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On the Regular Sturm-Liouville Transform

Antonio G. García and Miguel A. Hernández-Medina

Departamento de Matemáticas, Universidad Carlos III de Madrid,
28911-*Leganés, Spain*
Departamento de Matemática Aplicada, E.T.S.I.T. Universidad Politécnica de Madrid,
28940-*Madrid, Spain*

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1. Introduction

The Whittaker-Shannon-Kotel'nikov Sampling Theorem, hereafter WSK Theorem, states that any function $f \in L^2(\mathbb{R})$, bandlimited to $[-\pi,\pi]$, i.e. such that the support of its Fourier transform is contained in $[-\pi,\pi]$ (equivalently $f(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widehat{f}(\omega)e^{it\omega}\,d\omega$, where $\widehat{f}$ stands for the Fourier transform of f) may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ on the integers as

$$f(z) = \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{sinc}\,(z-n),$$

where sinc denotes the cardinal sine $\mathrm{sinc}\,(z) = \sin \pi z/\pi z$ [4, 10, 13]. The choice of the interval $[-\pi,\pi]$ is arbitrary. The same result applies to any compact interval $[-\pi\sigma,\pi\sigma]$ taking the samples in $\{n/\sigma\}$ and replacing π with π/σ in the cardinal sines.

This theorem and its numerous offspring have been proved in many different ways, e.g. using Fourier expansions, the Poisson summation formula, contour integrals, etc. (see, for instance, [4]). But the most elegant proof is probably the one due to Hardy, using that the inverse Fourier transform $\mathcal{F}^{-1}$ is an isometry from $L^2[-\pi,\pi]$ onto the Paley-Wiener space $PW_\pi = \{f \in L^2(\mathbb{R})\cap\mathcal{C}(\mathbb{R})\colon\ \mathrm{supp}\ \widehat{f} \subseteq [-\pi,\pi]\}$. Any value $f(t_n)$ of f is the inner product in $L^2[-\pi,\pi]$ of $\widehat{f}$ and the complex exponential $e^{-it_n\omega}$. Furthermore, the classical Paley-Wiener Theorem shows that PW_π coincides with the space of entire functions of exponential type at most π whose restriction to the real axis is square integrable, i.e.

$$PW_\pi = \{f \in \mathcal{H}(\mathbb{C}) \,:\, |f(z)| \leq Ae^{\pi|z|},\ f|_\mathbb{R} \in L^2(\mathbb{R})\}.$$

The Paley-Wiener space PW_π, inverse image space of $L^2[-\pi, \pi]$ through $\mathcal{F}^{-1}$, is a reproducing kernel Hilbert space, hereafter RKHS, whose reproducing kernel is the function $k(z,\omega) = \operatorname{sinc}(z - \overline{\omega})$, i.e.

$$f(\omega) = \langle f(z), \operatorname{sinc}(z - \overline{\omega})\rangle_{PW_\pi}\,, \quad f \in PW_\pi\,.$$

The key point in Hardy's proof is that an expansion converging in $L^2[-\pi,\pi]$ is transformed by $\mathcal{F}^{-1}$ into another expansion which converges in the topology of PW_π. This implies, in particular, that it converges uniformly on compact sets of the complex plane (to be precise, it converges on horizontal strips of $\mathbb{C}$) [10]. Choosing the first expansion in such a way that the coefficients are samples of f or of some function related to f (its derivatives, its Hilbert transform, etc.) provides different sampling theorems for functions in PW_π. This Fourier duality technique can also be applied to the multidimensional case, or to the so-called multi-band case of functions whose Fourier transform has support on the union of a finite number of disjoint sets of finite Lebesgue measure (see [4] for more details).

