EXTRACTA MATHEMATICAE Vol. **14**, Núm. 3, 295 – 327 (1999)

An Elementary Approach to Some Questions in Higher Order Smoothness in Banach Spaces

M. Fabian [1,*] and V. Zizler [2,†]

[1,2] *Mathematical Institute, Czech Academy of Sciences*
Žitná 25, 11567 *Praha* 1, *Czech Republic*
[2] *Faculty of Economics, W. Churchila* 3, 130 00 *Praha* 3, *Czech Republic*

(Research paper presented by J.A. Jaramillo)

AMS *Subject Class.* (1991): 46B20, 46E30, 46G05 *Received April* 16, 1998

1. Introduction

It is known that a Banach space X is isomorphic to a Hilbert space if both X and X^* admit C^2-smooth bumps [25]. Recall that under *bump* we understand a real valued function with bounded and nonempty support. It is also known that ℓ_p spaces do not admit any bump function with the order of smoothness better than that of their canonical norm [19]. There is an interplay between the geometry of Banach spaces and higher order smooth variational principles. We refer the reader to [8, Chap. 5], [17] and to the references in these works for further information and quotations.

The known proofs of these results usually use some advanced methods from the Banach space theory, like Kwapień theorem on a type-cotype characterization of spaces isomorphic to Hilbert spaces, Bessaga-Pelczyński result on spaces not containing copies of c_0 etc.

In this paper we give simple, elementary, and selfcontained proofs to some of these results.

For this purpose we further develop a method of line integral convolutions [15] in order to directly obtain a quadratic form needed for constructing an inner product (Theorem 6 below) and prove a nonconvex version of the Šmulyan duality lemma (Lemma 2 below). We also give an alternative proof

* Supported by grants AV 101-97-02 and GAČR 201-98-1449.
† Supported by grants GAUK 186, GAUK 1/1998, and GAČR 201-98-1449.

to a version of Stegall's variational principle [28], [29] that we need for our proofs (Theorem 5 below). Finally, we recall an elementary proof of the fact that Banach spaces isomorphic to a Hilbert space are separably determined (Theorem 9 below) [22].

Using these tools, we prove, in particular, that X is isomorphic to a Hilbert space if both X and X^* admit smooth bumps with locally Lipschitzian derivative (Theorem 10 below), that ℓ_p (p not an even integer) does not admit any bump with Taylor expansion of order p, and that ℓ_p, with $p < 2$, does not admit any continuous twice Gâteaux smooth bump (Theorem 13 below). Finally we prove a polynomial variational principle for Banach spaces that admit a separating polynomial (Theorem 21 below).

An advantage of our approach is that it shows an interplay of the higher order smoothness and some other geometric properties of spaces in a transparent way.

We focus on an explanation of the main points in these phenomena rather than on obtaining the most general forms of the results.

2. Tools

We will follow the notation standard in Banach space theory. In particular, if $(X, \|\cdot\|)$ is a Banach space, then its closed unit ball is denoted by B_X. If x^* is an element of the dual X^* and $x \in X$, the value of x^* at x is denoted by $\langle x^*, x\rangle$. We shall always assume that X is a subspace of its second dual X^{**}. Let $f : X \to (-\infty, +\infty]$ be a function such that $f(x) < +\infty$ for at least some $x \in X$. For $x \in X$ we define (Moreau-Rockafellar) *subdifferential* $\partial f(x)$ of f at x as the set (possibly empty) of all ξ in the dual X^* such that

$$f(x+h) - f(x) \geq \langle \xi, h\rangle \qquad \text{for all } h \in X.$$

If f is convex, then, clearly, f is Gâteaux smooth at x if and only if $\partial f(x)$ is a singleton; then $\partial f(x)$ consists of the Gâteaux derivative $f'(x)$ of f at x. From the separation theorem we get that $\partial f(x) \neq \emptyset$ if f is continuous at x and f is convex. The (Legendre-Fenchel) *conjugate* f^* to f is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : \ x \in X\}, \qquad x^* \in X^*.$$

Thus f^* is a convex function from X^* to $(-\infty, +\infty]$. We also put $f^{**} = (f^*)^*$. If $g : X^* \to (-\infty, +\infty]$, we put for $x \in X$

$$g_*(x) = \sup\{\langle x^*, x\rangle - g(x^*) : \ x^* \in X^*\}.$$

Lemma 1. *Let $f : X \to \mathbb{R}$ be a function, $x \in X$, and $x^* \in X^*$. Then $x^* \in \partial f(x)$ if and only if $f^*(x^*) + f(x) = \langle x^*, x\rangle$ and this implies that $x \in \partial f^*(x^*)$ and $f^{**}(x) = f(x)$. If f is convex and continuous, then $f^{**}{}_{|X} = f$.*

Proof. Let $x^* \in \partial f(x)$. Then

$$\begin{aligned} f^*(x^*) &= \sup\{\langle x^*, y\rangle - f(y) :\ y \in X\} \\ &= -f(x) + \langle x^*, x\rangle - \inf\{f(y) - f(x) - \langle x^*, y - x\rangle :\ y \in X\} \\ &= -f(x) + \langle x^*, x\rangle. \end{aligned}$$

If this identity holds and $y \in X$, then

$$\begin{aligned} f(y) - f(x) =& f(y) + f^*(x^*) - \langle x^*, x\rangle \\ \geq& f(y) + \langle x^*, y\rangle - f(y) - \langle x^*, x\rangle = \langle x^*, y - x\rangle; \end{aligned}$$

so $x^* \in \partial f(x)$. If the identity holds and $y^* \in X^*$, then

$$\begin{aligned} f^*(y^*) - f^*(x^*) =& f^*(y^*) + f(x) - \langle x^*, x\rangle \\ \geq& \langle y^*, x\rangle - f(x) + f(x) - \langle x^*, x\rangle = \langle y^* - x^*, x\rangle; \end{aligned}$$

so $x \in \partial f^*(x^*)$. Further, the identity yields

$$f(x) = \langle x^*, x\rangle - f^*(x^*) \leq f^{**}(x).$$

The reverse inequality follows from a general fact that $f^{**}{}_{|X} \leq f$. Indeed, for $z \in X$ we have

$$\begin{aligned} f^{**}(z) &= \sup\{\langle y^*, z\rangle - f^*(y^*) :\ y^* \in X^*\} \\ &\leq \sup\{\langle y^*, z\rangle - (\langle y^*, z\rangle - f(z)) :\ y^* \in X^*\} = f(z). \end{aligned}$$

Assume that f is convex and continuous. Then the Hahn-Banach theorem guarantees that $\partial f(x) \neq \emptyset$ for every $x \in X$ and thus $f^{**}(x) = f(x)$ by the first part of the proof. ∎

Given a Banach space $(Z, \|\cdot\|)$, a function $f : Z \to (-\infty, +\infty]$, $z_0 \in \operatorname{dom} f := \{z \in Z :\ f(z) < +\infty\}$, $z_0^* \in Z^*$ and $t \geq 0$, put

$$\begin{aligned} \alpha(f, z_0, z_0^*, t) &= \sup\{f(z) - f(z_0) - \langle z_0^*, z - z_0\rangle :\ z \in Z,\ \|z - z_0\| = t\}, \\ \beta(f, z_0, z_0^*, t) &= \inf\{f(z) - f(z_0) - \langle z_0^*, z - z_0\rangle :\ z \in Z,\ \|z - z_0\| = t\}. \end{aligned}$$

LEMMA 2. *Let* $f : Z \to (-\infty, +\infty]$ *be a function (not necessarily convex),* $z_0 \in \operatorname{dom} f$ *and* $z_0^* \in \partial f(z_0)$. *Then* $\beta(f^*, z_0^*, z_0, s) \geq 0$ *and*

$$\alpha(f, z_0, z_0^*, t) + \beta(f^*, z_0^*, z_0, s) \geq ts \qquad \text{for all } t, s \geq 0.$$

If, in addition, f *is Fréchet smooth at* z_0, *then* $\beta(f^*, z_0^*, z_0, s) \to 0$ *implies* $s \downarrow 0$.

Proof. That $\beta(f^*, z_0^*, z_0, s) \geq 0$ follows from Lemma 1. Take $s \geq 0$ and consider a $z^* \in Z^*$, with $\|z^* - z_0^*\| = s$. Using Lemma 1, for $z^* \in Z^*$ we can estimate

$$\begin{aligned} f^*(z^*) - f^*(z_0^*) - \langle z^* - z_0^*, z_0\rangle & \\ &= \sup\{\langle z^*, z\rangle - f(z) : \ z \in Z\} + f(z_0) - \langle z^*, z_0\rangle \\ &= -\inf\{(f(z) - f(z_0) - \langle z_0^*, z - z_0\rangle) + \langle z_0^* - z^*, z - z_0\rangle : \ z \in Z\} \\ &\geq -\inf\{\alpha(f, z_0, z_0^*, \|z - z_0\|) + \langle z_0^* - z^*, z - z_0\rangle : \ z \in Z\} \\ &= -\inf\{\alpha(f, z_0, z_0^*, t) - \|z_0^* - z^*\| t : \ t \geq 0\} \\ &= \sup\{-\alpha(f, z_0, z_0^*, t) + st : \ t \geq 0\}. \end{aligned}$$

The inequality is thus proved.

Assume that f is Fréchet smooth at z_0. This is equivalent with

$$\lim_{t \downarrow 0} \alpha(f, z_0, z_0^*, t)/t = 0.$$

Assume there is $\Delta > 0$ and $s_i > \Delta$, $i = 1, 2, \ldots$, such that $\beta(f^*, z_0^*, z_0, s_i) < \frac{1}{i}$ for all $i \in \mathbb{N}$. Then

$$\frac{1}{i} > \beta(f^*, z_0^*, z_0, s_i) > s_i t - \alpha(f, z_0, z_0^*, t) \qquad \text{for all } t > 0 \text{ and for all } i \in \mathbb{N}.$$

Hence, if s_0 is a cluster point of the sequence (s_i), we have $s_0 \geq \Delta > 0$ and $s_0 t - \alpha(f, z_0, z_0^*, t) \leq 0$ for all $t > 0$. This is impossible as $\lim_{t\downarrow 0} \alpha(f, z_0, z_0^*, t)/t = 0$. ∎

LEMMA 3. *Let* $f : Z \to (-\infty, +\infty]$ *be a proper lower semicontinuous (not necessarily convex) function with* $\inf f > -\infty$ *and assume that the conjugate function* f^* *is Fréchet smooth at* $z_0^* \in Z^*$. *Then the derivative* $z_0 := (f^*)'(z_0^*)$ *belongs to* Z *and*

$$f^{**}(z_0) = f(z_0). \tag{1}$$

Proof. We note that for $0 \neq t \in \mathbb{R}$ the functions

$$g_t(h^*) := \frac{1}{t}\big[f^*(z_0^* + th^*) - f^*(z_0^*)\big], \qquad h^* \in Z^*,$$

are weak* lower semicontinuous and for $t \to 0$ converge to z_0 uniformly on the unit ball of Z^*. It follows z_0 is thus weak* lower semicontinuous there. And since z_0 is linear, z_0 is weak* continuous. Hence $z_0 \in Z$ by the Banach-Dieudonné theorem (see e.g. [18, Theorem 222]).

In order to prove (1), put for $z \in Z$

$$\psi(z) = \inf\Big\{ \sum_{i=1}^m \alpha_i f(z_i) \ : \alpha_i \geq 0,\ z_i \in Z,\ i = 1, \ldots, m, \\ \sum_{i=1}^m \alpha_i = 1,\ \sum_{i=1}^m \alpha_i z_i = z,\ m \in \mathbb{N}\Big\}.$$

We note that ψ is a convex function minorizing f. As f^{**} is the supremum of all affine continuous functions minorizing f, we have ${f^{**}}_{|Z} \leq \psi$.

