EXTRACTA MATHEMATICAE Vol. 14, Núm. 3, 373 – 377 (1999)

Chain-Finite Operators and Locally Chain-Finite Operators

Teresa Bermúdez and Antonio Martinón

Departamento de Análisis Matemático, Universidad de La Laguna, 38271-La Laguna (Tenerife), Spain

(Research announcement presented by M. González)

AMS *Subject Class.* (1991): 47A11, 47B99 *Received May* 18, 1999

1. Introduction and Preliminaries

The problem we are concerned with in this research announcement is the algebraic characterization of chain-finite operators (global case) and of locally chain-finite operators (local case).

In the global case, recall that a bounded linear operator T on a Banach space X ($T \in L(X)$) is a *chain-finite* operator, denoted by $T \in CF(X)$, if there exists a non negative integer k such that $N(T^k) = N(T^{k+1})$ and $R(T^k) = R(T^{k+1})$, where $N(T)$ and $R(T)$ denote the kernel and the range of T, respectively. The smallest non negative integer k for which this occurs will be denoted by $l(T)$. The following characterizations of chain-finite operators are well-known. Given $T \in L(X)$, T is chain-finite operator with $l(T) = k$ if and only if 0 is a pole of the resolvent operator $(\lambda - T)^{-1}$ of T of order k [8, Theorem V.10.1 & V.10.2]. Moreover, T is a chain-finite operator if and only if

$$X = N(T^k) \oplus R(T^k) \tag{1}$$

for some $k \in \mathbb{N}$ [6, Proposition 38.4].

In [5], González and Onieva prove the following algebraic property: if $T \in CF(X)$, then there exists a positive integer k and an operator $B \in L(X)$ such that

$$T^k B T^k = T^k \text{ and } TB = BT. \tag{2}$$

The following condition is similar and apparently weaker than (2)

$$T^kBT^k = T^k \text{ and } T^kB = BT^k. \tag{3}$$

Also Laursen and Mbekhta [7] prove that T is chain-finite operator with $l(T) \leq 1$ if and only if T is relatively regular and commutes with some generalized inverse, namely there exists $S \in L(X)$ such that $T = TST$ and $ST = TS$.

In the local case, taking into account [1, Remark 1.5]

$$\sigma(Tx, T) \subset \sigma(x, T) \subset \sigma(Tx, T) \cup \{0\},$$

where $\sigma(x, T)$ denotes the local spectrum of T at x, we can easily derive the following chain of inclusions for the local spectra

$$\sigma(x, T) \supset \sigma(Tx, T) \supset \cdots \supset \sigma(T^kx, T) \supset \ldots, \tag{4}$$

where 0 is the only point which may make these subsets different. Hence there is at most one inclusion in (4) which is not an equality. Then it is said that T is a locally chain-finite operator at x if the chain given in (4) breaks. Namely, given $T \in L(X)$ and $x \in X$, we say that T is a *locally chain-finite operator* at x with $l(T, x) = k > 0$ if $\sigma(T^{k-1}x, T) \neq \sigma(T^kx, T)$ and with $l(T, x) = 0$ if $0 \notin \sigma(x, T)$ [3, Definition 4.1]. This notion is a localization of the concept of chain-finite operator: if T satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP), then T is a chain-finite operator if and only if T is a locally chain-finite operator at x for every $x \in X$ [3, Theorem 4.2]. Moreover, locally chain-finite operators are related with the facts that 0 is a pole of the local resolvent function and that the vector has a unique decomposition similar to (1). Indeed, given $T \in L(X)$ and $x \in X$, if T has SVEP and $0 \in \sigma(x, T)$ then by [2, Theorem 1], 0 is a pole of order k of the local resolvent function if and only if

$$0 \in \sigma(T^{k-1}x, T) \setminus \sigma(T^kx, T); \tag{5}$$

equivalently, there exists a unique decomposition $x = x_1 + x_2$ such that $x_1 \in N(T^k) \setminus N(T^{k-1})$ and $\sigma(x_2, T) = \sigma(x, T) \setminus \{0\}$ [3, Theorem 3.3].

Given $T \in L(X)$, a complex number λ belongs to the *resolvent* set $\rho(T)$ of T if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote $\sigma(T) = \mathbb{C} \setminus \rho(T)$ the *spectrum* of T. The resolvent map $R(., T) : \rho(T) \longrightarrow L(X)$ is analytic, hence the following equation has an analytic solution on $\rho(T)$

$$(\mu - T)w(\mu) = x, \tag{6}$$

given by $w(\mu) = R(\mu, T)x$ for every $\mu \in \rho(T)$ and $x \in X$. This function may admit an analytic extension for some $x \in X$. So, we say that a complex number λ belongs to the *local resolvent set* of T at x, denoted $\rho(x, T)$, if there exists an analytic function $w : U \longrightarrow X$, defined on a neighborhood U of λ, which satisfies (6) for every $\mu \in U$. The *local spectrum* of T at x is the complement $\sigma(x, T) := \mathbb{C} \setminus \rho(x, T)$.

Since w is not necessarily unique, a complementary property is needed to prevent ambiguity. An operator $T \in L(X)$ satisfies the SVEP if $h \equiv 0$ is the unique analytic solution of $(\lambda - T)h(\lambda) = 0$ on any open subset of the plane with values in X. If T satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_T$ defined on $\rho(x, T)$ satisfying (6), which is called the *local resolvent function* of T at x. See [4] for further details.

2. Chain-finite operators

The following result proves that conditions (2) and (3) are algebraic characterizations of a chain-finite operator.