One direction in which the WSK Theorem has been generalized is replacing the kernel function, $e^{i\lambda\omega}$, by a more general kernel $K(\omega,\lambda)$ leading to the following generalization by Kramer [1, 5]: Let $K(\omega,\lambda)$ be a function, continuous in λ such that, as a function of ω, $K(\omega,\lambda) \in L^2(I)$ for every real number λ, where I is an interval on the real line. Assume that there exists a sequence of real numbers $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\{K(\omega,\lambda_n)\}_{n\in\mathbb{Z}}$ is a complete orthogonal sequence of functions of $L^2(I)$. Then for any f of the form

$$f(\lambda) = \int_I F(\omega)K(\omega,\lambda)\,d\omega\,,$$

where $F \in L^2(I)$, we have

$$f(\lambda) = \sum_{n=-\infty}^{\infty} f(\lambda_n)S_n(\lambda)\,,$$

with

$$S_n(\lambda) = \frac{\int_I K(\omega,\lambda)\overline{K(\omega,\lambda_n)}\,d\omega}{\int_I |K(\omega,\lambda_n)|^2\,d\omega}\,.$$

The series (2) converges uniformly wherever $\|K(\cdot,\lambda)\|_{L^2(I)}$ is bounded.

In particular, if $I=[-\pi,\pi]$, $K(\omega,\lambda)=e^{i\lambda\omega}$ and $\{\lambda_n=n\}_{n\in\mathbb{Z}}$, we get the WSK sampling theorem.

One way to generate kernels $K(\omega,\lambda)$ and sampling points $\{\lambda_n\}_{n\in\mathbb{Z}}$ is to consider Sturm-Liouville boundary-value problems [3, 11, 12]. The kernel will be the function $\phi(x,\lambda)$ which generates the eigenfunctions of the problem taking $\lambda=\lambda_n$, $n\in\mathbb{N}$. Thus, we obtain the so-called Sturm-Liouville type transform, term first coined in [14].

The aim of this paper is twofold: firstly, to apply the integral transform theory which appears in [8] characterizing the space of output functions from a linear integral transform as a RKHS, and secondly, to obtain a Fourier-type duality to be used in order to obtain the sampling theorem associated with the regular Sturm-Liouville transform. For sampling theorems in the framework of the RKHS see [7].

2. Preliminaries

Consider the regular Sturm-Liouville problem

$$\begin{aligned} -y''+q(x)y &= \lambda y, \quad x\in[a,b],\ q\in\mathcal{C}[a,b] && (1)\\ y(a)\cos\alpha+y'(a)\sin\alpha &= 0, && (2)\\ y(b)\cos\beta+y'(b)\sin\beta &= 0\,. && (3)\end{aligned}$$

The problem (1)–(3) defines a self-adjoint operator [2, p. 141] with discrete spectrum [9]. The eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ are real, and following [9, pp. 12 and ff.], simple and bounded from below. Furthermore, the associate eigenfunctions form an orthogonal basis of $L^2(a,b)$.

Let $\phi(x,\lambda)$ and $\xi(x,t)$ be the solutions of (1) verifying

$$\begin{aligned} \phi(a,\lambda)&=\sin\alpha\,, \quad \phi'(a,\lambda)=-\cos\alpha\,,\\ \xi(b,\lambda)&=\sin\beta\,, \quad \xi'(b,\lambda)=-\cos\beta\,.\end{aligned}$$

The function $\phi(x,\lambda)$ verifies the boundary condition (2) for all λ, and consequently, λ_n will be an eigenvalue if and only if $\phi(x,\lambda_n)$ fulfills the boundary condition (3). Therefore, $\{\phi(x,\lambda_n)\}_{n=0}^{\infty}$ will be the eigenfunctions of the problem (1)–(3).

The wronskian W of ϕ and ξ is defined as

$$W(\phi(\cdot,\lambda),\xi(\cdot,\lambda))=\begin{vmatrix}\phi(x,\lambda) & \xi(x,\lambda)\\ \phi'(x,\lambda) & \xi'(x,\lambda)\end{vmatrix}.$$

The following result may be found in [9, pp. 7-11 and 19]