First, we show that $(z_0, f^{**}(z_0))$ lies in the closure of $\operatorname{epi}\psi := \{(z,t) \in Z \times \mathbb{R} :\ \psi(z) \geq t\}$. Assume this not true. Since $\operatorname{epi}\psi$ is a convex set, there are $(\xi, s) \in Z^* \times \mathbb{R}$ and $\infty < c < d < +\infty$ such that

$$\text{(2)} \qquad \langle \xi, z_0 \rangle + s f^{**}(z_0) < c < d < \langle \xi, z \rangle + st \qquad \text{for all } (z,t) \in \operatorname{epi}\psi.$$

Take $z \in \operatorname{dom} f$. Then $(z,t) \in \operatorname{epi}\psi$ for all large $t \in \mathbb{R}$ and so (2) implies that $s \geq 0$. Assume for a while that $s = 0$. Then (2) yields

$$\langle \xi, z - z_0 \rangle > d - c\ (> 0) \qquad \text{for all } z \in \operatorname{dom} f.$$

For $n = 1, 2, \ldots$ define an (affine continuous) function

$$g_n(z) = -n\langle \xi, z - z_0 \rangle + n(d-c) + \inf f, \quad z \in Z.$$

Observe that g_n minorizes f and hence also the restriction of f^{**} to Z. Thus, in particular,

$$f^{**}(z_0) \geq \lim_{n\to\infty} g_n(z_0) = \lim_{n\to\infty} n(d-c) + \inf f = +\infty.$$

However, by Lemma 1, $f^{**}(z_0) = \langle z_0^*, z_0 \rangle - f^*(z_0^*) \in \mathbb{R}$, a contradiction. Therefore $s > 0$. Then, from (2) we have

$$f(z) \geq \langle \frac{1}{s}\xi, z_0 - z \rangle + f^{**}(z_0) + \frac{1}{s}(d-c) \qquad \text{for all } z \in Z.$$

The function on the right hand side of this inequality is affine and continuous. Thus $f(z)$ can be replaced by $f^{**}(z)$ here and, in particular we get that $+\infty > f^{**}(z_0) \geq f^{**}(z_0) + \frac{1}{s}(d-c)$, which is impossible. Thus we have proved that $(z_0, f^{**}(z_0))$ belongs to the closure of $\operatorname{epi}\psi$.

Now we are ready to prove that $f(z_0) \leq f^{**}(z_0)$. Fix a $\Delta > 0$. From the lower semicontinuity of f at z_0 we find $0 < \delta < \Delta$ such that $f(z) > f(z_0) - \Delta$ whenever $z \in Z$, $\|z - z_0\| < \delta$. From Lemma 2, we find $0 < \gamma < \Delta$ such that $\beta(f^{**}, z_0, z_0^*, s) < \gamma$ implies $s < \delta$. As $(z_0, f^{**}(z_0))$ is in the weak closure of $\operatorname{epi}\psi$, there is $(z, t) \in \operatorname{epi}\psi$ such that

$$t - f^{**}(z_0) - \langle z_0^*, z - z_0\rangle < \gamma.$$

Find $m \in \mathbb{N}$, $\alpha_i \geq 0$ and $z_i \in Z$, $i = 1, \ldots, m$ such that $\sum_{i=1}^m \alpha_i = 1$, $\sum_{i=1}^m \alpha_i z_i = z$, and

$$\sum_{i=1}^m \alpha_i f(z_i) - f^{**}(z_0) - \Big\langle z_0^*, \sum_{i=1}^m \alpha_i z_i - z_0 \Big\rangle < \gamma.$$

From this it follows that there is $i \in \{1, \ldots, m\}$ such that

$$f(z_i) - f^{**}(z_0) - \langle z_0^*, z_i - z_0\rangle < \gamma.$$

Hence $\beta(f^{**}, z_0, z_0^*, \|z_i - z_0\|) < \gamma$ and so $\|z_i - z_0\| < \delta$. Thus

$$\begin{aligned} f(z_0) &< f(z_i) + \delta < \gamma + f^{**}(z_0) + \langle z_0^*, z_i - z_0\rangle + \delta \\ &< 2\Delta + f^{**}(z_0) + \|z_0^*\|\Delta. \end{aligned}$$

Letting $\Delta \downarrow 0$, we get $f(z_0) \leq f^{**}(z_0)$. This proves (1). ∎

LEMMA 4. *Let $g : Z^* \to (-\infty, +\infty]$ be a proper lower semicontinuous (not necessarily convex) function, with $\inf g > -\infty$. Assume that $g_* : Z \to (-\infty, +\infty]$ is Fréchet smooth at $z_0 \in Z$ and denote $z_0^* = (g_*)'(z_0)$. Then*

$$g(z_0^*) = (g_*)^*(z_0^*).$$

Proof. Proceed in the spirit of the proof of Lemma 3. Interchange the rôle of Z and Z^*. Instead of $Z \times \mathbb{R}$ with the norm topology, consider $Z^* \times \mathbb{R}$ with the weak* topology. ∎

A Banach space is called an *Asplund space* if every convex continuous function on it is Fréchet smooth at the points of a dense subset (see e.g. [8], [26], [12]). The following is a version of the Stegall variational principle [28], [10], [29], [26, Corollary 5.22].

THEOREM 5. *Let Z be an Asplund space. Let $\varphi : Z^* \to (-\infty, +\infty]$ be a proper lower semicontinuous (not necessarily convex) function such that* $\inf \varphi > -\infty$ *and* $\liminf_{\|z^*\|\to\infty} \varphi(z^*)/\|z^*\| > 0$. *Then, given $\epsilon > 0$, there are $z_0 \in Z$, $\|z_0\| < \epsilon$ and $z_0^* \in Z^*$ such that $z_0 \in \partial\varphi(z_0^*)$; actually, more can be said:*

$$\varphi(z^*) - \varphi(z_0^*) - \langle z^* - z_0^*, z_0\rangle \geq \gamma(\|z^* - z_0^*\|) \qquad \text{for all } z^* \in Z^*,$$

where $\gamma : (0, +\infty) \to [0, +\infty]$ is such that $s \downarrow 0$ if $\gamma(s) \to 0$. In particular,

$$\varphi(z_0^* + h^*) + \varphi(z_0^* - h^*) - 2\varphi(z_0^*) \geq 0 \qquad \text{for all } h^* \in Z^*.$$

Proof. Put for $z \in Z$

$$f(z) = \sup\{\langle z^*, z\rangle - \varphi(z^*) : \ z^* \in Z^*\}.$$

Then $f : Z \to (-\infty, +\infty]$, f is a convex function, which is bounded on a neighbourhood of the origin. Indeed, from the assumptions, there are $a > 0$ and $b > 0$ such that $\varphi(z^*) > a\|z^*\|$ whenever $z^* \in Z^*$ and $\|z^*\| > b$. We can then easily see that

$$f(z) \leq \max(0, b\|z\| - \inf \varphi) \qquad \text{whenever } z \in Z \text{ and } \|z\| < a.$$

Since the function φ is proper, f is locally bounded below. Hence f is convex and continuous on a neighbourhood of 0.

Since Z is an Asplund space, f is Fréchet smooth at some $z_0 \in Z$, with $\|z_0\| < \min(a, \epsilon)$. Put $z_0^* = f'(z_0)$. In the notation of Lemma 2, we have

$$f^*(z^*) - f^*(z_0^*) - \langle z^* - z_0^*, z_0\rangle \geq \beta(f^*, z_0^*, z_0, \|z^* - z_0^*\|) \qquad \text{for all } z^* \in Z^*.$$

Put $\gamma(s) = \beta(f^*, z_0^*, z_0, s)$, $s \geq 0$. Then, by Lemma 2, $\gamma(\cdot) \geq 0$ and $\gamma(s) \to 0$ implies $s \downarrow 0$. Further, it is easy to check that $\varphi \geq f^*$ and Lemma 4 says that $\varphi(z_0^*) = f^*(z_0^*)$. Hence $\varphi(z^*) - \varphi(z_0^*) - \langle z^* - z_0^*, z_0\rangle \geq \gamma(\|z^* - z_0^*\|)$ for all $z^* \in Z^*$. ∎

Remark. Stegall's principle can also be formulated in a more general setting, when the function φ is defined on a dentable Banach space (see [28], [26, Corollary 5.22]). Recall that a Banach space X is *dentable* (which is equivalent to the Radon-Nikodým property) if for every bounded set $M \subset X$ and every $\epsilon > 0$ there are $\xi \in X^*$ and $a > 0$ such that the slice

$$\{x \in M : \ \langle \xi, x\rangle > \sup\langle \xi, M\rangle - a\}$$

has norm-diameter less than ϵ (see e.g. [8], or [26]). The proof proceeds as that of Theorem 5. We only need Collier's result that the dual to a dentable space is a weak* Asplund space (see [26, page 94]).

In what follows, $\mathcal{G}^1$-*smoothness* and $\mathcal{F}^1$-*smoothness* will mean Gâteaux and Fréchet smoothness. Let X be a Banach space, $x \in X$, and f a real valued function defined on a neighbourhood of x. We say that f is $\mathcal{G}^2$-*smooth* at x if f is $\mathcal{G}^1$-smooth at the points of a neighbourhood of x and there exists a bounded bilinear form $B : X \times X \to \mathbb{R}$ such that

$$\lim_{\tau \to 0} \sup_{h \in B_X} \left| \frac{1}{\tau}[f'(x+\tau k)h - f'(x)h] - B(h,k) \right| = 0$$

for all $k \in X$. Then we denote $f''(x) = B$. If the above limit is uniform with respect to $k \in B_X$, we say that f is $\mathcal{F}^2$-*smooth* at x. For $n = 3, 4, \ldots$ the $\mathcal{G}^n$- and $\mathcal{F}^n$-*smoothness* and the symbol $f^{(n)}$, the *n-th derivative* of f, are defined by induction. For $n = 1, 2, \ldots$ the $\mathcal{C}^n$-*smoothness* means the $\mathcal{F}^n$-smoothness plus that the n-th derivative is a continuous mappping from X to the Banach space of n-linear forms on X. It should be noted that our definition of G^n-smoothness coincides with the usual one, see, e.g. [24].

The following theorem is devoted to constructing $\mathcal{G}^2$-smoothness from Lipschitzness of the first derivative. A "norm" variant of it can be found in [15].

THEOREM 6. *Assume that a separable Banach space $(Z, \|\cdot\|)$ admits an $\mathcal{F}^1$-smooth bump $b : Z \to \mathbb{R}$ whose derivative is (locally) Lipschitz. Then there exists a bump $f : Z \to \mathbb{R}$ such that*

(i) *f is $\mathcal{F}^1$-smooth with (locally) Lipschitz derivative, and*
(ii) *f is $\mathcal{G}^2$-smooth.*

Proof. First assume that b' is globally Lipschitz on Z. Since b is a bump, b is also Lipschitz on Z. Let L be a Lipschitz constant of both b and b'. By enlarging L, if necessary, we may assume that $\sup\{\|b'(z)\| : z \in Z\} < L$.

Let $S = \{h_j : j \in \mathbb{N}\}$ be a countable set which is contained and dense in the unit ball of Z. Assume moreover that $S = -S$ and that $S + S \subset 2S$. Denote $Q = [-1, 1]^{\mathbb{N}}$,

$$T = \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{8}, \frac{1}{8}\right] \times \left[-\frac{1}{16}, \frac{1}{16}\right] \times \ldots,$$

and

$$K = \Big\{ \sum_{j=1}^{\infty} t_j h_j : t = (t_1, t_2, \ldots) \in T \Big\}.$$

Note that Q, T and K are compact spaces. Let $\varphi : \mathbb{R} \to [0, +\infty)$ be a C^1-smooth function, with Lipschitz derivative φ', with support in $[-\frac{1}{2}, \frac{1}{2}]$, and such that $\int_{\mathbb{R}} \varphi = 1$. For $m, i \in \mathbb{N},\ i \leq m$, and $t = (t_1, t_2, \ldots) \in Q$ we define

$$\psi_m(t) = \sum_{j=1}^{m} t_j h_j, \qquad \varphi_m(t) = \prod_{k=1}^{m} 2^k \varphi(2^k t_k),$$
$$\varphi_m^i(t) = \prod_{k \neq i}^{m} 2^k \varphi(2^k t_k) \cdot 2^{2i} \varphi'(2^i t_i).$$

Note that $\|\psi_m(t)\| \leq \sum_{j=1}^{m} 2^{-j-1} < \frac{1}{2}$ whenever $\varphi_m(t) \neq 0$. Let μ be the measure on Q obtained as the product of countably many Lebesgue measures on $[-1, 1]$.

