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On the Boundary of a Polyhedral Banach Space

V.P. Fonf

Department of Mathematics, Ben-Gurion University of the Negev
P.O. Box 653, 84105 *Beer-Sheva, Israel*

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1. Introduction

All Banach spaces under consideration are real and infinite-dimensional (unless otherwise specified).

Definition 1.1. Let X be a Banach space. A subset $B \subset S_{X^*}$ of the unit sphere of the dual space is called a boundary of X if for each $x \in X$ there is an $f \in B$ such that $f(x) = ||x||$.

It is not difficult to see that the whole sphere S_{X^*} and the set $\operatorname{ext} B_{X^*}$ of extreme points of the unit ball B_{X^*} of the dual space are boundaries (in the first case this is just the Hahn-Banach theorem and in the second, the Krein-Milman theorem). In general a boundary must not contain all extreme points but it must contain a so called w^*-exposed points.

Definition 1.2. Let X be a Banach space. A functional $f_0 \in S_{X^*}$ is called a w^*-exposed point of B_{X^*} if there is an $x_0 \in S_X$ such that $f_0(x_0) = 1 > f(x_0)$ for each $f \in B_{X^*}$, $f \neq f_0$. Moreover $f_0 \in S_{X^*}$ is called w^*-strongly exposed if $||.|| - \lim f_n = f_0$ whenever $\{f_n\} \subset B_{X^*}$ and $\lim f_n(x_0) = 1$. If an x_0 as above is taken from $S_{X^{**}}$, the functional f_0 is called exposed (resp. strongly exposed) point of B_{X^*}.

It is clear that each boundary B contains the (possibly empty) set $w^* - \exp B_{X^*}$ of all w^*-exposed points. If $B = w^* - \exp B_{X^*}$ then B is a *minimal* boundary. We shall see that this is a case for polyhedral spaces.

Definition 1.3. A Banach space X is called polyhedral [4] if the unit ball of each of its finite-dimensional subspace is a polytope.

Let us introduce the following notation. For a functional $f \in S_{X^*}$ put

$$\Gamma_f = \{x \in X \,:\, f(x) = 1\}\,, \qquad \gamma_f = \Gamma_f \cap S_X\,.$$

THEOREM 1.4. ([1]) *Let X be a polyhedral Banach space with the density character w.*

(i) *The set $B = w^* - \operatorname{str}\exp B_{X^*}$ is a (minimal) boundary for X and moreover for each $f \in B$, $\operatorname{int}_{\Gamma_f} \gamma_f \neq \emptyset$ and thus $\operatorname{card} B \leq w$ (in particular, if X is separable then B is countable).*

(ii) *Actually*

$$\operatorname{card} B = w\,. \tag{1.1}$$

Part (i) of the theorem plays an important role in the study of polyhedral Banach spaces (see [1, 2, 3]). While part (ii) (which is trivial for a separable X) still has no applications. The proof of Theorem 1.4 which appeared in [1] was complicated (and a proof of part (ii) was not presented in [1]). Later in 1990 the author considerably simplified the proof and a new proof circulated as a non-widespread preprint. The main purpose of this paper is to present this proof. In addition we give some new properties of polyhedral Banach spaces.

We divide the proof of Theorem 1.4 into two parts. In Section 1 we prove the main part (i) and part (ii) will be proved in Section 2.

2. A Minimal Boundary

The proof of the first part of Theorem 1.4 is clearly inspired by [5] but the tool we use (an opening of two subspaces of a Banach space) is different.

Recall that the "opening" of two subspaces L and M of a given Banach space X is just the Hausdorff distance between their unit spheres:

$$\theta(L, M) = \max\{\sup\{d(x, S_M) \,:\, x \in S_L\},\ \sup\{d(y, S_L) \,:\, y \in S_M\}\}\,.$$

The set of all subspaces of a Banach space equiped with this metric is a complete metric space.

Denote $|A|$ the cardinality of a set A. We start with two lemmas. The first is just a consequence of the definition of $\theta(L, M)$.