Theorem 1. *Let $T \in L(X)$ and let k be a positive integer. The following assertions are equivalent:*

(a) *There exists $B \in L(X)$ such that $T^k B T^k = T^k$ and $BT = TB$.*

(b) *There exists $B \in L(X)$ such that $T^k B T^k = T^k$ and $BT^k = T^k B$.*

(c) *$T \in CF(X)$ and $l(T) \leq k$.*

Remark 1. By Theorem 1, it is clear that $T \in CF(X)$ with $l(T) \leq k$ if and only if $T^k \in CF(X)$ with $l(T^k) \leq 1$. Using that $T \in CF(X)$ with $l(T) = k$ if and only if 0 is a pole of the resolvent operator of T of order k, we have that 0 is a pole of the resolvent operator of T of order less than or equal to k if and only if 0 is a pole of the resolvent operator of T^k of order less than or equal to 1.

As an immediate consequence of Theorem 1 we get the following result of Laursen and Mbekhta [7, Theorem 3].

Corollary 1. *Let $T \in L(X)$. The following assertions are equivalent:*

1. *There exists $B \in L(X)$ such that $TBT = T$ and $BT = TB$.*
2. *$X = N(T) \oplus R(T)$.*

3. Locally chain-finite operators

The following proposition is a useful property to the remainder results.

Proposition 1. *Assume that $T \in L(X)$ has the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. Then T is locally chain-finite operator with $l(T,x) \leq k$ if and only if T^k is locally chain-finite at x with $l(T^k,x) \leq 1$.*

Next, we give a sufficient condition of locally chain-finite operators.

Theorem 2. *Assume that $T \in L(X)$ has the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $n \in \mathbb{N} = \{1, 2, \dots\}$, then T is locally chain-finite at x with $l(T,x) \leq k$.*

Corollary 2. *Assume that $T \in L(X)$ has the SVEP and let k be a positive integer. Then T is chain-finite operator with $l(T) \leq k$ if and only if there exists $B \in L(X)$ such that $T^k B^n T^k = B^{n-1} T^k$ for all $n \in \mathbb{N}$.*

Remark 2. In the proof of Corollary 2, we do not need the hypothesis of the SVEP to show the necessity of the condition that characterizes chain-finite operators. On the contrary, this hypothesis cannot be neglected to establish that the condition is sufficient.

Example 1. Let T be the left shift operator on $\ell_2(\mathbb{N})$, i.e. $T(x_1, x_2, \dots) := (x_2, x_3, \dots)$ and B the right shift operator, i.e. $B(x_1, x_2, \dots) := (0, x_1, x_2, \dots)$. Then T has not the SVEP. Moreover, T is not chain-finite operator and $TB^nT = B^{n-1}T$ for all $n \in \mathbb{N}$.

With some additional hypotheses we have the converse of Theorem 2.

Proposition 2. *Let $T \in L(X)$ with the SVEP such that 0 is an isolated point of $\sigma(T)$, let k be a positive integer and let $x \in X \setminus \{0\}$. Then T is locally chain-finite operator at x with $l(T,x) \leq k$ if and only if there exists $B \in L(X)$ such that $T^k B^n T^k x = B^{n-1} T^k x$ for all $n \in \mathbb{N}$.*

In the next proposition, we give a necessary condition for an operator to be locally chain-finite operators similar to the necessary condition of chain-finite operators given in Theorem 1.

Proposition 3. *Let $T \in L(X)$ with the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If T is locally chain-finite operator at x with $l(T,x) \leq k$, then there exists $B \in L(X)$ such that $T^kBT^kx = T^kx$ and $TBx = BTx$.*

Corollary 3. *Assume that $T \in L(X)$ has the SVEP, let k be a positive integer and let $x \in X \setminus \{0\}$. If there exists $B \in L(X)$ such that $T^kB^nT^kx = B^{n-1}T^kx$ for all $n \in \mathbb{N}$, then there exists $S \in L(X)$ which $T^kST^kx = T^kx$ and $TSx = STx$.*

The necessary condition given in Proposition 3 is not a sufficient condition.

Example 2. Let T be the right shift operator on $\ell_2(\mathbb{N})$, B the left shift operator and $x := (0, 1, 0, \dots)$. Then T is not locally chain-finite operator at x, $TBTx = Tx$ and $TBx = BTx$.

References

[1] Bartle, R., Spectral decomposition of operators in Banach spaces, *Proc. London Math. Soc.,* **(3) 20** (1970), 438–450.

[2] Bermúdez, T., González, M., Martinón, A., Stability of the local spectrum, *Proc. Amer. Math. Soc.,* **125** (1997), 417–425.

[3] Bermúdez, T., González, M., Martinón, A., On the poles of the local resolvent, *Math. Nachr.,* **193** (1998), 19–26.

[4] Colojoara, I., Foias, C., "Theory of Generalized Spectral Operators", Gordon and Breach, New York, 1968.

[5] González, M., Onieva, V.M., On the convergence of Neumann series in Banach spaces, in "Actas XII Jornadas Luso-Espanholas Mat. Vol.II", Univ. Minho, Braga, (1987) , 335–338.

[6] Heuser, H.G., "Functional Analysis", Wiley, New York, 1982.

[7] Laursen, K.B., Mbekhta, M., Closed range multipliers and generalized inverses, *Stud. Math.,* **107** (1993), 127-135.

[8] Taylor, A.C., Lay, D.C., "Introduction to Functional Analysis" (2nd edition), Wiley, New York, 1980.