For $m \in \mathbb{N}$ define

$$f_m(z) = \int_Q b(z - \psi_m(t)) \varphi_m(t) \mathrm{d}\mu(t), \qquad z \in Z.$$

Note that the integrand here is continuous on (the compact) Q. Hence $f_m(z)$ is well defined for every $z \in Z$. From the mean value theorem, because $\int_Q \varphi_m(t) \mathrm{d}\mu(t) = 1$, we have

$$\begin{aligned}
&\left| \frac{1}{\tau} \big[f_m(z + \tau h) - f_m(z) \big] - \int_Q b'(z - \psi_m(t)) h \varphi_m(t) \mathrm{d}\mu(t) \right| \\
&\quad \leq \int_Q \left| \Big(\frac{1}{\tau} \big[b(z + \tau h - \psi_m(t)) - b(z - \psi_m(t)) \big] - b'(z - \psi_m(t)) h \Big) \right| \varphi_m(t) \mathrm{d}\mu(t) \\
&\quad \leq L |\tau| \|h\|^2;
\end{aligned}$$

$$\left| \frac{1}{\tau} \big[f_m(z + \tau h) - f_m(z) \big] - \int_Q b'(z - \psi_m(t)) h \varphi_m(t) \mathrm{d}\mu(t) \right| \leq L |\tau| \|h\|^2; \tag{3}$$

for all $0 \neq \tau \in \mathbb{R},\ z, h \in Z$, and $m \in \mathbb{N}$. Hence f_m is $\mathcal{F}^1$-smooth on Z and

$$f_m'(z) h = \int_Q b'(z - \psi_m(t)) h \varphi_m(t) \mathrm{d}\mu(t), \qquad z, h \in Z.$$

Let q be a Lipschitz constant of φ'. The substitution $t_i - \tau \mapsto t_i$ yields

$$\begin{aligned}\Big|\frac{1}{\tau}\big[f'_m(z+\tau h_i)h - f'_m(z)h\big] - \int_Q b'(z-\psi_m(t))h\varphi^i_m(t)\mathrm{d}\mu(t)\Big| \\ = \Big|\int_Q b'(z-\psi_m(t))h\prod_{k\neq i}^{m} 2^k\varphi(2^k t_k)\Big(\frac{2^i}{\tau}\big[\varphi(2^i(t_i+\tau)) \\ - \varphi(2^i t_i)\big] - 2^{2i}\varphi'(2^i t_i)\Big)\mathrm{d}\mu(t)\Big| \\ \leq L\|h\|q2^{3i}|\tau|;\end{aligned}$$

$$\begin{aligned}\Big|\frac{1}{\tau}\big[f'_m(z+\tau h_i)h - f'_m(z)h\big] - \int_Q b'(z-\psi_m(t))h\varphi^i_m(t)\mathrm{d}\mu(t)\Big| \\ \leq L\|h\|q2^{3i}|\tau|\end{aligned} \tag{4}$$

for all $0 \neq \tau \in \mathbb{R},\ z \in Z,\ h \in Z$, and $i, m \in \mathbb{N},\ i \leq m$.

Further, we can easily estimate for $z, z' \in Z,\ h \in Z$, and $m_1, m_2 \in \mathbb{N},\ m_1 < m_2$,

$$\big|f_{m_1}(z) - f_{m_2}(z)\big| \leq L\sum_{m_1+1}^{m_2} 2^{-j-1}, \tag{5}$$

$$\begin{aligned}\Big|\int_Q b'(z-\psi_{m_1}(t))h\varphi_{m_1}(t)\mathrm{d}\mu(t) - \int_Q b'(z-\psi_{m_2}(t))h\varphi_{m_2}(t)\mathrm{d}\mu(t)\Big| \\ \leq L\|h\|\sum_{m_1+1}^{m_2} 2^{-j-1},\end{aligned} \tag{6}$$

$$\begin{aligned}\Big|\int_Q b'(z-\psi_{m_1}(t))h\varphi_{m_1}(t)\mathrm{d}\mu(t) - \int_Q b'(z'-\psi_{m_1}(t))h\varphi_{m_1}(t)\mathrm{d}\mu(t)\Big| \\ \leq L\|h\|\|z-z'\|,\end{aligned} \tag{7}$$

$$\begin{aligned}\Big|\int_Q b'(z-\psi_{m_1}(t))h\varphi^i_{m_1}(t)\mathrm{d}\mu(t) - \int_Q b'(z-\psi_{m_2}(t))h\varphi^i_{m_2}(t)\mathrm{d}\mu(t)\Big| \\ \leq L\|h\|\sum_{m_1+1}^{m_2} 2^{-j-1}\cdot 2^i\int_{\mathbb{R}}|\varphi'(s)|\mathrm{d}s.\end{aligned} \tag{8}$$

The estimates (5), (6), (8) allow us to define

$$f(z) := \lim_{m\to\infty} f_m(z), \qquad z \in Z,$$

$$g(z)(h) := \lim_{m\to\infty} \int_Q b'(z-\psi_m(t))h\varphi_m(t)\mathrm{d}\mu(t) \ (= \lim_{m\to\infty} f'_m(z)h), \quad z,h \in Z,$$

$$d(z)(h,h_i) := \lim_{m\to\infty} \int_Q b'(z-\psi_m(t))h\varphi^i_m \mathrm{d}\mu(t), \qquad z,h \in Z,\ i \in \mathbb{N}.$$

It is easy to check that f is a bump on Z. Moreover, the first limit is uniform with respect to z from Z, the second and the third limit are uniform with respect to z from Z and h from a bounded set.

Letting $m \to \infty$ in (3) then yields

$$\left|\frac{1}{\tau}[f(z+\tau h)-f(z)] - g(z)(h)\right| \le L|\tau|\|h\|^2, \qquad 0 \ne \tau \in \mathbb{R},\ z,h \in Z.$$

Clearly, $g(z)(\cdot)$ is linear and $|g(z)(h)| \le L\|h\|$. Hence f is $\mathcal{F}^1$-smooth on Z and

$$f'(z)h = g(z)(h) = \lim_{m\to\infty} f'_m(z)h, \qquad z,h \in Z.$$

Letting $m \to \infty$ in (7) then gives that f' is Lipschitz, with Lipschitz constant L. Letting $m \to \infty$ in (4) yields

$$\left|\frac{1}{\tau}[f'(z+\tau h_i)h - f'(z)h] - d(z)(h,h_i)\right| \le L2^{3i}\|h\|q|\tau|, \quad 0 \ne \tau \in \mathbb{R},\ z,h \in Z,$$

where q is the Lipschitz constant of φ'. So

$$\lim_{\tau\to 0} \frac{1}{\tau}\big[f'(z+\tau h_i)h - f'(z)h\big] = d(z)(h,h_i) \tag{9}$$

uniformly for z from Z and h from a bounded set. For $k \in B_Z$, $z \in Z$, $h \in B_Z$, and $\tau,\sigma \in \mathbb{R}\backslash\{0\}$ we have

$$\begin{aligned}&\left|\frac{1}{\tau}[f'(z+\tau k)h - f'(z)h] - \frac{1}{\sigma}[f'(z+\sigma k)h - f'(z)h]\right| \\ &\qquad \le 2L\|k-h_i\| + \left|\frac{1}{\tau}[f'(z+\tau h_i)h - f'(z)h] - \frac{1}{\sigma}[f'(z+\sigma h_i)h - f'(z)h]\right|\end{aligned}$$

for all $i \in \mathbb{N}$. Hence

$$\lim_{\tau\to 0} \frac{1}{\tau}[f'(z+\tau k)h - f'(z)h]$$

exists for every $k \in Z$ uniformly with respect to $z \in Z$ and $h \in B_Z$. Denote this limit by $d(z)(h,k)$. Moreover, as f' is Lipschitz, with Lipschitz constant

L, we have $|d(z)(h,k)| \leq L\|h\|\|k\|$ for all $z,h,k \in Z$. Clearly, $d(z)(\cdot,k)$ is linear. In order to prove that f is $\mathcal{G}^2$-smooth, it remains to show the linearity of $d(z)(h,\cdot)$. Trivially, $d(z)(h,\lambda k) = \lambda d(z)(h,k)$ for all $\lambda \in \mathbb{R}$. Hence it remains to verify that

$$d(z)(h,u+v) = d(z)(h,u) + d(z)(h,v) \qquad \text{for all } u,v \in B_Z. \tag{10}$$

Second, assume first that $u = h_i,\ v = h_j$ for some $i,j \in \mathbb{N}$. The substitutions $t_i - \tau \mapsto t_i,\ t_j - \tau \mapsto t_j$ give for $m \geq \max\{i,j\}$

$$\begin{aligned}
&\frac{1}{\tau}\big[f_m'(z+\tau(h_i+h_j))h - f_m'(z)h\big] \\
&\quad = \int_Q b'(z-\psi_m(t))h \times \prod_{k\neq i,j}^{m} 2^k\varphi(2^k t_k)\frac{1}{\tau}\big[2^i\varphi(2^i(t_i+\tau))2^j\varphi(2^j(t_j+\tau)) \\
&\qquad - 2^i\varphi(2^i t_i)2^j\varphi(2^j t_j)\big]\,\mathrm{d}\mu(t) \\
&\quad \to \int_Q b'(z-\psi_m(t))h\varphi_m^i(t)\mathrm{d}\mu(t) + \int_Q b'(z-\psi_m(t))h\varphi_m^j(t)\mathrm{d}\mu(t) \qquad \text{as } \tau \to 0
\end{aligned}$$

uniformly with respect to $m \in \mathbb{N}$. Hence, letting $m \to \infty$, we get

$$\Big|\frac{1}{\tau}\big[f'(z+\tau(h_i+h_j))h - f'(z)h\big] - d(z)(h,h_i) - d(z)(h,h_j)\Big| \to 0 \qquad \text{as } \tau \to 0.$$

Thus

$$d(z)(h,h_i+h_j) = d(z)(h,h_i) + d(z)(h,h_j). \tag{11}$$

Now, as f' is Lipschitz, $|d(z)(h,k_1) - d(z)(h,k_2)| \leq L\|h\|\|k_1 - k_2\|$ for all $k_1,k_2 \in B_Z$. Thus, (11) implies (10). Therefore f is $\mathcal{G}^2$-smooth on Z and $f''(z)(h,k) = d(z)(h,k)$ for every $z,h,k \in Z$.

Second, assume that b' is locally Lipschitz. As the set K is compact, for every $z \in Z$ there is an open set $z \in U \subset Z$ such that the mapping b' is Lipschitz on the set $U - K$. Using this fact, we can easily check that the whole argument above, with minor changes, works, and so, the resulting bump f has locally Lipschitz derivative and is $\mathcal{G}^2$-smooth. ∎

Remarks. 1. The second derivative $f''(z)(\cdot,\cdot)$ *is in fact symmetric*. Indeed, an integration by parts yields that $\int_Q b'(z-\psi_m(t))h_i\varphi_m^j(t)\mathrm{d}\mu(t) = \int_Q b'(z-\psi_m(t))h_j\varphi_m^i(t)\mathrm{d}\mu(t)$ and so $f''(z)(h_i,h_j) = f''(z)(h_j,h_i)$. whenever $i,j \in \mathbb{N}$. Hence $f''(z)(h,k) = f''(k,h)$ for all $h,k \in Z$.

2. In the above argument, we proved that f *is in fact (locally) uniformly* $\mathcal{G}^2$*-smooth on* Z, that is, $\lim_{\tau\to 0}\sup_{h\in B_Z}\left|\frac{1}{\tau}[f'(z+\tau k)h - f'(z)h] - f''(z)(h,k)\right| = 0$ *(locally) uniformly on* Z *for every* $k\in Z$.

3. The above argument works if the bump b is replaced by a smooth equivalent norm $\|\cdot\|$ such that $\|\cdot\|'$ is (locally) Lipschitz on $Z\backslash B_Z$. The function f is then convex nonnegative, and coercive (i.e., $f(x)\to +\infty$ if $x\in X$ and $\|x\|\to+\infty$). So, an appropriate implicit function theorem and some extra effort yield an equivalent $\mathcal{F}^1$-smooth norm, with (locally) Lipschitz derivative, and which is (locally) uniformly $\mathcal{G}^2$-smooth, see [15] and [1].