LEMMA 2.1. ([1]) *Let X be a polyhedral Banach space and L be a finite-dimensional subspace of X. Then there exists an $\varepsilon = \varepsilon(L) > 0$ such that for each subspace $M \subset X$ with $\theta(L, M) < \varepsilon(L)$, $|\operatorname{ext} B_{M^*}| \geq |\operatorname{ext} B_{L^*}|$.*

Proof. Denote $|\operatorname{ext} B_{L^*}| = 2m$ and suppose to the contrary that there exists a sequence $\{M_n\}$ of (finite-dimensional) subspaces of X with $\theta - \lim M_n = L$ such that $|\operatorname{ext} B_{M_n^*}| \le 2(m-1)$ for all n. Without loss of generality we may assume that for all n $|\operatorname{ext} B_{M_n^*}| = 2(m-k)$ for some $k \ge 1$. Let $\operatorname{ext} B_{M_n^*} = \{\pm \hat{f}_i^n\}_{i=1}^{m-k}$, $n = 1, 2, \dots$, $Y = [L \cup (\cup_{n=1}^{\infty} M_n)]$. Denote f_i^n a Hahn-Banach extension of $\hat{f}_i^n$ from M_n on Y, $i = 1, \dots, m-k$, $n = 1, 2, \dots$. By passing to a subsequence we may assume that for each $i = 1, \dots, m-k$ there exists a limit $f_i = w^* - \lim_n f_i^n \in B_{Y^*}$. Take any functional $\hat{g} \in B_{L^*}$ and let g be a Hahn-Banach extension of $\hat{g}$ on Y. For each $n = 1, 2, \dots$ we have a representation

$$g|_{M_n} = \sum_{i=1}^{m-k} a_i^n \hat{f}_i^n, \qquad \sum_{i=1}^{m-k} |a_i^n| \le 1 .$$

Without loss of generality we may assume that for all $i = 1, \dots, m-k$ there exists a limit $a_i = \lim_n a_i^n$. Clearly, $\sum_{i=1}^{m-k} |a_i| \le 1$ and

$$\sum_{i=1}^{m-k} a_i f_i = w^* - \lim_n \sum_{i=1}^{m-k} a_i^n f_i^n .$$

Take any vector $x \in S_L$ and by using $\theta - \lim M_n = L$, find a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in S_{M_n}$, such that $||.|| - \lim x_n = x$. We have

$$\hat{g}(x) = g(x) = \lim_n g(x_n) = \lim_n \left(\sum_{i=1}^{m-k} a_i^n f_i^n \right) (x_n) = \left(\sum_{i=1}^{m-k} a_i f_i \right) (x)$$

which gives $\hat{g} = \sum_{i=1}^{m-k} a_i f_i|_L$. By taking into account $\sum_{i=1}^{m-k} |a_i| \le 1$, we conclude that $\operatorname{ext} B_{L^*} \subset \{\pm f_i\}_{i=1}^{m-k}$, $k \ge 1$, a contradiction with $|\operatorname{ext} B_{L^*}| = 2m$. ∎

The second lemma (which is of some independent interest) is related to Mazur's theorem on smooth points.

Definition 2.2. ([1]) A subspace $M \subset X$ of a Banach space X is called smooth if each smooth point of the sphere S_M is a smooth point of the whole sphere S_X. For a finite-dimensional subspace M of a polyhedral space X this is equivalent to the following: M is smooth iff each $f \in \operatorname{ext} B_{M^*}$ has a unique extension to a norm one functional on the whole space X.

Lemma 2.3. *Let X be a polyhedral Banach space and L be a two-dimensional subspace of X. Then for each $\varepsilon > 0$ there exists a (two-dimensional)* *smooth subspace M with $\theta(L,M)<\varepsilon$ and such that if $\{\pm f_i|_M\}_{i=1}^n=\operatorname{ext} B_{M^*}$, $f_i\in S_{E^*}$, $i=1,\dots,n$, then $\{\pm f_i|_L\}_{i=1}^n\supset \operatorname{ext} B_{L^*}$.*

Proof. Suppose that L is not smooth. Then there exist two functionals $f,g\in S_{X^*}$ with $f\neq g$ but $f|_L=g|_L\in \operatorname{ext} B_{L^*}$. Take any $y\in E$ with $f(y)\neq g(y)$ and put $G=\operatorname{span}\{L,y\}$. G is a 3-dimensional polyhedral space and a simple consideration shows that there exists a 2-dimensional subspace $L_1\subset G$ such that:

(1) $\theta(L,L_1)<\min\{\varepsilon(L)/4\,,\,\varepsilon/4\}$, (the function $\varepsilon(L)$ comes from Lemma 2.1).