Lemma 2.1. *$W(\lambda) \doteq W(\phi(\cdot,\lambda),\xi(\cdot,\lambda))$ is independent of $x \in [a,b]$; it is an entire function of order $1/2$, its zeros are real, simple and located at, and only at, the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. When $k \to \infty$ we have*

$$\sqrt{\lambda_k} = \frac{k\pi}{b-a} + O\left(\frac{1}{k}\right).$$

We also have

$$W(\lambda) = -\cos\beta\phi(b,\lambda) - \sin\beta\phi'(b,\lambda). \tag{4}$$

Since $W(\lambda)$ is an entire function of orden $1/2$ with simple zeros in $\{\lambda_n\}_{n=0}^{\infty}$, Hadamard's Factorization Theorem [10] asserts that

$$W(\lambda) = CP(\lambda) \tag{5}$$

where $C \in \mathbb{C}$ and

$$P(\lambda) = \prod_{n=0}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \qquad \text{if } 0 \notin \{\lambda_n\}_{n=0}^{\infty} \tag{6}$$

$$P(\lambda) = \lambda\prod_{n=1}^{\infty}\left(1 - \frac{\lambda}{\lambda_n}\right), \quad \text{if } \lambda_0 = 0. \tag{7}$$

The function $\phi(x,\lambda)$ fulfills all the requirements in Kramer's Theorem. Therefore, the function $F(\lambda) = \langle f, \overline{\phi(\cdot,\lambda)}\rangle_{L^2(a,b)}$, with $f \in L^2(a,b)$, can be recovered through its samples in the eigenvalues of (1)-(3)

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) S_n(\lambda)\,,$$

where

$$S_n(\lambda) = \alpha_n^{-2}\int_a^b \overline{\phi(x,\lambda_n)}\phi(x,\lambda)\,dx$$

and the constants α_n are the normalizing factors for the eigenfunctions of the problem (1)-(3), i.e. $\alpha_n = \|\phi(\cdot,\lambda_n)\|$. The convergence of the series is absolute and uniform on subsets $D \subset \mathbb{C}$ where $\|\phi(\cdot,\lambda)\|$ is bounded.

We define
$$\rho(\lambda) = \sum_{\lambda_n \leq \lambda} \alpha_n^{-2}.$$
This non decreasing function will define a positive measure $d\rho(\lambda)$ on $\mathbb{R}$ in the Lebesgue–Stieltjes sense. We define
$$B_\rho^{a,b} = \left\{ F : \mathbb{C} \to \mathbb{C} : \ F(\lambda) = \int_a^b f(x)\phi(x,\lambda)\, dx \text{ with } f \in L^2(a,b) \right\}.$$
We know [9] that $\phi(x,\lambda) = O(e^{(x-a)\sqrt{|\lambda|}})$ as $|\lambda| \to \infty$, uniformly in x. Using Cauchy-Schwarz's inequality in
$$F(\lambda) = \int_a^b f(x)\phi(x,\lambda)\, dx$$
we obtain the inequality
$$|F(\lambda)| \leq A e^{(b-a)\sqrt{|\lambda|}}\,,$$
and therefore, the functions in $B_\rho^{a,b}$ are entire functions of order $1/2$ and type at most $b-a$.

DEFINITION 2.1. We define the *regular Sturm-Liouville transform* associated with the problem (1)–(3) as the application $\tau : L^2(a,b) \longrightarrow B_\rho^{a,b}$ given by
$$[\tau(f)](\lambda) = F(\lambda) \doteq \int_a^b f(x)\phi(x,\lambda)\, dx, \qquad \text{for } f \in L^2(a,b).$$

In the next section we prove that this application τ is an isometry mapping the Hilbert space $L^2(a,b)$ onto $B_\rho^{a,b}$.

3. THE SPACE $B_\rho^{a,b}$

Let r be the application defined by $r(F) \doteq F|_{\mathbb{R}}$ for $F \in B_\rho^{a,b}$. We denote by T the composition $r\tau$. We have the following result

THEOREM 3.1. *The linear application T is an isometry from $L^2(a,b)$ onto $L^2_\rho(\mathbb{R})$. Furthermore, if $F = T(f)$ then*
$$f(x) = \int_{-\infty}^{\infty} F(\lambda)\phi(x,\lambda)\, d\rho(\lambda)\,. \tag{8}$$

Proof. Let $f \in L^2(a,b)$ and $F(\lambda) = [T(f)](\lambda)$. Since $\{\phi(x,\lambda_n)\}_{n=0}^{\infty}$ is an orthogonal basis in $L^2(a,b)$ we have

$$f(x) = \sum_{n=0}^{\infty} \langle f, \phi(\cdot,\lambda_n)\rangle \frac{\phi(x,\lambda_n)}{\alpha_n^2} = \int_{\mathbb{R}} F(\lambda)\phi(x,\lambda)\, d\rho(\lambda)$$

with convergence in $L^2(a,b)$, thus (8) is satisfied.