3. Spaces Isomorphic to a Hilbert Space

For motivation, we will start with a simple proof of the following result.

Theorem 7. *Assume that a Banach space* $(X,\|\cdot\|)$ *is such that the norm* $\|\cdot\|$ *and its dual norm are both* $\mathcal{F}^2$*-smooth on* $X\backslash\{0\}$ *and* $X^*\backslash\{0\}$*, respectively. Then* X *is isomorphic to a Hilbert space.*

Before starting on the proof we remark that easy finite dimensional examples show that the norm of a Banach space X together with its dual norm may be twice differentiable and X itself be a non Hilbert space.

Proof. Put $f=\frac{1}{2}\|\cdot\|^2$. We can easily see that $f^*=\frac{1}{2}\|\cdot\|^2$ where $\|\cdot\|$ means the norm on X^* dual to $\|\cdot\|$. Clearly, f and f^* are $\mathcal{F}^2$-smooth on $X\backslash\{0\}$ and $X^*\backslash\{0\}$, respectively. Take any $0\neq x_0\in X$ and denote $x_0^*=f'(x_0)$. By Lemma 1, we have that $x_0=(f^*)'(x_0^*)$. The Mean value theorem yields $c>0,\ \delta>0$ such that

$$f^*(x_0^*+h^*)-f^*(x_0^*)-\langle h^*,x_0\rangle\le c\|h^*\|^2 \qquad \text{whenever } h^*\in X^*,\ \|h^*\|<\delta.$$

Then, using the notation introduced prior to Lemma 2,

$$\alpha(f^*,x_0^*,x_0,t)\le ct^2 \qquad \text{if } 0\le t<\delta.$$

Hence, by Lemma 2,

$$\beta(f^{**},x_0,x_0^*,s)\ge\frac{1}{4c}s^2 \qquad \text{if } 0\le s<2c\delta.$$

By Lemma 3, $f^{**}(x_0)=f(x_0)$. Thus

$$\beta(f,x_0,x_0^*,s)\ge\frac{1}{4c}s^2 \qquad \text{if } 0\le s<2c\delta.$$

This means that

$$f(x_0+h)-f(x_0)-\langle x_0^*,h\rangle \geq \frac{1}{4c}\|h\|^2 \qquad \text{whenever } h\in X,\ \|h\|<2c\delta.$$

Using l'Hospital's rule, we get

$$f''(x_0)(h,h)\geq \frac{1}{2c}\|h\|^2 \qquad \text{for all } h\in X.$$

As $f''(x_0)$ is a bounded bilinear form,

$$h\mapsto \big[f''(x_0)(h,h)\big]^{1/2}, \qquad h\in X,$$

is an equivalent Hilbertian norm on X. ∎

Remarks. 1. We note that X is isomorphic to a Hilbert space if $\|\cdot\|^2$ is $\mathcal{G}^2$-smooth at 0 (see e.g. [8, page 184]).

2. From the above argument it follows that it was enough to assume in Theorem 7 that $\|\cdot\|$ is $\mathcal{G}^2$-smooth on $X\backslash\{0\}$ and the dual norm $\|\cdot\|$ is $\mathcal{F}^1$-smooth on $X^*\backslash\{0\}$ with pointwise Lipschitz derivative (see [11]).

3. Theorem 7 with $\mathcal{F}^2$ replaced by $\mathcal{C}^2$ was proved in [4], [27] and [30].

If a norm on a Banach space X is $\mathcal{F}^2$-smooth away from the origin and another equivalent norm on X^* is $\mathcal{F}^2$-smooth away from the origin, then the above argument, of course, does not work. Yet such an X is still isomorphic to a Hilbert space. Actually, the norms can be replaced by bumps: (Note that the existence of a norm with some smoothness implies the existence of a bump with the same smoothness; see the proof of Corollary 14.)

More precisely, we have the following result.

THEOREM 8. *Assume that a Banach space X as well as its dual X^* admit a $\mathcal{F}^2$-smooth bump. Then X is isomorphic to a Hilbert space.*

Proof. First of all, note that X is an Asplund space (see e.g. [18, Corollary 369]). Let $f:X\to\mathbb{R}$, $g:X^*\to\mathbb{R}$ be $\mathcal{F}^2$-smooth bumps. Assume that $f(0)\neq 0$ and put $\psi=f^{-2}$ (i.e., $\psi(x)=+\infty$ if $f(x)=0$). Then we can easily verify that the conjugate function ψ^* is convex, locally bounded, and hence continuous on all of X^*, see [26, Proposition 1.6]. Apply Theorem 5 to $Z:=X$ and $\varphi:=-\psi^*+g^{-2}$. We get $x_0^*\in X^*$ such that for all $h^*\in X^*$

$$-\psi^*(x_0^*+h^*)+g^{-2}(x_0^*+h^*)-\psi^*(x_0^*-h^*)+g^{-2}(x_0^*-h^*)+2\psi^*(x_0^*)-2g^{-2}(x_0^*)\geq 0.$$

Hence $g(x_0^*) \neq 0$ and for all $h \in X^*$

$$\psi^*(x_0^*+h^*)+\psi^*(x_0^*-h^*)-2\psi^*(x_0^*) \le g^{-2}(x_0^*+h^*)+g^{-2}(x_0^*-h^*)-2g^{-2}(x_0^*).$$

We can easily see that g^{-2} is $\mathcal{F}^2$-smooth at x_0^*. Thus, there are $c > 0$ and $\delta > 0$ such that

$$g^{-2}(x_0^*+h^*)+g^{-2}(x_0^*-h^*)-2g^{-2}(x_0^*) \le c\|h^*\|^2 \qquad \text{for all } h^* \in X^*,\ \|h^*\| < \delta.$$

Hence

$$\psi^*(x_0^*+h^*)+\psi^*(x_0^*-h^*)-2\psi^*(x_0^*) \le c\|h^*\|^2 \qquad \text{for all } h^* \in X^*,\ \|h^*\| < \delta.$$

As ψ^* is convex and continuous, the above inequality implies that ψ^* is Fréchet smooth at x_0^*. Denote $x_0 = \psi^{*\prime}(x_0^*)$. Thus $\alpha(\psi^*, x_0^*, x_0, t) \le ct^2$ for $0 < t < \delta$. (We use the notation introduced prior to Lemma 2.) Using Lemma 3, we get that x_0 belongs to X, and Lemma 2 gives that

$$ct^2 + \beta(\psi^{**}, x_0, x_0^*, s) \ge ts \qquad \text{for all } 0 < t < \delta.$$

Hence, using Lemma 3,

$$\beta(\psi, x_0, x_0^*, s) \ge \frac{1}{4c}s^2 \qquad \text{for all } 0 < s < 2c\delta.$$

Thus

$$\psi(x_0+h) - \psi(x_0) - \langle x_0^*, h\rangle \ge \frac{1}{4c}\|h\|^2 \qquad \text{whenever } h \in X \text{ and } \|h\| < 2c\delta.$$

Observe that ψ is $\mathcal{F}^2$-smooth at x_0 since f is such and $\psi = f^{-2}$. Hence, l'Hospital's rule used twice gives

$$\psi''(x_0)(h,h) \ge \frac{1}{2c}\|h\|^2 \qquad \text{for all } h \in X.$$

As $\psi''(x_0)$ is a bounded bilinear form on X, we can conclude that

$$h \mapsto \left[\psi''(x_0)(h,h)\right]^{1/2}, \qquad h \in X$$

is an equivalent Hilbertian norm on X. ∎

Remarks. 1. The above argument implies that it is enough to assume in Theorem 8 that X has a continuous and $\mathcal{G}^2$-smooth bump and X^* has an $\mathcal{F}^1$-smooth bump with pointwise Lipschitz derivative at every point.

2. Theorem 8 with $\mathcal{F}^2$ replaced by $\mathcal{C}^2$ was proved by Meškov [25].

The result below easily follows from Kwapień's theorem [20]. Here we present an elementary proof of it due to Lindenstrauss [22].

THEOREM 9. *Assume that every separable subspace of a Banach space $(X, \|\cdot\|)$ is isomorphic to a Hilbert space. Then X is isomorphic to a Hilbert space.*

Proof. For a separable subspace Y of X, let c_Y denote the supremum of all $c > 0$ such that there exists a bilinear form $Q : Y \times Y \to \mathbb{R}$ satisfying

$$c\|y\|^2 \leq Q(y, y) \leq \|y\|^2 \qquad \text{for all } y \in Y.$$

We observe that

$$c_0 := \inf\{c_Y : \ Y \text{ is a separable subspace of } X\}$$

is a positive number. Indeed, assume, by contrary, that for every $n \in \mathbb{N}$ there is a separable subspace Y_n of X such that $c_{Y_n} < \frac{1}{n}$. Let Y be the closure of $\bigcup_{n=1}^{\infty} Y_n$; this will again be a separable subspace. Then

$$c_Y \leq c_{Y_n} < \frac{1}{n} \qquad \text{for all } n \in \mathbb{N},$$

a contradiction with $c_Y > 0$.

For every separable subspace Y of X we find a bilinear form $Q_Y : Y \times Y \to \mathbb{R}$ such that

$$\frac{1}{2}c_0\|y\|^2 \leq Q_Y(y, y) \leq \|y\|^2 \qquad \text{for all } y \in Y$$

and define then $\widetilde{Q}_Y : X \times X \to \mathbb{R}$ by

$$\widetilde{Q}_Y(x_1, x_2) = \begin{cases} Q_Y(x_1, x_2) & \text{if } (x_1, x_2) \in Y \times Y, \\ 0 & \text{if } (x_1, x_2) \in (X \times X)\backslash(Y \times Y). \end{cases}$$

Let $\mathcal{Y}$ denote the family of all separable subspaces of X and endow it by the relation "$\subset$". Then $(\mathcal{Y}, \subset)$ is a directed set. Consider the net $\big(\widetilde{Q}_Y(\cdot,\cdot)_{|B_X \times B_X} : Y \in \mathcal{Y}\big)$. Since it lies in the compact space $[-1, 1]^{B_X \times B_X}$, it has a convergent subnet, say $\big(\widetilde{Q}_{Y_\alpha}(\cdot,\cdot)_{|B_X \times B_X}\big)$. Using the bilinearity of Q_{Y_α}, we get that

$$Q(x_1, x_2) := \lim_{\alpha} \widetilde{Q}_{Y_\alpha}(x_1, x_2)$$

exists for all $x_1, x_2 \in X$. Clearly, Q is a bilinear form on all of X and

$$\frac{1}{2}c_0\|x\|^2 \le Q(x,x) \le \|x\|^2 \qquad \text{for all } x \in X.$$

Therefore X is isomorphic to a Hilbert space. ∎

THEOREM 10. ([11]) *Assume that a Banach space X admits an $\mathcal{F}^1$-smooth bump with locally Lipschitz derivative and its dual X^* admits an $\mathcal{F}^1$-smooth bump with pointwise Lipschitz derivative. Then X is isomorphic to a Hilbert space.*

Proof. Let Y be a separable subspace of X. According to Theorem 9, it is enough to show that Y is isomorphic to a Hilbert space. By Theorem 6, we can construct a continuous bump $f : Y \to \mathbb{R}$ which is $\mathcal{G}^2$-smooth. By a shift, we may assume that $f(0) \neq 0$. Let $g : X^* \to \mathbb{R}$ be an $\mathcal{F}^1$-smooth bump such that its derivative is pointwise Lipschitz. Define $\psi : X \to (-\infty, +\infty]$ by

$$\psi(x) = \begin{cases} f^{-2} & \text{if } x \in Y, \\ +\infty & \text{if } x \in X\backslash Y. \end{cases}$$

Further we copy the proof of Theorem 8 (taking into account Remark following Theorem 5) until we obtain

$$\psi(x_0+h) - \psi(x_0) - \langle x_0^*, h\rangle \ge \frac{1}{4c}\|h\|^2 \qquad \text{whenever } h \in X \text{ and } \|h\| < 2c\delta.$$

Using the definition of ψ, we get that $x_0 \in Y$ and

$$f^{-2}(x_0+h) - f^{-2}(x_0) - \langle x_0^*, h\rangle \ge \frac{1}{4c}\|h\|^2 \qquad \text{whenever } h \in Y \text{ and } \|h\| < 2c\delta.$$

It is easy to check that f^{-2} is $\mathcal{G}^2$ smooth at x_0. Then, applying l'Hospital's rule twice, we get that

$$(f^{-2})''(x_0)(h,h) \ge \frac{1}{2c}\|h\|^2 \qquad \text{for all } h \in Y.$$

As $(f^{-2})''(x_0)$ is a bounded bilinear form on X, we can conclude that

$$h \mapsto \left[(f^{-2})''(x_0)(h,h)\right]^{1/2}, \qquad h \in Y,$$

is an equivalent Hilbertian norm on Y. ∎

Remarks. 1. In Theorem 10, we can replace the word "bump" by "norm". Indeed, there is a standard procedure of constructing a bump sharing the same smoothness as the given norm has; see the proof of Corollary 14.

