this system are showed in (4) of Figure 1.

(5) For system (5), $x^2 + y^2 - 1 = 0$ is always an invariant set. Since on this curve there are no critical points for $\lambda > 0$, a saddle node point for $\lambda = 0$ ((1,0)) and two critical points for $\lambda < 0$ (a node $(1 + \lambda, \sqrt{1 - (1 + \lambda)^2})$ and a saddle $(1+\lambda, -\sqrt{1 - (1 + \lambda)^2})$) we have that a saddle loop bifurcation appears for $\lambda = 0$. Note that the loop is precisely the invariant set $\{(x, y) \mid x^2 + y^2 - 1 = 0\} - \{(1, 0)\}$.

Using again Proposition 1 and the usual techniques, (5) of Figure 1 follows. This system is already considered in [1, p. 433], but there, there is no a study of the number of limit cycles that it has.

(6) For system (6) the only finite critical point is the origin. Furthermore the infinite critical points are given by the directions $x^2 + y^2 - 4xy = 0$. Studying these critical points we obtain that the infinity is a repeller. Using this property and Proposition 2 phase portraits (6) of Figure 1 follow.

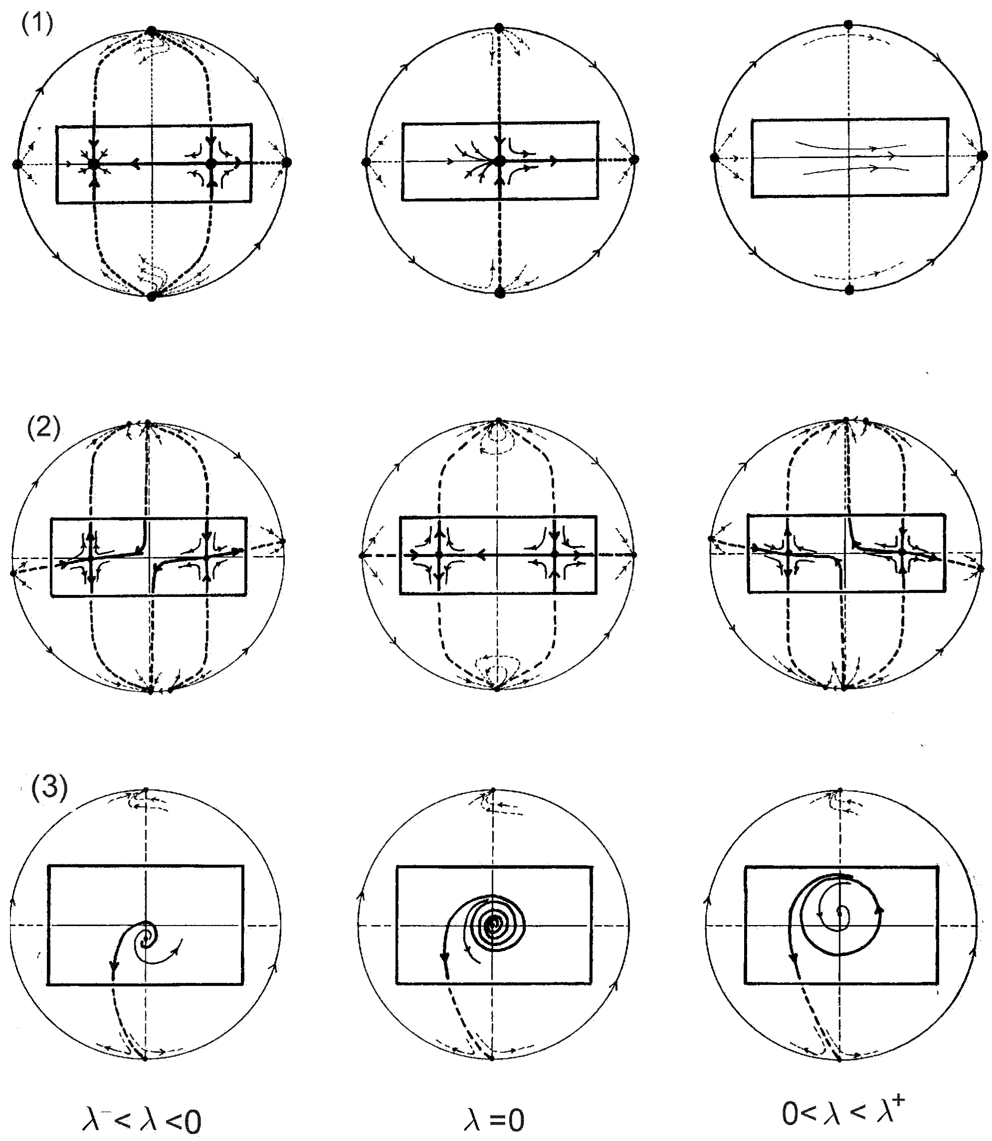

Figure 1. Phase portraits on the Poincaré sphere for the six families considered.