Furthermore

$$\begin{aligned}\int_{\mathbb{R}} |F(\lambda)|^2\, d\rho(\lambda) = \sum_{n=0}^{\infty} |F(\lambda_n)|^2 \alpha_n^{-2} &= \sum_{n=0}^{\infty} |\langle f, \phi(\cdot,\lambda_n)\rangle|^2 \alpha_n^{-2} \\ &= \int_a^b |f(x)|^2\, dx,\end{aligned}$$

where we have used Parseval's equality.

To prove that T is surjective, let $F \in L^2_\rho(\mathbb{R})$. Defining

$$f(x) = \int_{\mathbb{R}} F(\lambda)\phi(x,\lambda)\, d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{F(\lambda_n)}{\alpha_n^2}\phi(x,\lambda_n)\,,$$

this function belongs to $L^2(a,b)$ and $\tau(f) = F$. ∎

The above theorem shows that if $F(\lambda) \in B_\rho^{a,b}$ then its restriction to the real line belongs to $L^2_\rho(\mathbb{R})$, and every function in $L^2_\rho(\mathbb{R})$ can be extended to a function in $B_\rho^{a,b}$. Thus, $B_\rho^{a,b}$ is a Hilbert space of entire functions endowed with the inner product

$$\langle F, G\rangle_{B_\rho^{a,b}} \doteq \int_{\mathbb{R}} F(\lambda)\overline{G(\lambda)}\, d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{F(\lambda_n)\overline{G(\lambda_n)}}{\alpha_n^2}$$

for any $F, G \in B_\rho^{a,b}$, or

$$\langle F, G\rangle_{B_\rho^{a,b}} = \int_a^b f(x)\overline{g(x)}dx\,,$$

where $\tau(f) = F$ and $\tau(g) = G$. Furthermore, we have found a characterization of the image space $\tau(L^2(a,b)) = B_\rho^{a,b}$ through the regular Sturm-Liouville transform as

$$B_\rho^{a,b} = \{F \in \mathcal{H}(\mathbb{C}),\ \text{with } |F(\lambda)| \leq Ae^{(b-a)\sqrt{|\lambda|}} \text{ and } F|_{\mathbb{R}} \in L^2_\rho(\mathbb{R})\},$$

with

$$\|F\|^2_{B^{a,b}_\rho} = \int_{\mathbb{R}} |F(\lambda)|^2 d\rho(\lambda) = \sum_{n=0}^{\infty} \frac{|F(\lambda_n)|^2}{\alpha_n^2} = \int_a^b |f(x)|^2 dx\,,$$

where $\tau(f) = F$.

The inversion formula (8) is given by means of the σ-finite, purely atomic measure $d\rho(\lambda)$ whose support is $\{\lambda_n\}$. As we will see in the next section, this inversion formula is important from a theoretical point of view. However, one can obtain an inversion formula involving a continuous measure using other techniques. See [15] for the details.

Now we prove that $B^{a,b}_\rho$ is a RKHS.