2. With more effort, we can obtain that X is isomorphic to a Hilbert space if X and X^* both admit $\mathcal{F}^1$-smooth bumps (equivalent norms) with pointwise Lipschitz derivatives [7]. Extra tools needed in the proof of this statement are the Baire category theorem and a Day's technique of constructing uniformly rotund norms.

3. A further elaboration of 2. (see [24]) plus anderwerff's result that continuous $\mathcal{G}^2$-smooth bumps imply Asplundness [33] yields: X is isomorphic to a Hilbert space if X and X^* both admit a continuous $\mathcal{G}^2$-smooth bump (equivalent norm). Note that the "norm" variant of this statement follows directly, according to [3, Proposition 2.2], from 2.

4. A first result in the flavour of Theorem 10 was established in [22]. A consequence of Kadec-Lindenstrauss study of moduli of convexity and smoothness, and convergence of series yields: If a Banach space X admits an unconditional basis and an equivalent norm such that this norm and the dual norm are smooth and have Lipschitz derivatives on unit spheres, then X is isomorphic to a Hilbert space.

4. Higher Order Smoothness in ℓ_p Spaces

We say that $P : X \to \mathbb{R}$ is a *polynomial* of degree $n \in \mathbb{N}$ if there is $a \in \mathbb{R}$ and for $i = 1, 2, \dots, n$ there are continuous i-linear forms $B_i : X^i \to \mathbb{R}$ such that

$$P(x) = a + B_1(x) + B_2(x,x) + \cdots + B_n(x, \dots, x) \qquad \text{for all } x \in X.$$

Let $1 \leq p < +\infty$ be a real number. Let $\varphi : X \to (-\infty, +\infty]$ be a function and $x \in X$, with $\varphi(x) < +\infty$. We say that φ is *$\mathcal{T}^p$-smooth* at x if there is a polynomial $P : X \to \mathbb{R}$ of degree at most p such that $P(0) = 0$ and

$$\varphi(x+h) = \varphi(x) + P(h) + o(\|h\|^p) \qquad \text{for all } h \in X.$$

If $\Omega \subset X$ is an open set, we say that φ is $\mathcal{T}^p$-smooth on Ω if it is $\mathcal{T}^p$-smooth at every $x \in \Omega$.

Lemma 11. *Let $\varphi : X \to (-\infty, +\infty]$ be a function and $x \in \operatorname{dom}\varphi$. Let $\alpha : \mathbb{R} \to (\infty, +\infty]$ be a function C^∞-smooth at $\varphi(x)$.*

(i) ([5, Chapitre 1, Théorème 5.6.3]) *If $p \in \mathbb{N}$ and if φ is $\mathcal{F}^p$-smooth at x, then φ is $\mathcal{T}^p$-smooth at x.*

(ii) *If $1 < p < +\infty$ and φ is $\mathcal{T}^p$-smooth at x, then $\alpha \circ \varphi$ is $\mathcal{T}^p$-smooth at x.*

(iii) *If φ is continuous and $\mathcal{G}^2$-smooth at x, then $\alpha \circ \varphi$ is $\mathcal{G}^2$-smooth at x.*

Proof. (i) For $h \in X$ denote

$$P(h) = \varphi'(x)(h) + \frac{1}{2}\varphi''(x)(h,h) + \cdots + \frac{1}{p!}\varphi^{(p)}(x)(h,\ldots,h).$$

Applying l'Hospital's rule $p-1$ times, we get

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{1}{p\cdots 2t}\Big(\varphi^{(p-1)}(x+th)(h,\ldots,h) - \varphi^{(p-1)}(x)(h,\ldots,h) \\ &\quad - \varphi^{(p)}(x)(h,\ldots,h,th)\Big) \\ &= \lim_{t\to 0} \frac{\mathrm{d}^{p-1}}{\mathrm{d}t^{p-1}}\big(\varphi(x+th) - P(th)\big)\Big/\frac{\mathrm{d}^{p-1}}{\mathrm{d}t^{p-1}}\big(t^p\big) \ = \ \cdots \\ &= \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(x+th) - P(th)\big)\Big/\frac{\mathrm{d}}{\mathrm{d}t}\big(t^p\big) \\ &= \lim_{t\to 0} \frac{1}{t^p}\big(\varphi(x+th) - \varphi(x) - P(th)\big). \end{aligned}$$

Noting that all the limits above are uniform with respect to $h \in B_X$, we can conclude that φ is $\mathcal{T}^p$-smooth at x.

(ii) Write $\varphi(x+h) = \varphi(x) + P(h) + o(\|h\|^p)$, $h \in X$, where P is a polynomial of degree at most p, with $P(0) = 0$. Find $n \in \mathbb{N}$ such that $n > p$. Then

$$\begin{aligned} \alpha(\varphi(x+h)) &= \alpha(\varphi(x)) + \alpha'(\varphi(x))(P(h) + o(\|h\|^p)) + \cdots \\ &\qquad + \frac{1}{n!}\alpha^{(n)}(\varphi(x))\big(P(h) + o(\|h\|^p)\big)^n + o\big(|P(h) + o(\|h\|^p)|^n\big). \end{aligned}$$

An inspection of this formula gives that there exists a polynomial Q of degree at most p such that

$$\alpha(\varphi(x+h)) = \alpha(\varphi(x)) + Q(h) + o(\|h\|^p) \qquad \text{for all } h \in X.$$

(iii) Assume that φ is continuous and $\mathcal{G}^2$-smooth at x. We can easily check that $\alpha \circ \varphi$ is then Gâteaux smooth at x, with $(\alpha\circ\varphi)'(x) = \alpha'(\varphi(x))\varphi'(x)$.

Also, for $h, k \in X$ and $0 \neq \tau \to 0$ we have

$$\begin{aligned}\frac{1}{\tau}&\big[\alpha'(\varphi(x+\tau k))\varphi'(x+\tau k)(h) - \alpha'(\varphi(x))\varphi'(x)(h)\big]\\ &= \frac{1}{\tau}\alpha'(\varphi(x+\tau k))\big[\varphi'(x+\tau k)(h) - \varphi'(x)(h)\big]\\ &\quad + \frac{1}{\tau}\big[\alpha'(\varphi(x+\tau k)) - \alpha'(\varphi(x))\big]\varphi'(x)(h)\\ &\to \alpha'(\varphi(x))\varphi''(x)(h,k) + \alpha''(\varphi(x))\varphi'(x)(k)\varphi'(x)(h).\end{aligned}$$

Note that this convergence is uniform with respect to $h \in B_X$. Therefore $\alpha \circ \varphi$ is $\mathcal{G}^2$-smooth at x. ∎

Let $1 \leq p < +\infty$ and Γ be an infinite set. The symbol $\ell_p(\Gamma)$ means the (Banach) space of all $x = (x_\gamma)_{\gamma\in\Gamma} \in \mathbb{R}^\Gamma$ such the norm $\|x\|_p := \big(\sum_{\gamma\in\Gamma} |x_\gamma|^p\big)^{1/p}$ is finite. For $\gamma \in \Gamma$ let e_γ denote the vector having 1 at the γ's position and 0 at all the other positions. If $\Gamma = \mathbb{N}$ we write just ℓ_p instead of $\ell_p(\Gamma)$.

LEMMA 12. *Let $1 < p < +\infty$.*

(i) [2, Lemma 1] *If $P : \ell_p \to \mathbb{R}$ is a polynomial of degree less than p, with $P(0) = 0$, then*

$$\lim_{i\to\infty} P(e_i) = 0.$$

(ii) *If Γ is an uncountable set and $P : \ell_p(\Gamma) \to \mathbb{R}$ is a polynomial of degree less than p, with $P(0) = 0$, then $P(e_\gamma) = 0$ for all but countably many $\gamma \in \Gamma$.*

Proof. (i) Since $e_i \to 0$ weakly, $P(e_i) \to 0$ for every polynomial of degree 1, with $P(0) = 0$. Consider $p > 1$. Take an integer $1 < n < p$, and assume we have already verified this statement for all polynomials of degree less than n. Let P be a polynomial of degree n, with $P(0) = 0$. By contradiction, assume that $\limsup_{i\to\infty} |P(e_i)| =: 3a > 0$. Without loss of generality we may assume that $\limsup_{i\to\infty} P(e_i) = 3a$. We observe that

$$P(x+h) = P(x) + B(x,h) + P(h), \qquad \text{for all } x, h \in \ell_p,$$

where $B(x,\cdot)$ is a polynomial of degree less than n. We shall construct a sequence (x_i) of elements in ℓ_p, with finite support, as follows. Put $x_1 = e_1$. If x_i has been constructed, find $j \in \mathbb{N}$, not belonging to the support of x_i, so that $P(e_j) > 2a$ and $B(x_i, e_j) > -a$; it exists according to the induction assumption. Put $x_{i+1} = x_i + e_j$. Then

$$P(x_{i+1}) = P(x_i) + B(x_i, e_j) + P(e_j) > P(x_i) + a > \cdots > P(e_1) + ia.$$

Note that $\|x_i\|_p = i^{1/p}$ for all $i \in \mathbb{N}$. Therefore

$$\frac{P(x_i)}{\|x_i\|_p{}^n} > \frac{P(e_1) + (i-1)a}{i^{n/p}} = \frac{P(e_1) - a}{i^{n/p}} + i^{1-n/p}a \to +\infty \qquad \text{as } i \to \infty,$$

contradicting the fact that P is of degree n.

(ii) Assume this is not true. Then there surely exists $k \in \mathbb{N}$ such that the set $N := \{\gamma \in \Gamma : \; |P(e_\gamma)| > 1/k\}$ is infinite. Let $\{\gamma_n : \; n \in \mathbb{N}\}$ be a countable subset in N and let $T : \ell_p \to \ell_p(\Gamma)$ be the canonical embeding sending $(x_n)_{n\in\mathbb{N}}$ to $(y_\gamma)_{\gamma\in\Gamma}$, where $y_\gamma = x_n$ if $\gamma = \gamma_n$, $n \in \mathbb{N}$, and $y_\gamma = 0$ if $\gamma \in \Gamma \backslash \{\gamma_n : \; n \in \mathbb{N}\}$. Then $P \circ T$ is a polynomial on ℓ_p, with $|P \circ T(e_n)| > 1/k$ for all $n \in \mathbb{N}$, which contradicts (i). ∎

Theorem 13. (i) *If* $1 \le p < \infty$ *and* $\frac{p}{2} \notin \mathbb{N}$, *then the space* ℓ_p *does not admit any* $\mathcal{T}^p$*-smooth bump.*

(ii) *If* $1 \le p < 2$, *the space* ℓ_p *does not admit any continuous* $\mathcal{G}^2$*-smooth bump.*

(iii) *If* $p \in \mathbb{N}$ *and* $\frac{p}{2} \notin \mathbb{N}$, *and* Γ *is an uncountable set, then the space* $\ell_p(\Gamma)$ *does not admit any continuous* $\mathcal{G}^p$*-smooth bump.*

Proof. Let $b : \ell_p \to \mathbb{R}$ be a continuous bump. Apply Stegall's variational principle (Theorem 5) to $\varphi := b^{-2} - \|\cdot\|_p{}^p$ and to $Z := c_0$ if $p = 1$, and to $Z := \ell_q$, $q = \frac{p}{p-1}$, if $p > 1$. We get $x \in \ell_p$ such that

$$b^{-2}(x+h) + b^{-2}(x-h) - 2b^{-2}(x) \ge \|x+h\|_p{}^p + \|x-h\|_p{}^p - 2\|x\|_p{}^p \tag{12}$$

for all $h \in \ell_p$. Then $b(x) \neq 0$.