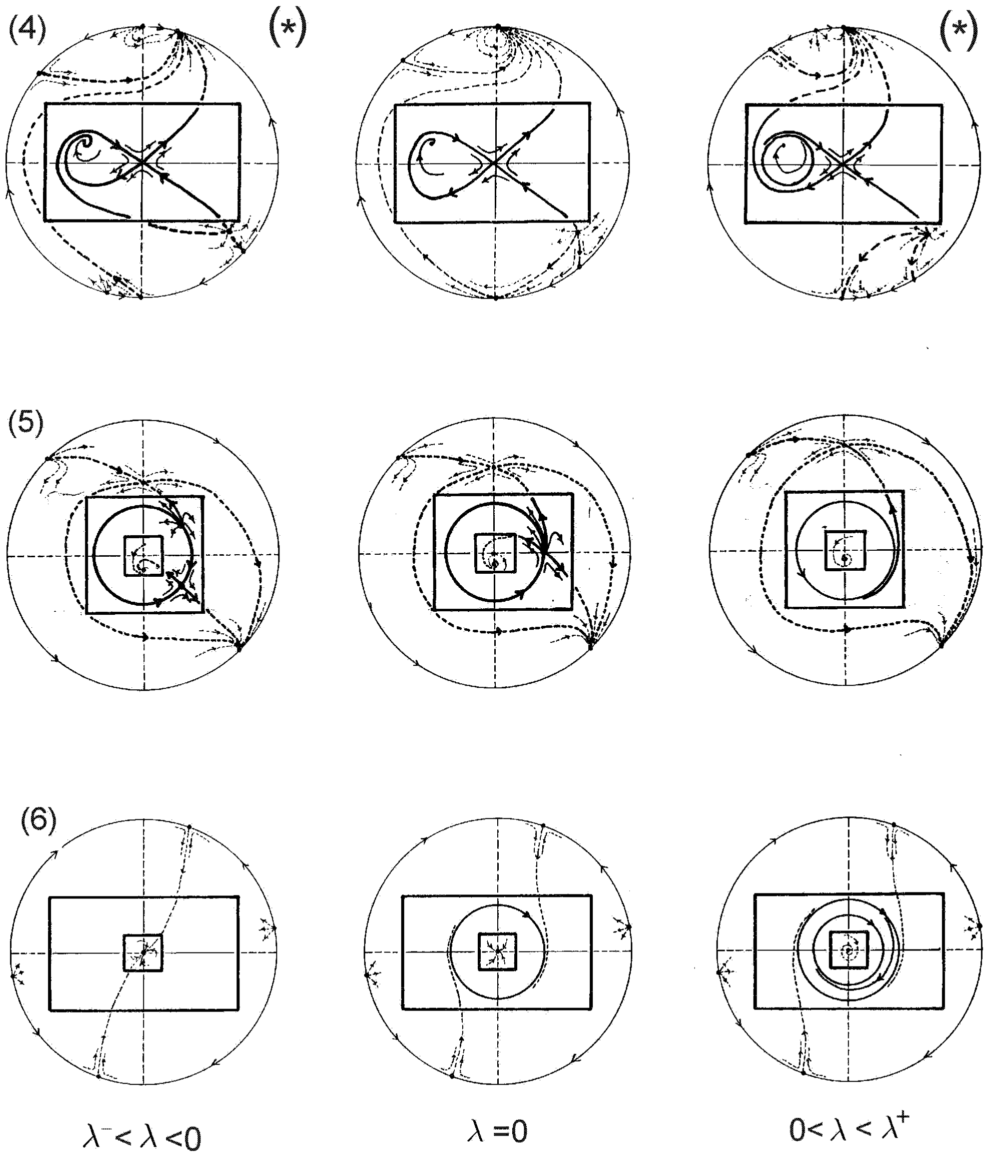

Figure 1. continued

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