THEOREM 3.2. *$B^{a,b}_\rho$ is a RKHS space with reproducing kernel*

$$k(\lambda,\mu) \doteq \langle \phi(\cdot,\lambda), \phi(\cdot,\mu)\rangle_{L^2(a,b)}. \tag{9}$$

Proof. Let $F \in B^{a,b}_\rho$ and $\lambda \in \mathbb{C}$. Defining $l_\lambda F \doteq F(\lambda)$ we have

$$|l_\lambda F| = |F(\lambda)| = \left| \int_a^b (\tau^{-1}F)(x)\phi(x,\lambda)\, dx \right| .$$

Applying Cauchy-Schwarz's inequality we obtain

$$\begin{aligned} |l_\lambda F| &\leq \|\tau^{-1}F\|_{L^2(a,b)} \|\phi(\cdot,\lambda)|_{L^2(a,b)} \\ &= \|F\|_{B_\rho} \|\phi(\cdot,\lambda)\|_{L^2(a,b)} \,. \end{aligned}$$

Thus, $B^{a,b}_\rho$ is a RKHS space since the point evaluation l_λ is a bounded linear functional on $B^{a,b}_\rho$ for each $\lambda \in \mathbb{C}$ [8, 10]. Taking $f = \tau^{-1}(F) \in L^2(a,b)$, we have

$$\begin{aligned} F(\lambda) &= \langle f, \overline{\phi(\cdot,\lambda)}\rangle_{L^2(a,b)} = \langle \tau f, \tau\overline{\phi(\cdot,\lambda)}\rangle_{B^{a,b}_\rho} \\ &= \langle F, \tau\overline{\phi(\cdot,\lambda)}\rangle_{B^{a,b}_\rho}, \end{aligned}$$

and therefore,

$$k(\lambda,\mu) = \langle \phi(\cdot,\lambda), \phi(\cdot,\mu)\rangle_{L^2(a,b)}$$

is the reproducing kernel of $B^{a,b}_\rho$. ∎

Since $B_\rho^{a,b}$ is a RKHS, we know that the convergence in the $B_\rho^{a,b}$ norm $\| \, \|_{B_\rho^{a,b}}$ implies pointwise convergence. Furthermore, if $|k(\lambda,\lambda)| \le M$, for each $\lambda \in D \subset \mathbb{C}$, the convergence will be uniform on D.

Lemma 3.1. *For any compact subset $\Omega \subset \mathbb{C}$ there exists a constant $M(\Omega)$ such that*

$$|k(\lambda,\lambda)| \le M(\Omega), \text{ for each } \lambda \in \Omega.$$

Proof. Using (9) we have

$$|k(\lambda,\lambda)| = \|\phi(\cdot,\lambda)\|^2_{L^2(a,b)}.$$

Since $\|\phi(\cdot,\lambda)\|_{L^2(a,b)} \le Be^{(b-a)\sqrt{|\lambda|}}$, we obtain

$$|k(\lambda,\lambda)| \le A^2 e^{2(b-a)\sqrt{|\lambda|}},$$

and the result follows. Therefore, convergence in $B_\rho^{a,b}$ implies uniform convergence on compact subsets of $\mathbb{C}$. ∎

4. Fourier-type duality associated with the regular Sturm-Liouville transform

The isometry τ from $L^2(a,b)$ onto $B_\rho^{a,b}$ enables us to transfer orthogonal and Riesz bases back and forth from one space to the other through τ or τ^{-1}, exactly like in the Fourier setting. For this reason, we say that a Fourier-type duality exists associated with the regular Sturm-Liouville transform.

Corollary 4.1. $\{\varphi_n(\lambda)\}_{n=0}^\infty \doteq \tau(\{\phi(\cdot,\lambda_n)\}_{n=0}^\infty) = \{k(\lambda,\lambda_n)\}_{n=0}^\infty$ *is an orthogonal basis of the Hilbert space* $B_\rho^{a,b}$.

The following theorem ensures that any function in $B_\rho^{a,b}$ can be recovered through its samples on the eigenvalues of the problem (1)-(3) by means of an interpolatory Lagrange-type series.

Theorem 4.1. (Sampling Theorem in $B_\rho^{a,b}$) *Any $F \in B_\rho^{a,b}$ can be expanded as*

$$F(\lambda) = \sum_{n=1}^\infty F(\lambda_n) S_n(\lambda)\,, \tag{10}$$

where

$$S_n(\lambda) = \frac{P(\lambda)}{(\lambda - \lambda_n)P'(\lambda_n)}\,,$$

and $P(\lambda)$ is given by (6) or (7). The convergence is absolute and uniform on compact subsets of $\mathbb{C}$.