(i) Let $1 \le p < \infty$ and assume b is $\mathcal{T}^p$-smooth. Lemma 11 (ii) guarantees that b^{-2} is $\mathcal{T}^p$-smooth at x. Then (12) yields

$$P(h) + o(\|h\|_p{}^p) \ge \|x+h\|_p{}^p + \|x-h\|_p{}^p - 2\|x\|_p{}^p \qquad \text{for all } h \in \ell_p \tag{13}$$

where P is a polynomial of even degree, say n, with $n < p$ (as $p/2 \notin \mathbb{N}$) and such that $P(0) = 0$. (If Q is a polynomial of odd degree, then

$h \mapsto Q(h) + Q(-h)$ is a polynomial of degree 1 less.) Fix $t > 0$ and take in (13) $h = te_i$, $i = 1, 2, \dots$. Then, by Lemma 12 (ii),

$$\begin{aligned} o(t^p) &= \lim_{i\to\infty} \big(P(te_i) + o(\|te_i\|_p{}^p)\big) \\ &\geq \lim_{i\to\infty} \big(\|x + te_i\|_p{}^p + \|x - te_i\|_p{}^p - 2\|x\|_p{}^p\big) = 2t^p, \end{aligned}$$

which is impossible.

(ii) Let $1 \leq p < 2$ and assume that b is $\mathcal{G}^2$-smooth. Lemma 11 (iii) guarantees that b^{-2} is $\mathcal{G}^2$-smooth at x. L'Hospital rule used twice yields that

$$\lim_{t\to 0} \frac{1}{t^2}\Big(b^{-2}(x+th)+b^{-2}(x-th)-2b^{-2}(x)\Big) = \big(b^{-2}\big)''(x)(h,h) \qquad \text{for all } h \in \ell_p.$$

Hence, from (12) we have

$$\limsup_{t\to 0} \frac{1}{t^2}\Big(\|x + th\|_p{}^p + \|x - th\|_p{}^p - 2\|x\|_p{}^p\Big) \leq \big(b^{-2}\big)''(x)(h,h) \tag{14}$$

for all $h \in \ell_p$. (Note that this lim sup is not uniform with repsect to $h \in B_{\ell_p}$.) As in the proof of [3, Proposition 2.2], put

$$\varphi(h) = \sup_{0<t\leq 1} \frac{1}{t^2}\Big(\|x + th\|_p{}^p + \|x - th\|_p{}^p - 2\|x\|_p{}^p\Big), \qquad h \in \ell_p.$$

From (14) we can see that $\varphi(h) < +\infty$ for every $h \in \ell_p$ and that $\varphi : \ell_p \to \mathbb{R}$ is a lower semicontinuous, symmetric, and convex function. A Baire category argument easily yields a nonempty open set $U \subset \ell_p$ and $c > 0$ so that $\sup \varphi(U) < c$. From the symmetry and convexity of φ we then conclude that φ is bounded by c on a neighbourhood of 0, say $\varphi(h) \leq c$ whenever $h \in \ell_p$ and $\|h\|_p \leq \delta$, for some $\delta > 0$. Thus, in particular,

$$\|x+t\delta e_i\|_p{}^p+\|x-t\delta e_i\|_p{}^p-2\|x\|_p{}^p \leq t^2 c \qquad \text{for all } i \in \mathbb{N} \text{ and for all } t \in [0,1].$$

Hence, letting $i \to \infty$ here, we get $2t^p\delta^p \leq t^2 c$ for all $t \in [0,1]$, which is impossible for $1 \leq p < 2$.

(iii) Let $p \in \mathbb{N}$ be odd and assume that b is a continuous $\mathcal{G}^p$-smooth bump on $\ell_p(\Gamma)$. Applying Stegall's variational principle as above, we get $x \in \ell_p(\Gamma)$ such that $b(x) \neq 0$ and (12) holds for all $h \in \ell_p(\Gamma)$. Since b is $\mathcal{G}^p$-smooth, an argument used in the proof of Lemma 11 (i) yields a polynomial $Q : \ell_p(\Gamma) \to \mathbb{R}$ of degree at most p, with $Q(0) = 0$, such that for every $h \in \ell_p(\Gamma)$

$$\frac{1}{|t|^p}\big(b(x + th) - b(x) - Q(th)\big) \to 0 \qquad \text{as } t \to 0.$$

Then, as in the proof of Lemma 11 (ii), we can find a polynomial $P : \ell_p(\Gamma) \to \mathbb{R}$ of degree at most p, with $P(0) = 0$, such that for every $h \in \ell_p(\Gamma)$

$$\frac{1}{|t|^p}\big(b^{-2}(x+th) - b^{-2}(x) - P(th)\big) \to 0 \qquad \text{as } t \to 0.$$

Then, putting $R(h) = P(h) + P(-h)$, $h \in \ell_p(\Gamma)$, R will be a polynomial of degree less than p, and for every $h \in \ell_p(\Gamma)$ there is a function ω_h, with $\lim_{t\to 0} \omega_h(t)/|t|^p = 0$, such that

$$b^{-2}(x+th) + b^{-2}(x-th) - 2b^{-2}(x) = R(th) + \omega_h(t), \qquad t \in \mathbb{R}.$$

Now, realizing that the support of x is at most countable, and using Lemma 12 (ii) (countably many times), we find $\gamma \in \Gamma$ so that $x_\gamma = 0$ and $R(\frac{1}{n}e_\gamma) = 0$ for all $n \in \mathbb{N}$. Taking then $h = \frac{1}{n}e_\gamma$ in (12), we get

$$\omega_{e_\gamma}\big(\frac{1}{n}\big) = R\big(\frac{1}{n}e_\gamma\big) + \omega_{e_\gamma}\big(\frac{1}{n}\big) \geq \big\|x + \frac{1}{n}e_\gamma\big\|_p^{\ p} + \big\|x - \frac{1}{n}e_\gamma\big\|_p^{\ p} - 2\|x\|_p^{\ p} = \frac{2}{n^p},$$

which is impossible for $n \to \infty$. ∎

Remarks. 1. Theorem 13 (i) is proved in [9]. A slightly weaker form of it goes back to Kurzweil [19].

2. Once we know that ℓ_p, $1 \leq p < 2$, is not isomorphic to a Hilbert space, then 13 (ii) also follows from [8, Theorem V.1.1] and from Remark 1 after Theorem 8.

3. Theorem 13 (iii) was proved in [23] using farthest points.

4. We in fact proved that the space $\ell_p(\Gamma)$, with real $1 \leq p < \infty$ and uncountable Γ, does not admit any continuous bump with directional Taylor expansion of degree p.

COROLLARY 14. (i) *If $1 \leq p < \infty$ and $\frac{p}{2} \notin \mathbb{N}$, then the space ℓ_p does not admit any equivalent norm which is $\mathcal{T}^p$-smooth on $\ell_p\backslash\{0\}$.*

(ii) *If $1 \leq p < 2$, the space ℓ_p does not admit any equivalent norm which is $\mathcal{G}^2$-smooth on $\ell_p\backslash\{0\}$.*

(iii) *If $p \in \mathbb{N}$ and $\frac{p}{2} \notin \mathbb{N}$, and Γ is an uncountable set, then the space $\ell_p(\Gamma)$ does not admit any equivalent norm which is $\mathcal{G}^p$-smooth on $\ell_p(\Gamma)\backslash\{0\}$.*

Proof. Assume that such a norm, say $\|\cdot\|$, exists. Take a $\mathcal{C}^\infty$-smooth function $\alpha : \mathbb{R} \to [0,1]$ such that $\alpha(1) = 1$ and $\alpha(t) = 0$ whenever $t \leq \frac{1}{2}$ or $t \geq 2$. Then $\alpha \circ \|\cdot\|$ is a bump and, according to Lemma 11, has the same smoothness as $\|\cdot\|$ had. Now Theorem 13 finishes the proof. ∎

Remarks. 1. Corollary 14 (ii) was proved in [15]; Corollary 14 (iii) was proved in [32].

2. There is a more direct proof of Corollary 14: Applying Stegall's principle (Theorem 5) to $\varphi = \|\cdot\|^{p+1} - \|\cdot\|_p{}^p$, we get $x \in \ell_p$ so that

$$\|x+h\|^{p+1} + \|x-h\|^{p+1} - 2\|x\|^{p+1} \geq \|x+h\|_p{}^p + \|x-h\|_p{}^p - 2\|x\|_p{}^p$$

for all $h \in X$. Then, applying Lemma 11 (ii) - (iii), we further proceed as in the proof of Theorem 13.

COROLLARY 15. *The space ℓ_p for $p \in \mathbb{N}$, $p/2 \notin \mathbb{N}$, does not admit any $\mathcal{F}^p$-smooth bump or equivalent norm.*

Proof. Apply Lemma 11 (i) together with Theorem 13 (i) and Corollary 14 (i). ∎

A function $\varphi : X \to \mathbb{R}$ is called *pointwise Lipschitz smooth* on an open set $\Omega \subset X$ if it is $\mathcal{F}^1$-smooth on Ω and for every $x \in \Omega$ there are $c > 0$, $\delta > 0$ such that $\|\varphi'(x+h) - \varphi'(x)\| \leq c\|h\|$ whenever $h \in X$ and $\|h\| < \delta$.

COROLLARY 16. *For $1 \leq p < 2$ the space ℓ_p does not admit any pointwise Lipschitz smooth bump or equivalent norm.*

Proof. Observe that this type of smoothness implies the $\mathcal{T}^p$-smoothness for $1 \leq p < 2$. Thus, Corollary 14 (i) applies. ∎

COROLLARY 17. *If $1 \leq p \leq +\infty$ and $p \neq 2$, then the space ℓ_p is not isomorphic to a Hilbert space.*

Proof. Assume $1 \leq p < 2$. If ℓ_p were isomorphic to a Hilbert space, then it would admit an equivalent $\mathcal{C}^\infty$-smooth norm (away from 0). However, this contradicts Corollary 16. If $2 < p < +\infty$ and ℓ_p were isomorphic to a Hilbert space, then so would be its dual ℓ_p^*, which is isometric to ℓ_q, with $q = \frac{p}{p-1}$. However, $1 < q < 2$ and this contradicts the first case. The space ℓ_∞ is not an Asplund space. ∎

Remark. For positive results about the smoothness of L_p spaces we refer to [8, Theorem V.1.1].

We conclude this section by one fact related to Theorem 13 (ii).

THEOREM 18. *For $1 \le p < 2$ the canonical norm $\|\cdot\|_p$ on ℓ_p is nowhere $\mathcal{G}^2$-smooth.*

Proof. Let $p = 1$. Assume that $\|\cdot\|_1$ is $\mathcal{G}^2$-smooth at some $x \in \ell_p$. Then, by [3, Proposition 2.2], $\|\cdot\|_1$ is $\mathcal{F}^1$-smooth. But for $t > 0$

$$\begin{aligned}\sup_{i\in\mathbb{N}} \big(\|x+te_i\|_1 + &\|x-te_i\|_1 - 2\|x\|_1\big) \\ &\ge \limsup_{i\to\infty} \big(\|x+te_i\|_1 + \|x-te_i\|_1 - 2\|x\|_1\big) \\ &\ge \limsup_{i\to\infty} \big(|x_i+t| + |x_i-t| - 2|x_i|\big) = 2t,\end{aligned}$$

a contradiction.