(2) $|\operatorname{ext} B_{L_1^*}|>|\operatorname{ext} B_{L^*}|$.

(3) For each subset $T\subset S_{X^*}$ with $\{f|_{L_1}:\ f\in T\}\supset \operatorname{ext} B_{L_1^*}$ we have $\{f|_L:\ f\in T\}\supset \operatorname{ext} B_{L^*}$.

If L_1 is not smooth then starting with L_1 and repeating the procedure, we construct a 2-dimensional subspace L_2 which has the properties (1) – (3), where the following substitutions are made: L by L_1, L_1 by L_2, $\min\{\varepsilon(L)/4$, $\varepsilon/4\}$ by $\min\{\varepsilon(L)/8,\ \varepsilon(L_1)/8,\ \varepsilon/8\}$, and so on. If in some step we get a smooth subspace L_n, the lemma is proved. Otherwise we construct a sequence of 2-dimensional subspaces $\{L_n\}_1^\infty$ with the following properties:

(a) For each integers $n\geq k$, $\theta(L_k,L_n)<\varepsilon(L_k)$.

(b) For each integer n, $\theta(L_n,L_{n+1})<\varepsilon/2^{n+2}$.

(c) For each integer k, $|\operatorname{ext} B_{L_k^*}|\geq 2k$.

From (b) it follows that $\theta-\lim L_n=L_0$ exists and from (a), (c) and Lemma 2.1 it follows that $|\operatorname{ext} B_{L_0^*}|=\infty$, a contradiction. ∎

Remark 2.4. To have a full similarity between Lemma 2.3 and the Mazur theorem we should prove that if a polyhedral space X is separable then the set of all two-dimensional smooth subspaces of X is a dense G_δ*-subset* of the set of all two-dimensional subspaces of X. We will prove this (even in a more general setting) in Corollary 2.6 below.

Proof of part (i) *of Theorem* 1.4. Let Φ be the set of all 2-dimensional smooth subspaces of X. Put

$$B=\{f\in S_{X^*}:\ \ f|_L\in \operatorname{ext} B_{L^*},\ L\in\Phi\}\,.$$

By Lemma 2.3, B is a boundary for X. Let $f\in B$, $L\in\Phi$ and $x\in S_L\setminus \operatorname{ext} S_L$ be such that $f(x)=1$. Since L is smooth it follows that for each $y\in \operatorname{Ker} f$ the point x is not an extreme point of $S_{\operatorname{span}\{x,y\}}$. By the Baire Category Theorem we get that $\operatorname{int}_{\Gamma_f}\gamma_f\neq\emptyset$ and, in particular $\operatorname{card} B\leq w$. ∎

Remark 2.5. (i) The above consideration shows that a point $x \in S_X$ is smooth iff $x \in \operatorname{int}_{\Gamma_f} \gamma_f$ for some $f \in B$.

(ii) A nice alternative proof of part (i) of Theorem 1.4, which uses the Mazur Theorem directly, is given in [7].

COROLLARY 2.6. *If a polyhedral space X is separable then the set of all finite-dimensional smooth subspaces of X is a dense G_δ-subset of the set of all finite-dimensional subspaces of X.*

Proof. Let B be a minimal (countable) boundary of X. Put

$$A = \cup_{f,g\in B} \operatorname{Ker}(f-g).$$

By using Remark 2.5 (i) it is not difficult to see that each subspace $L \subset X$ which contains at least one point from $X \setminus A$, is smooth. Since each set $\operatorname{Ker}(f-g)$ is closed and nowhere dense in X, by the Baire Category Theorem we have that the set $X \setminus A$ is dense in X. From the above consideration the θ-density of smooth subspaces (not just finite-dimensional) easily follows.