Proof. We know that $\{\varphi_n(\lambda)\}_{n=0}^{\infty}$ is an orthogonal basis of $B_\rho^{a,b}$. Thus, for each $F \in B_\rho^{a,b}$ we have

$$\begin{aligned} F(\lambda) &= \sum_{n=0}^{\infty} \langle F, \varphi_n\rangle_{B_\rho} \frac{\varphi_n(\lambda)}{\|\varphi_n\|_{B_\rho}^2} \\ &= \sum_{n=0}^{\infty} \langle \tau^{-1}F, \tau^{-1}\varphi_n\rangle_{L^2(a,b)} \frac{\varphi_n(\lambda)}{\|\phi(\cdot,\lambda_n)\|_{L^2(a,b)}^2}. \end{aligned}$$

Since $\tau^{-1}\varphi_n = \phi(x,\lambda_n)$, then $\langle \tau^{-1}F, \tau^{-1}\varphi_n\rangle_{L^2(a,b)} = F(\lambda_n)$. The proof will be complete once we identify the sampling functions

$$S_n(\lambda) \doteq \frac{\varphi_n(\lambda)}{\|\phi(\cdot,\lambda_n)\|_{L^2(a,b)}^2}\,.$$

For the sake of completeness we include here the proof which appears in [4] or [13].

The functions $\phi(x,\lambda)$ y $\phi(x,\lambda_n)$ are solutions of (1). Then,

$$(\lambda - \lambda_n)\phi(x,\lambda)\phi(x,\lambda_n) = [\phi(x,\lambda)\phi'(x,\lambda_n) - \phi'(x,\lambda)\phi(x,\lambda_n)]'.$$

Integrating

$$(\lambda - \lambda_n)\int_a^b \phi(x,\lambda)\phi(x,\lambda_n)\,dx = \phi(b,\lambda)\phi'(b,\lambda_n) - \phi'(b,\lambda)\phi(b,\lambda_n)\,. \qquad (11)$$

If $\sin\beta \neq 0$, having in mind that $\phi(x,\lambda_n)$ verifies the boundary condition (3), using (4) we have

$$W(\lambda)\phi(b,\lambda_n) = \sin\beta[\phi(b,\lambda)\phi'(b,\lambda_n) - \phi'(b,\lambda)\phi(b,\lambda_n)]\ .$$

Therefore

$$\int_a^b \phi(x,\lambda)\phi(x,\lambda_n)\,dx = \frac{W(\lambda)}{\lambda - \lambda_n}\frac{\phi(b,\lambda_n)}{\sin\beta}\,,$$

and if $\lambda \to \lambda_n$,

$$\|\phi(\cdot,\lambda_n)\|^2_{L^2(a,b)} = W'(\lambda_n)\frac{\phi(b,\lambda_n)}{\sin\beta}\,.$$

Now, using (5),

$$\frac{\varphi_n(\lambda)}{\|\phi(\cdot,\lambda_n)\|^2_{L^2(a,b)}} = \frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)}\,. \tag{12}$$

If $\sin\beta = 0$, by (4) $W(\lambda)\phi'(b,\lambda_n) = -\cos\beta\phi(b,\lambda)\phi'(b,\lambda_n)$, and by (3) we can write (11) as

$$\int_a^b \phi(x,\lambda)\phi(x,\lambda_n)\,dx = -\frac{W(\lambda)}{\lambda-\lambda_n}\frac{\phi'(b,\lambda_n)}{\cos\beta}\,.$$

Proceeding as before we obtain again (12).