Let $p > 1$. Put $q = \frac{p}{p-1}$; then $2 < q < +\infty$. Put $Z = \ell_q$ and $f(z) = \frac{1}{q}\|z\|_q{}^q$, $z \in \ell_q$. Then Z^* is isometric with ℓ_p and we can easily calculate the conjugate $f^*(z^*) = \frac{1}{p}\|z^*\|_p{}^p$, $z^* \in \ell_p$. Assume that $\|\cdot\|_p$ is $\mathcal{G}^2$-smooth at some $z_0^* \in \ell_p$. Then so is f^*. Denote $z_0 = f^{*\prime}(z_0^*)$. Then $z_0^* \in \partial f(z_0)$ by Lemma 1 as ℓ_q is reflexive. By [8, Theorem V.1.1(iii)], f is $\mathcal{F}^2$-smooth at z_0. Hence there is $c > 0$ so that

$$f(z) - f(z_0) - \langle z_0^*, z - z_0\rangle \le \|z - z_0\|_q{}^2 \qquad \text{for all } z \in \ell_q.$$

Now, using Lemma 2 and the symbols introduced prior it, we can estimate for $s \ge 0$

$$\beta(f^*, z_0^*, z_0, s) \ge \{ts - \alpha(f, z_0, z_0^*, t) :\ t \ge 0\} \ge \sup\{ts - ct^2 :\ t \ge 0\} = \frac{1}{4c}s^2.$$

Therefore

$$f^*(z_0^* + th^*) - f^*(z_0^*) - \langle th^*, z_0\rangle \ge \frac{1}{4c}t^2\|h^*\|_p{}^2$$

for all $h^* \in \ell_p$ and all $t \in \mathbb{R}$. Now, as f^* is $\mathcal{G}^2$-smooth at z_0^*, l'Hospital rule used twice gives

$$\frac{1}{2}f^{*\prime\prime}(z_0^*)(h^*, h^*) \ge \frac{1}{4c}\|h^*\|_p{}^2 \qquad \text{for all } h^* \in \ell_p.$$

In this way we obtained an equivalent Hilbertian norm on ℓ_p, which is impossible by Corollary 17. ∎

Remark. There is a statement more general than Theorem 18: *Assume that a Banach space X is not isomorphic to a Hilbert space, and that it* *admits an equivalent norm* $\|\cdot\|$ *having modulus of convexity* (see, e.g., [8, Definition IV.1.4]) *of power type 2. Then this norm is nowhere* $\mathcal{G}^2$*-smooth.* Its proof starts by applying [14, Lemma 5] which yields, for every $x \in X$, a closed hyperplane $H \subset X$ and $c > 0$ such that

$$\|x + h\| - \|x\| \geq c\|h\|^2$$

whenever $h \in H$ and $\|h\|$ is small enough. Then we can finish the proof as that of Theorem 18.

5. Higher Order Smooth Variational Principles and Geometry of Spaces

Let $\mathcal{S}$ be any "reasonable" class of smoothness like "$\mathcal{G}^k$-smooth", "$\mathcal{F}^k$-smooth", "$\mathcal{C}^k$-smooth", $k \in \mathbb{N}, \dots$. We say that a Banach space X admits an $\mathcal{S}$ *variational principle* if for every lower semicontinuous bounded below function $f : X \to (-\infty, +\infty]$ there are an $\mathcal{S}$ function $g : X \to \mathbb{R}$ and a point $x_0 \in X$ such that $f(x_0) = g(x_0)$ and $f \geq g$. Similarly we define the polynomial variational principle. We require then that the function g is a polynomial on X.

Let us recall that smooth variational principles as presented e.g. in [8] require the completeness of spaces of differentiable functions involved. This usually involves a kind of Lipschitz property of the derivatives and excludes the direct use of C^∞ smoothness. This drawback is overcome by Stegall's variational principle (Theorem 5). In this section we will discuss this approach to this problem.

Theorem 19. *Assume that a Banach space* X *is dentable and admits an* $\mathcal{S}$ *bump. Then* X *admits an* $\mathcal{S}$ *variational principle.*

Proof. Let $b : X \to \mathbb{R}$ be an $\mathcal{S}$ bump. Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous bounded below function. Applying Stegall's principle mentioned in Remark following Theorem 5 to the function $\varphi := f + b^{-2}$, we get $\xi \in X^*$ and $x_0 \in X$ such that

$$f(x) + b^{-2}(x) - f(x_0) - b^{-2}(x_0) - \langle \xi, x - x_0 \rangle \geq 0 \qquad \text{for all } x \in X.$$

Then, necessarily, $b(x_0) \neq 0$. For $x \in X$ denote

$$\psi(x) = -b^{-2}(x) + f(x_0) + b^{-2}(x_0) + \langle \xi, x - x_0 \rangle$$

and put $g = \alpha \circ \psi$ where $\alpha : [-\infty, +\infty) \to \mathbb{R}$ is a nondecreasing $\mathcal{C}^\infty$-smooth function such that

$$\alpha(t) = \begin{cases} t & \text{if } t \geq \inf f, \\ \inf f - 1 & \text{if } t \leq \inf f - 1. \end{cases}$$

It is easy to see that then $f(x_0) = g(x_0)$ and $f \geq g$. And, as $\mathcal{S}$ is reasonable, we assume that $b \in \mathcal{S}$ implies $g \in \mathcal{S}$ (see [5, Chapitre 1, Théorème 5.4.2]). The proof is finished. ∎

Remarks. 1. In the cases when $\mathcal{S}$ is "$\mathcal{F}^1$-smooth" [13], or "$\mathcal{G}^1$-smooth with bounded derivative", or "$\mathcal{F}^1$-smooth with bounded derivative" [8, Theorem I.2.3], or "$\mathcal{C}^1$-smooth with bounded derivative" [8, Proposition I.2.7], then the assumption of the dentability in the above theorem may be dropped. The arguments are then different. The validity of the $\mathcal{C}^1$-smooth variational principle without the assumption of dentability is an open question.

2. The smooth variational principles considered in [8, Section I.2] speak about the so called strong minimum. When we use the full conclusion of Theorem 5 (involving the function γ), we also get the strong minimum in Theorem 19.

The assumption of dentability is not a loss of generality for the $\mathcal{C}^k$-smooth variational principle with $k > 1$. This can be seen in the following result.

THEOREM 20. ([13]) *Given an integer $k > 1$, a Banach space X admits a $\mathcal{C}^k$-smooth variational principle if and only if it admits a $\mathcal{C}^k$-smooth bump and is superreflexive (or dentable or just $c_0 \not\subset X$).*

Proof. Necessity. Apply the $\mathcal{C}^k$-smooth variational principle to the function $f = \|\cdot\|^{-1}$. We get a $\mathcal{C}^k$-smooth function $g : X \to \mathbb{R}$ and $x_0 \in X$ such that $\|x_0\|^{-1} = g(x_0)$ and

$$\frac{1}{\|x\|} \geq g(x) \qquad \text{for all } x \in X;$$

thus $x_0 \neq 0$. We observe that if $x \in X$ and $\|x\| > 2\|x_0\|$, then

$$g(x) \leq \frac{1}{\|x\|} < \frac{1}{2\|x_0\|}.$$

Put $b = \alpha \circ g$, where $\alpha : \mathbb{R} \to \mathbb{R}$ is a nondecreasing $\mathcal{C}^k$-smooth function such that

$$\alpha(t) = \begin{cases} t & \text{if } t \geq \frac{1}{\|x_0\|}, \\ 0 & \text{if } t \leq \frac{1}{2\|x_0\|}. \end{cases}$$

Then b is a $\mathcal{C}^k$-smooth bump on X [5, Chapitre 1, Théorème 5.4.2].

As to the superreflexivity of X, it is enough to show that every separable subspace of X is superreflexive. Indeed, this follows from the very definition of the superreflexivity [8, page 133] and from the fact that the reflexivity is separably determined (see e.g. [18, page 182]). So, let Y be a separable subspace of X. Provide Y with an equivalent locally uniformly rotund norm $|\cdot|$ (see e.g. [8, Theorem II.2.6(i)]) and put

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } x \in Y, \\ +\infty & \text{if } \in X \backslash Y. \end{cases}$$

Applying the $\mathcal{C}^k$-smooth variational principle to this f, we find a $\mathcal{C}^k$-smooth function $g : X \to \mathbb{R}$ and $x_0 \in X$ so that $f(x_0) = g(x_0)$ and $f \geq g$. Hence $0 \neq x_0 \in Y$. Find $\eta \in Y^*$, $|\eta| = 1$, such that $\langle \eta, x_0 \rangle = |x_0|$ and put $H = \eta^{-1}(0)$. Consider $h_n \in H$, $n = 1, 2 \ldots$, such that $|x_0 + h_n| \to |x_0|$. Then

$$2|x_0| = \langle \eta, 2x_0 + h_n \rangle \leq |2x_0 + h_n| \leq |x_0| + |x_0 + h_n| \to 2|x_0| \qquad \text{as } n \to \infty,$$

and so

$$2|x_0|^2 + 2|x_0 + h_n|^2 - |x_0 + x_0 + h_n|^2 \to 0 \qquad \text{as } n \to \infty.$$

Now, using the definition of the locally uniform rotundity (see, e.g., [18, page 96]), we get that $|h_n| \to 0$. Since g is $\mathcal{C}^k$-smooth and $k > 1$, there is $\delta > 0$ such that the derivative g' is Lipschitz on the ball around x_0 with radius 2δ. From the above, we find $\gamma > 0$ such that $h \in H$ and $|h| > \delta$ implies that $|x_0 + h| > |x_0| + \gamma$. Hence

$$g(x_0 + h) \leq \frac{1}{|x_0 + h|} < \frac{1}{|x_0| + \gamma} \qquad \text{whenever } h \in H \text{ and } |h| > \delta.$$

Consider a nondecreasing $\mathcal{C}^2$-smooth function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\alpha(t) = \begin{cases} t & \text{if } t \geq \frac{1}{|x_0|}, \\ 0 & \text{if } t \leq \frac{1}{|x_0| + \gamma}. \end{cases}$$

Finally, define

$$b(h) = \begin{cases} 0 & \text{if } h \in H \text{ and } |h| > \delta, \\ \alpha(g(x_0 + h)) & \text{if } h \in H \text{ and } |h| < 2\delta. \end{cases}$$

We can easily verify that b is a $\mathcal{C}^1$-smooth bump function on H, with Lipschitz derivative. Then, by [8, Theorem V.3.2], H, and so Y, is suprereflexive.

Finally, recall that the (super)reflexivity implies the dentability of the space [26, Theorem 2.12, Lemma 2.18] and the latter implies that $c_0 \not\subset X$ since the unit ball in c_0 is not dentable.

Sufficiency. Let X admit a $\mathcal{C}^k$-smooth bump, with $k > 1$, and $c_0 \not\subset X$. Then, by [8, Theorems V.3.4 and V.3.2], X admits a $\mathcal{C}^1$-smooth bump with Lipschitz derivative, and we already know that this implies the superreflexivity of X. Now, Theorem 19 completes the proof. ∎

Recall that a polynomial P on a Banach space X is called *separating* if $P(0) = 0$ and there is $c > 0$ such that $P(h) > c$ for all h from the unit sphere of X.

THEOREM 21. *For a Banach space $(X, \|\cdot\|)$ the following statements are equivalent:*

(i) *X admits the $\mathcal{C}^\infty$-smooth variational principle;*

(ii) *X admits the $\mathcal{C}^k$-smooth variational principle for every $k \in \mathbb{N}$;*

(iii) *X admits a $\mathcal{C}^\infty$-smooth bump and is superreflexive (or dentable, or just $c_0 \not\subset X$);*

(iv) *X admits a $\mathcal{C}^k$-smooth bump for every $k \in \mathbb{N}$ and is superreflexive (or dentable, or just $c_0 \not\subset X$);*

(v) *X admits a separating polynomial ;*

(vi) *X admits the polynomial variational principle.*

Proof. (i) ⇒ (ii) and (iii) ⇒ (iv) are trivial.

(i) ⇒(iii) and (ii) ⇒(iv): see the proof of the necessity part in Theorem 19.