Next, for each two functionals $f, g \in B$ and for each positive integer n, we define the set $A_{f,g,n}$ of finite-dimensional subspaces L of X as follows

$$A_{f,g,n} = \{L \subset X \ : \ \exists x_0 \in S_L,\ f|_{(x_0+1/nB_X)\cap S_L} = g|_{(x_0+1/nB_X)\cap S_L} = 1\}\,.$$

It is not difficult to check that each set $A_{f,g,n}$ is θ-closed and a finite-dimensional subspace $M \subset X$ is smooth iff M does not belong to any $A_{f,g,n}$. Since the family $\{A_{f,g,n}\}$ is countable, it follows that the set of all smooth finite-dimensional subspaces of X is a G_δ-set. ∎

3. A CARDINALITY

In this section we establish that the cardinality of a minimal boundary is *equal* to the density character of the space, i.e. (1.1). To this end we *explicitly* construct a subset D of a polyhedral Banach space X with $|D| = |B|$ (where B is a minimal boundary of X) and such that $\operatorname{cl}\operatorname{co} D = B_X$. We conclude the paper by Theorem 3.9 which also implies $\operatorname{card} B = w$ (but non-constructively).

We start with some auxiliary results.

PROPOSITION 3.1. ([6]) *Let B be a boundary of a Banach space X. Then for each functional $f \in B_{X^*}$ there exists a Borel probability measure ν supported on $w^* - \operatorname{cl} B$ which represents f.*

Definition 3.2. We say that a Banach space X has the property (*) if there is a 1-norming subset $B \subset S_{X^*}$ such that each w^*-limit point of B with the norm 1 (if any) does not attain its norm on the unit ball B_X.

Remark 3.3. It is not difficult to check that in the definition above B is actually a boundary and a Banach space X which has the property (*) is polyhedral.

Proposition 3.4. ([2]) *Let X be a separable polyhedral space and $B = \{h_i\}_{i=1}^{\infty}$ be a boundary of X. Let $\varepsilon > 0$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ be any sequence of positive numbers, $0 < \varepsilon_i < \varepsilon$, which tends to zero. Define a new norm on X as follows*

$$|||x||| = \sup\{|(1+\varepsilon_i)h_i(x)| \,:\, i = 1, 2, \dots\}, \qquad x \in X\,.$$

Then the new norm is ε-isomorphic to the original one and the space $Y = (X, |||.|||)$ has the property (*).

Proof. Put $B_1 = \{\pm(1+\varepsilon_i)h_i\}_1^{\infty}$. Then

$$|||x||| = \sup\{h(x) :\; h \in B_1\}\,.$$

It is clear that for each $x \in X$ we have

$$||x|| < |||x||| \leq (1+\varepsilon)||x||\,. \tag{3.1}$$

Let f be any w^*-limit point of the set B_1. Since $\varepsilon_i \to 0$, $\;f \in B_{X^*}$. Suppose that for some $x \in X$ with $|||x||| = 1$ we have $f(x) = 1$. Since $f \in B_{X^*}$ it follows that $||x|| \geq f(x) = 1$. Therefore $||x|| \geq |||x|||$, which contradicts the first (strict) inequality in (3.1). Thus, no w^*-limit point of the set B_1 with the norm one attains its norm $|||.|||$ and hence Y has property (*). ∎

Proposition 3.5. *Let Y be a polyhedral Banach space with property (*), $B = \{h_i :\; i \in J\}$ be a corresponding boundary of Y and $D \subset S_Y$ be such that for each $\sigma \subset J$, $|\sigma| < \infty$ with $\cap_{i\in\sigma}\gamma_{h_i} \neq \emptyset$, holds $D \cap \cap_{i\in\sigma}\gamma_{h_i} \neq \emptyset$. Then $\operatorname{cl}\operatorname{co} D = B_Y$.*