The absolute convergence in (10) follows from the unconditional character of any orthonormal basis and the fact that convergence in a RKHS implies pointwise convergence. The uniform convergence is a consequence of Lemma 3.1. ∎

Let us illustrate all these results with an example, *the finite continuous cosine transform*:

Consider the regular Sturm-Liouville problem

$$\begin{aligned} &-y'' = -\lambda y, \quad x\in[0,\pi],\\ &y'(0) = y'(\pi) = 0\,. \end{aligned}$$

In this case, $\phi(x,\lambda) = \cos\sqrt{\lambda}x$ and therefore

$$[\tau(f)](\lambda) = F(\lambda) = \int_0^\pi f(x)\cos\sqrt{\lambda}x\,dx, \qquad \text{for } f\in L^2(0,\pi).$$

The eigenvalues are $\lambda_n = n^2$, $n\in\mathbb{N}_0 \doteq \mathbb{N}\cup\{0\}$ and the eigenfunctions are $\{\cos nx\}_{n\in\mathbb{N}_0}$. As a consequence,

$$\rho(\lambda) = \begin{cases} \frac{2}{\pi}\left([\sqrt{\lambda}]+1\right) & \text{if } \lambda\ge 0\\ 0 & \text{if } \lambda<0 \end{cases}$$

where $[\,\cdot\,]$ denotes the integer part of a real number.

The reproducing kernel, $k(\lambda,\mu)$, is given by

$$k(\lambda,\mu)=\int_0^\pi \cos\sqrt{\lambda}s\,\overline{\cos\sqrt{\mu}s}\,ds\,,$$

and therefore

$$k(\lambda,\mu)=\frac{\sqrt{\lambda}\sin\sqrt{\lambda}\pi\,\cos\sqrt{\overline{\mu}}\pi-\sqrt{\overline{\mu}}\cos\sqrt{\lambda}\pi\,\sin\sqrt{\overline{\mu}}\pi}{\lambda-\overline{\mu}}.$$

The functions

$$\varphi_n(\lambda)=k(\lambda,n^2)=\frac{(-1)^n\sqrt{\lambda}\sin\sqrt{\lambda}\pi}{\lambda-n^2},\quad n\in\mathbb{N}_0\,,$$

constitute an orthogonal basis and the function F can be recovered through its samples in the points n^2 as

$$F(\lambda)=F(0)\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}F(n^2)\frac{(-1)^n\sqrt{\lambda}\sin\sqrt{\lambda}\pi}{\lambda-n^2}\,.$$

Since the reproducing kernel k is equal to

$$k(\lambda,\mu)=\sum_{n=0}^{\infty}\frac{1}{\alpha_n^2}\varphi_n(\lambda)\overline{\varphi_n(\mu)},$$

(see [8, 10]), for $\lambda,\mu\geq 0$ we obtain the formula

$$\begin{aligned}\frac{\sin\sqrt{\lambda}\pi}{\sqrt{\lambda}\pi}\left(\frac{\sin\sqrt{\mu}\pi}{\sqrt{\mu}\pi}\right) &+ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sqrt{\lambda}\sin\sqrt{\lambda}\pi}{\lambda-n^2}\left(\frac{\sqrt{\mu}\sin\sqrt{\mu}\pi}{\mu-n^2}\right)\\ &= \frac{\sqrt{\lambda}\sin\sqrt{\lambda}\pi\,\cos\sqrt{\mu}\pi-\sqrt{\mu}\cos\sqrt{\lambda}\pi\,\sin\sqrt{\mu}\pi}{\lambda-\mu}.\end{aligned}$$

5. The discrete regular Sturm-Liouville transform

Let $l_\alpha^2 \doteq \left\{\{a_n\}_{n=0}^\infty\subset\mathbb{C}\colon \sum_{n=0}^\infty\frac{|a_n|^2}{\alpha_n^2}<\infty\right\}$, endowed with the inner product

$$\langle\{a_n\},\{b_n\}\rangle_{l_\alpha^2}\doteq\sum_{n=0}^{\infty}\frac{a_n\overline{b_n}}{\alpha_n^2}\,. \tag{13}$$

Following [10], we can see that $\{\lambda_n\}$ is a *complete interpolating sequence* for $B_\rho^{a,b}$, i.e. the set of all sequences $\{F(\lambda_n)\}$ where F ranges over $B_\rho^{a,b}$ coincides with l_α^2, and the interpolation problem

$$F(\lambda_n) = a_n\,, \quad n \in \mathbb{N}_0$$

where $F \in B_\rho^{a,b}$ has exactly one solution provided $\{a_n\} \in l_\alpha^2$.