(iv) ⇒ (v). Assume X admits a $\mathcal{C}^k$-smooth bump for every $k \in \mathbb{N}$ and $c_0 \not\subset X$. From the proof of Theorem 20 we already know that X is superreflexive. According to the result of Pisier (see e.g. [8, Theorem IV.4.8]), X admits an equivalent norm $|\cdot|$ such that its modulus of rotundity $\delta(\epsilon) \geq c\epsilon^q$, $\epsilon > 0$, where $c > 0$ and $q \geq 2$ are constants. Fix an integer $k > q$ and let $b : X \to \mathbb{R}$

be a $\mathcal{C}^k$-smooth bump. Applying Stegall's principle (Theorem 5) to $b^{-2} - |\cdot|$, we find $\xi \in X^*$ and $x_0 \in X$ such that

$$b^{-2}(x) - |x| - b^{-2}(x_0) + |x_0| - \langle \xi, x - x_0 \rangle \geq 0 \qquad \text{for all } x \in X.$$

Hence $b(x_0) \neq 0$ and

$$b^{-2}(x_0 + h) + b^{-2}(x_0 - h) - 2b^{-2}(x_0) \geq |x_0 + h| + |x_0 - h| - 2|x_0| \quad \text{for all } h \in X. \tag{15}$$

By replacing b by $|x_0|^{-2} b$, we may and do assume that $|x_0| = 1$. Find $\eta \in X^*$ such that $|\eta| = 1$ and $\langle \eta, x_0 \rangle = |x_0|$, and put $H = \eta^{-1}(0)$. We claim that

$$|x_0 + h| > 1 + c2^{-q}|h|^q \qquad \text{for all } h \in H, \ |h| \leq 2c^{1/q}. \tag{16}$$

Indeed, fix $h \in H$, $|h| \leq 2c^{1/q}$, and denote $\Delta = c2^{-q}|h|^q$. Then $\Delta \leq 1$. If $|x_0 + h| > 1 + \Delta$, we are done. Assume that $|x_0 + h| \leq 1 + \Delta$. Put $x_1 = \frac{x_0}{1+\Delta}$, $x_2 = \frac{x_0+h}{1+\Delta}$; then $|x_1| \leq 1$, $|x_2| \leq 1$. According to the definition of $\delta(\epsilon)$ (see e.g. [8, page 130]),

$$1 - \frac{1}{2}|x_1 + x_2| \geq \delta(|x_1 - x_2|) \geq c|x_1 - x_2|^q = c\frac{|h|^q}{(1+\Delta)^q} = \frac{2^q \Delta}{(1+\Delta)^q} \geq \Delta.$$

On the other hand,

$$1 - \frac{1}{2}|x_1 + x_2| \leq 1 - \frac{1}{2}\langle \eta, x_1 + x_2 \rangle = 1 - \frac{1}{1+\Delta} = \frac{\Delta}{1+\Delta},$$

a contradiction. This proves (16).

Note that b^{-2} is $\mathcal{C}^k$-smooth and hence $\mathcal{T}^k$-smooth at x_0 by Lemma 11 (i). Let P be a polynomial of degree k such that

$$b^{-2}(x_0 + h) = b^{-2}(x_0) + P(h) + o(|h|^k) \qquad \text{for all } h \in X.$$

Then, by (15) and (16),

$$P(h) + P(-h) + o(|h|^k) \geq 2^{1-q}c|h|^q \qquad \text{for all } h \in H, \ |h| \leq 2c^{1/q}.$$

Hence, there are $0 < \gamma \leq 1$ and $d > 0$ such that

$$Q(h) := P(h) + P(-h) > d \qquad \text{whenever } h \in H \text{ and } |h| = \gamma.$$

We observe that there exists $n \in \mathbb{N}$ and for $i = 1, 2, \ldots, n$ there exists a continuous $2i$-linear form P_{2i} such that

$$Q(h) = P_2(h, h) + P_4(h, h, h, h) + \cdots + P_{2n}(h, \ldots, h) \qquad \text{for all } h \in X.$$

Put

$$\widetilde{Q}(h) = P_2(h, h)^2 + P_4(h, h, h, h)^2 + \cdots + P_{2n}(h, \ldots, h)^2, \qquad h \in X,$$

this will also be a polynomial and $\widetilde{Q}(0) = 0$. Fix an arbitrary $h \in X$, $|h| = 1$. We find $i \in \{1, \ldots, n\}$ so that $P_{2i}(\gamma h, \ldots, \gamma h) > \frac{d}{n}$. Then

$$\widetilde{Q}(h) \geq \frac{1}{\gamma}\gamma^{4i} P_{2i}(h, \ldots, h)^2 = \frac{1}{\gamma} P_{2i}(\gamma h, \ldots, \gamma h)^2 > \frac{d^2}{\gamma n^2}.$$

This means that $\widetilde{Q}$ is a separating polynomial on H. Finally, $(h, t) \mapsto t^2 + Q(h)$, $(h, t) \in H \times \mathbb{R} \equiv X$ is a separating polynomial on X.

(v) $\Rightarrow$ (vi). Let P be a polynomial on X and $c > 0$ be such that $P(x) \geq c > 0 = P(0)$ whenever $x \in X$ and $\|x\| = 1$. Write

$$P(x) = P_1(x) + P_2(x, x) + \cdots + P_n(x, \ldots, x), \qquad x \in X,$$

where P_i are continuous i-linear forms on X and put

$$\widetilde{P}(x) = P_1(x)^2 + P_2(x, x)^2 + \cdots + P_n(x, \ldots, x)^2, \qquad x \in X.$$

This will also be a polynomial on X. Fix an arbitrary $x \in X$, $\|x\| > 1$. We find $i \in \{1, \ldots, n\}$ such that $P_i\big(x/\|x\|, \ldots, x/\|x\|\big) > \frac{c}{n}$. Then

$$\widetilde{P}(x) \geq P_i(x, \ldots, x)^2 > \|x\|^{2i} \frac{c^2}{n^2} \geq \|x\|^2 \frac{c^2}{n^2}$$

and so $\frac{\widetilde{P}(x)}{\|x\|} \to +\infty$ as $\|x\| \to +\infty$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^2$-smooth bump such that $\alpha(0) \neq 0$. Then $\alpha \circ \widetilde{P}$ is a $\mathcal{C}^2$-smooth bump on X and an argument from the proof of Theorem 20 guarantess that X is superreflexive. Let $f : X \to (-\infty, +\infty]$ be a lower semicontinuous function, bounded below. Put $\varphi = f + \widetilde{P}$. As $\frac{\varphi(x)}{\|x\|} \to +\infty$ for $\|x\| \to +\infty$, we can use Theorem 5 for this φ. We get $x_0 \in X$ and $\xi \in X^*$ so that

$$f(x) + \widetilde{P}(x) - f(x_0) - \widetilde{P}(x_0) - \langle \xi, x - x_0 \rangle \geq 0 \qquad \text{for all } x \in X.$$

This means that X admits the polynomial variational principle.

(vi) $\Rightarrow$ (i). We need only observe that if P is a polynomial of degree $k \in \mathbb{N}$, then $P^{(k+1)} \equiv 0$, and hence P is $\mathcal{C}^\infty$-smooth. ∎

Remark. The equivalences (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) are proved in [14]; see also [6].

We finish this paper with a few questions that are open in this area.

QUESTIONS. 1. Does X admit a C^∞-smooth norm if it admits a separating polynomial?

2. Does X admit a C^∞-smooth norm if it admits C^k-smooth norms for every $k \in \mathbb{N}$?

3. Does a separable X admit a C^2-smooth norm if it admits a C^2-smooth bump?

Acknowledgements

The authors thank participants of the 26th Winter School on Abstract Analysis in Křišťanovice, and Spring School of Banach spaces, Paseky, Czech Republic for valuable discussions on the subject of this paper.

References

[1] BEDNAŘÍK, D. , On uniform Gâteaux smoothness of higher order on separable Banach spaces, *preprint.*

[2] BONIC, R. , FRAMPTON, J. , Smooth functions on Banach manifolds, *J. Math. Mech.* **15** (1966), 877–898.

[3] BORWEIN. J. , NOLL, D. , Second order differentiability of convex functions: A.D. Alexandrov's theorem in Hilbert space, *Trans. Amer. Math. Soc.* **342** (1994), 43–79.

[4] BONIC, R. , REIS, F. , A characterization of Hilbert space, *Anais Acad. Brasileira Cienc.* **38** (1966), 239–241.

[5] CARTAN, H. , "Calcul Differentiel, Formes Differentielles", Herman, Paris, 1967.

[6] DEVILLE, R. , A characterization of $\mathcal{C}^\infty$-smooth Banach spaces, *Bull. London Math. Soc.* **22** (1990), 13–17.

[7] DEVILLE, R. , GODEFROY, G. , ZIZLER, V. , Smooth bump functions and geometry of Banach spaces, *Mathematika* **40** (1993), 305–321.

[8] DEVILLE, R. , GODEFROY, G. , ZIZLER, V. , "Smoothness and Renormings in Banach Spaces", Pitman Monographs and Surveys in Pure and Appl. Math. No. 64, New York, 1993.

[9] DEVILLE, R. , GONZALO, R. , JARAMILLO, J.A. , Renormings of $L^p(L^q)$, *Math. Proc. Cambridge Phil. Soc.* **126** (1999), 155–169.

[10] FABIAN, M. , On minimum principles, *Acta Polytechnica* **20** (1983), 109–118.

[11] FABIAN, M. , Lipschitz smooth points of convex functions and isomorphic characterizations of Hilbert spaces, *Proc. London Math. Soc.* **51** (1985), 113–126.

[12] FABIAN, M. , "Gâteaux Differentiability of Convex Functions and Topology, Weak Asplund Spaces", John Wiley and sons, Inc., New York, 1997.

[13] Fabian, M., Hájek, P., Vanderwerff, J., On smooth variational principles in Banach spaces, *J. Math. Anal. Appl.* **197** (1996), 153–172.

[14] Fabian, M., Preiss, D., Whitfield, J.H.M., Zizler, V., Separating polynomials on Banach spaces, *Quart. J. Math. Oxford* **40** (1989), 409–422.

[15] Fabian, M., Whitfield, J.H.M., Zizler, V., Norms with locally Lipschitzian derivatives, *Israel J. Math.* **44** (1983), 262–276.

[16] Figiel T., Pisier, G., Séries aléatoires dans les espaces uniformḿent convexes ou uniformément lisses, *C.R. Acad. Sci. Paris*, Ser. A, **279** (1974), 611–614.

[17] Gonzalo, R., Jaramillo, J.A., Separating polynomials on Banach spaces, *Extracta Math.* **12** (1997), 145–164.

[18] Habala, P., Hájek, P., Zizler, V., "Intoroduction to Banach Spaces I, II", Matfyzpress, Praha, 1996.

[19] Kurzweil, J., On approximation in real Banach spaces, *Studia Math.* **16** (1954), 124–129.

[20] Kwapień, S., Isomorphic characterization of inner product space by orthogonal series with vector valued coefficients, *Studia Math.* **44** (1972), 583–595.

[21] Lindenstrauss, J., On operators which attain their norms, *Israel J. Math.* **1** (1963), 139–148.

[22] Lindenstrauss, J., On the modulus of smoothness and divergent series in Banach spaces, *Michigan Math. J.* **10** (1963), 241–252.

[23] McLaughlin, D., Vanderwerff, J., Higher order Gâteaux smooth bump functions on Banach spaces, *Bull. Australian Math. Soc.* **51** (1995), 291–300.

[24] McLaughlin, D., Poliquin, R., Vanderwerff, J., Zizler, V., Second order Gâteaux differentiable bump functions and approximations in Banach spaces, *Canad. J. Math.* **45** (1993), 612–625.

[25] Meškov, V.Z., Smoothness properties in Banach spaces, *Studia Math.* **63** (1978), 111–123.

[26] Phelps, R.R., "Convex Functions, Monotone Operators and Differentiability", Lect. Notes Math. No. 1364, 2nd Ed., Springer-Verlag, Berlin, 1993.

[27] Rao, M.M., Characterization of Hilbert spaces by smoothness, *Nederl. Acad. v. W. A.* **71** (1967), 132–135.

[28] Stegall, Ch., Optimization of functions on certain subsets of Banach spaces, *Math. Annalen* **236** (1978), 171–176.

[29] Stegall, Ch., Optimization and differentiation in Banach spaces, *J. Linear Algebra Appl.* **84** (1986), 191–211.

[30] Sundaresan, K., Smooth Banach spaces, *Bull. Amer. Math. Soc.* **72** (1966), 520–521.

[31] Sundaresan, K., Swaminathan, S., "Geometry and Nonlinear Analysis in Banach Spaces", Lect. Notes Math. No. 131, Springer-Verlag, Berlin 1985.

[32] Troyanski, S., Gâteaux differentiable norms in L_p, *Math. Ann.* **287** (1990), 221–227.

[33] Vanderwerff, J., Second-order Gâteaux differentiability and an isomorphic characterization of Hilbert spaces, *Quart. J. Math. Oxford* **44** (1993), 249–255.