Proof. Let $f_0 \in S_{Y^*}$, $\;f_0(x_0) = 1$, $\;x_0 \in S_Y$. By Proposition 3.1 there is a measure ν on $w^* - \operatorname{cl} B$ which represents f_0. Clearly $\operatorname{supp}\nu \subset \{g \in B_{Y^*} : g(x_0) = 1\}$. It easily follows from the property (*) that the set

$$w^* - \operatorname{cl} B \cap \{g \in B_{Y^*} \,:\, g(x_0) = 1\}$$

is finite. Therefore there are a finite subset $\sigma \subset J$ and positive numbers $\{\alpha_i\}_{i\in\sigma}$, $\sum \alpha_i = 1$, such that $f_0 = \sum_{i\in\sigma} \alpha_i h_i$. It is clear that $h_i(x_0) = 1$ for each $i \in \sigma$, and hence $\cap_{i\in\sigma}\gamma_{h_i} \neq \emptyset$. By the condition of the proposition $D \cap \cap_{i\in\sigma}\gamma_{h_i} \neq \emptyset$. Thus we proved that for each functional $f_0 \in S_{Y^*}$ which attains its norm, there is $x \in D$ with $f_0(x) = 1$. By using the Bishop-Phelps theorem we conclude that the subset $D \subset S_Y$ is 1-norming and the Hahn-Banach theorem completes the proof. ∎

LEMMA 3.6. *Let Z be a polyhedral Banach space and A be any dense subset of Z. If $B \subset S_{Z^*}$ is a boundary of A (i.e. for each $x \in A$ there is $f \in B$ with $f(x) = ||x||$) then B is a boundary of Z.*

Proof. Let B_0 be a minimal boundary of Z and $h \in B_0$. Since A is dense in Z, there is $x_0 \in A$ with $x_0/||x_0|| \in \operatorname{int}_{\Gamma_h} \gamma_h$. It is clear that $h \in B$ and, thus $B \supset B_0$ which completes the proof. ∎

Proof of part (ii) *of Theorem* 1.4. Let $B = \{h_i\}_{i\in I}$ be a minimal boundary of X. From the properties of B which were proved in Section 2, it follows that $w \geq |I|$ and we shall prove the inverse inequality. Fix any decreasing sequence $\{\varepsilon_k\}_{k=1}^{\infty}$ of positive numbers which tends to 0. For any finite subset $\sigma = \{i_j\}_{j=1}^{n} \subset I$ and for any integer k we define the set

$$M(\sigma, k) = \operatorname{int} B_X \cap \cap_{j=1}^{n} \{x \in X \,:\, (1 + \varepsilon_k) h_{i_j}(x) = 1\}\,.$$

Let $x(\sigma, k)$ be any vector from $M(\sigma, k)$ if $M(\sigma, k) \neq \emptyset$ and $x(\sigma, k) = 0$ otherwise. Put

$$D = \{x(\sigma, k) \,:\, \sigma \subset I,\ |\sigma| < \infty,\ k \in N\}\,.$$

Clearly $|I| = |D|$. Our goal is to prove that $\operatorname{cl}\operatorname{co} D = B_X$ which will give the desired inequality $w \leq |I|$. Take any $z \in S_X$, $\varepsilon > 0$ and find $h_{i_0} \in B$ with $h_{i_0}(z) = 1$ and $k_0 \in N$ with $\varepsilon_{k_0} < \varepsilon$. Put $\sigma_0 = \{i_0\}$ and $x_0 = x(\sigma_0, k_0)$ (clearly that $x_0 \neq 0$). Denote $z_0 = (1 + \varepsilon_{k_0})^{-1} z$, $H_0 = \{\pm x_0, \pm z_0\}$, $L_1 = \operatorname{span}\{x_0, z_0\}$. Since X is polyhedral and B is a boundary of X, there exists a $\sigma_1 = \{i_j\}_{j=1}^{n_1} \subset I$ such that $\{h_i\}_{i\in\sigma_1}$ is a symmetric set and that $\{h_i|_{L_1}\}_{i\in\sigma_1} = \operatorname{ext} B_{L_1^*}$. Since $H_0 \subset \operatorname{int} B_X$ there is a $k_1 \in N$ such that for each $i \in \sigma_1$ $|h_i(x_0)| < 1/(1 + \varepsilon_{k_1})$ and $|h_i(z_0)| < 1/(1 + \varepsilon_{k_1})$. Put $F_1 = \operatorname{int} B_X \cap \cap_{j=1}^{n_1} \{x \in X :\ |h_{i_j}(x)| \leq 1/(1 + \varepsilon_{k_1})\}$ and for any ordered subset $\delta \subset \{1, 2, \ldots, n_1\}$ put $\sigma_\delta = \{i_j\}_{j\in\delta}$, $M_\delta = \operatorname{int} B_X \cap \cap_{j\in\delta} \{x \in X :\ \ h_{i_j}(x) = 1/(1 + \varepsilon_{k_1})\}$. We consider just those δ for which $M_\delta \cap F_1 \neq \emptyset$. For each such δ define $\hat{\delta}$ as the largest subset of $\{1, 2, \ldots, n_1\}$ which contains δ and for which $M_{\hat{\delta}} \cap F_1 \neq \emptyset$. It is not difficult to verify that $x(\sigma_{\hat{\delta}}, k_1) \in F_1$ and it is trivial that $x(\sigma_{\hat{\delta}}, k_1) \in M(\sigma_\delta, k_1)$. Put $H_1 = \{x(\sigma_{\hat{\delta}}, k_1) : \ \delta \subset \{1, 2, \dots, n_1\},\ M_\delta \cap F_1 \neq \emptyset\},\ L_2 = \operatorname{span}\{L_1, H_1\}$.