In fact, we have the following result

THEOREM 5.1. *Define* $\gamma : L^2(a,b) \longrightarrow l_\alpha^2$ *and* $\eta : B_\rho^{a,b} \longrightarrow l_\alpha^2$ *as* $\gamma(f) = \{\langle f, \phi(\cdot,\lambda_n)\rangle_{L^2(a,b)}\}_{n=0}^\infty$, $\eta(F) = \{F(\lambda_n)\}_{n=0}^\infty$. *Then,* γ *and* η *are isometric isomorphisms verifying* $\gamma = \eta\tau$.

Proof. It will be suficient to prove that η, γ are well defined, are isometries and $\tau = \eta^{-1}\gamma$. For $f \in L^2(a,b)$,

$$f(x) = \sum_{n=0}^\infty \langle f, \phi(\cdot,\lambda_n)\rangle_{L^2(a,b)} \frac{\phi(x,\lambda_n)}{\alpha_n^2}\,.$$

Using Parseval's equality

$$\|f\|_{L^2(a,b)}^2 = \sum_{n=0}^\infty \frac{|\langle f, \phi(\cdot,\lambda_n)\rangle_{L^2(a,b)}|^2}{\alpha_n^2} = \|\{\langle f, \phi(\cdot,\lambda_n)\rangle_{L^2(a,b)}\}\|_{l_\alpha^2}^2\,,$$

and γ is an isometry. The classical Riesz-Fischer Theorem assures that γ is surjective, and therefore an isomorphism.

On the other hand, η is a well defined isometry since for each $F \in B_\rho^{a,b}$

$$\|F\|_{B_\rho}^2 = \sum_{n=0}^\infty \frac{|F(\lambda_n)|^2}{\alpha_n^2} < \infty.$$

Let $\{a_n\}_{n=0}^\infty \in l_\alpha^2(\mathbb{N}_0)$ and $f \in L^2(a,b)$ be such that $f = \gamma^{-1}(a_n)$. Then $a_n = \langle f, \phi(\cdot,\lambda_n)\rangle_{L^2(a,b)}$, for each $n \in \mathbb{N}_0$, and taking $F = \tau f$ we conclude that $F \in B_\rho^{a,b}$ and $F(\lambda_n) = a_n$ for each $n \in \mathbb{N}_0$, proving that η is an isomorphism and $\tau = \eta^{-1}\gamma$. ∎

We may refer to γ as the *discrete regular Sturm-Liouville transform.*

Finally, we can apply this result in connection with the inverse Sturm-Liouville problem. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence of distinct real positive numbers, and let $\{\tau_n\}_{n\in\mathbb{N}}$ and $\{\rho_n\}_{n\in\mathbb{N}}$ both belong to l^2. Let a, b and c be constants, and suppose further that

$$\sqrt{\lambda_n} = \frac{n\pi}{b-a} + \frac{a}{n} + \frac{b}{n^3} + \frac{\tau_n}{n^3} \quad n \in \mathbb{N},$$

and that in the sequence

$$\alpha_n = \frac{2}{b-a} + \frac{c}{n^2} + \frac{\rho_n}{n^3}$$

each α_n is positive. Then, according with an important inverse result due to Levitan and Gasymov [6], there exists a regular Sturm-Liouville eigenvalue problem having eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$, and for which $\{\alpha_n\}_{n\in\mathbb{N}}$ are the normalizing factors for the eigenfunctions. Using this result, we can obtain the following uniqueness theorem.

Theorem 5.2. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^2 \alpha_n^{-2} < \infty$. There exists a unique entire function F of order $1/2$ and type at most $b-a$ such that $F(\lambda_n) = a_n$. Moreover, this function is given by the Lagrange-type interpolatory series*

$$F(\lambda) = \sum_{n=1}^{\infty} a_n \frac{P(\lambda)}{(\lambda-\lambda_n)P'(\lambda_n)} \, .$$

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