Since X is polyhedral and B is a boundary of X, there is a finite subset $\sigma_2 = \{i_{n_1+1}, i_{n_1+2}, \dots, i_{n_2}\} \subset I$ such that $\{h_i\}_{i\in\sigma_2}$ is a symmetric set and that $\{h_i|_{L_2}\}_{i\in\sigma_1\cup\sigma_2} \supset \operatorname{ext} B_{L_2^*}$. Take $k_2 > k_1$ so large that for each $x \in H_0 \cup H_1$ and for each $i \in \sigma_1 \cup \sigma_2$ $|h_i(x)| \leq 1/(1+\varepsilon_{k_2})$. Put $F_2 = F_1 \cap (\cap_{j=n_1+1}^{n_2}\{x \in X : |h_{i_j}(x)| \leq 1/(1+\varepsilon_{k_2})\})$. Now we construct the set H_2 in the same manner as H_1 in the previous step. And so on. Denote $Z = \operatorname{cl} \cup_{k=1}^\infty L_k$. By Lemma 3.6 $B_1 = \cup_{k=1}^\infty \{h_i : \ i \in \sigma_k\}$ is a boundary of Z. Let Y be a Banach space Z in the norm

$$|||x||| = \sup\{(1+\varepsilon_{k_l})h_{i_j}(x) \ : \ l = 1, 2, \dots,\ j = n_{l-1}+1, \dots, n_l,\ n_0 = 0\}.$$

By Proposition 3.4 Y has the property (*) and by the construction, the set $D_1 = D \cap Z$ satisfies the condition of Proposition 3.5. Hence $\operatorname{cl}\operatorname{co} D_1 = B_Y$. Since $z \in B_X$ and $\varepsilon > 0$ are arbitrary and $d(z, B_Y) < \varepsilon$, it follows that $\operatorname{cl}\operatorname{co} D = B_X$, which completes the proof of Theorem 1.4. ∎

By using a separating theorem it is easy to see that if B is a boundary of a Banach space X then $w^* - \operatorname{cl}\operatorname{co} B = B_{X^*}$. The following theorem (which was also proved independently by L. Vesely [7]) shows that for a polyhedral Banach space we can take the norm-closure instead of the w^*-closure. We start with a separable case. The proof in the general case is a reduction to the separable one.

Proposition 3.7. *Let X be a* separable *polyhedral Banach space and B is a boundary of X then $||.|| - \operatorname{cl}\operatorname{co} B = B_{X^*}$.*

Proof. We may assume that B is the minimal boundary. In particular, B is symmetric, i.e. $B = -B$. Let $f \in B_{X^*}$ and $\varepsilon > 0$. Use the notation of the proof of proposition 3.4. By using the Bishop-Phelps theorem for the space $(X, |||.|||)$, we find a functional $g \in B_{(X,|||.|||)^*}$ which attains its $|||.|||$-norm and so that $||f - g|| < \varepsilon$ (recall that $B_{X^*} \subset B_{(X,|||.|||)^*}$). From the proof of Proposition 3.5 it follows that $g = \sum_{i\in\sigma} \alpha_i(1+\varepsilon_i)h_i$ where σ is a finite subset of integers and $\sum_{i\in\sigma}|\alpha_i| \leq 1$. It is clear that the functional $(1+\varepsilon)^{-1}g \in \operatorname{co} B$ still approximates f which completes the proof. ∎

Remark 3.8. A weaker result: the dual space for a separable polyhedral Banach space is separable too, was proved in [1].

THEOREM 3.9. *Let X be a polyhedral Banach space and B be a boundary of X. Then $||.|| - \operatorname{cl}\operatorname{co} B = B_{X^*}$.*

Proof. Put $V = ||.|| - \operatorname{cl}\operatorname{co} B$ and suppose to the contrary that there exists $f_0 \in S_{X^*} \setminus V$. Take $F_0 \in S_{X^{**}}$ such that

$$\sup\{F_0(g) : g \in V\} < \alpha < F_0(f_0) . \tag{3.2}$$

By the Goldstein theorem there is an $x_1 \in S_X$ so that $|(F_0 - x_1)(f_0)| < 1$. Since B is a boundary of X there is an $h_1 \in B$ with $h_1(x_1) = 1$. Put $\sigma_0 = \{f_0\}$, $\sigma_1 = \{h_1\}$, $L_1 = \operatorname{span}\{x_1\}$ and by the Goldstein theorem find an $x_2 \in S_X$ with $\max\{|(F_0 - x_2)(h) : h \in \sigma_0 \cup \sigma_1\} < 1/2$. Put $L_2 = \operatorname{span}\{x_1, x_2\}$. Since X is polyhedral and B is a boundary of X, there is a finite subset $\sigma_2 \subset B$ with $\{h|_{L_2} : h \in \sigma_2\} = \operatorname{ext} B_{L_2^*}$. By using again the Goldstein theorem find an $x_3 \in S_X$ with $\max\{|(F_0 - x_3)(h) : h \in \cup_{k=0}^{2} \sigma_k\} < 1/3$. The further construction is clear. In this way we construct a sequence $\{x_i\}_{i=1}^{\infty} \subset S_X$ and a sequence $\{\sigma_k\}_{k=1}^{\infty}$ of finite subsets of B such that:

(a) If $L_k = \operatorname{span}\{x_i\}_{i=1}^{k}$ then $\{h|_{L_k} : h \in \sigma_k\} = \operatorname{ext} B_{L_k^*}$.

(b) For each $i = 1, 2, \dots$, $\max\{|(F_0 - x_i)(h)| : h \in \cup_{k=0}^{i-1} \sigma_k\} < 1/i$.

Put $E = [x_i]_{i=1}^{\infty}$, $B_1 = \cup_{k=1}^{\infty} \sigma_k$ and $B_2 = \{h|_E : h \in B_1\}$. By using (a) and Lemma 3.6 we conclude that B_2 is a boundary of E and, hence by Proposition 3.7 we have

$$||.|| - \operatorname{cl}\operatorname{co} B_2 = B_{E^*} . \tag{3.3}$$

Let $G_0 \in B_{E^{**}} \subset X^{**}$ be any w^*-limit point of the set $\{x_i\}_{i=1}^{\infty}$ (we consider E as a subspace of E^{**} and E^{**} as a subspace of X^{**}). Then by (b) and (3.2),

$$\sup\{G_0(h) : h \in B_1\} < \alpha < G_0(f_0) . \tag{3.4}$$

Now we use the following easily verified equality: for each $g \in X^*$, $G_0(g) = G_0(g|_E)$. Hence by (3.3) and (3.4), $\sup\{G_0(g) : g \in B_{E^*}\} < \alpha$. Thus $\sup\{G_0(t) : t \in B_{X^*}\} = \sup\{G_0(t|_E) : t \in B_{X^*}\} = \sup\{G_0(g) : g \in B_{E^*}\} < \alpha$, i.e. $||G_0|| < \alpha$. However $f_0 \in B_{X^*}$ and by (3.4), $G_0(f_0) > \alpha$, a contradiction which completes the proof. ∎

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