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Homotopy Representability of Brauer Groups*

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1. Introduction

The purpose of this paper is to present certain facts and results showing a way through which simplicial homotopy theory can be used in the study of Auslander-Goldman-Brauer groups of Azumaya algebras over commutative rings.

We observe here that any Galois extension $S \supseteq R$, with finite Galois group G, has associated a monoidal group-like groupoid, whose nerve defines a simplicial Kan-complex $\mathcal{P}ic(S,G)$. This complex, which we will call the *Picard complex*, allows us to give a "geometric" description of $Pic(S)$, the Picard G-module of isomorphism classes of invertible S-modules, and also of $U(S)$, the G-module of units of S, since the semidirect product $Pic(S) \rtimes G$ and $U(S)$ respectively, arise as the first (and only) two non-trivial homotopy groups of the complex $\mathcal{P}ic(S,G)$.

Furthermore, we show that there exists a (Kan) fibration $\wp : \mathcal{P}ic(S,G) \longrightarrow K(G,1)$, with the Eilenberg-MacLane complex $K(G,1)$ as base complex. Then, we observe that simplicial cross-sections for $\wp$ are in agreement with Kanzaki factor sets [11, 8], and therefore with generalized crossed product algebras. At this point, taking into account the Hattori theorem [8, 9], we prove that there exists an isomorphism

$$Br(S/R) \cong \Gamma\left({}^{\mathcal{P}(S,G)}/_{K(G,1)}\right)$$

between the Brauer group of equivalence classes of Azumaya R-algebras split by S, and the group of fiber homotopy classes of cross-sections of $\wp$.

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It is remarkable that, from this homotopy representation of the relative Brauer groups, one can recover several well-known exact sequences linking the groups $Br(S/R)$, $Pic(S)$, $Pic(R)$, $U(S)$ and G as instances of exact sequences of groups obtained in simplicial homotopy theory.

2. The Picard categorical group

Monoidal categories and, in particular, categorical groups have been studied extensively in the literature and we refer to , [10], [12], [15] or [16] for the background. Let us recall that a monoidal category $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ consists of a category $\mathbb{G}$, a functor $\otimes : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$, an object I and natural isomorphisms

$$\begin{gathered} a = a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z) \\ l = l_X : I \otimes X \xrightarrow{\sim} X \quad , \quad r = r_X : X \otimes I \xrightarrow{\sim} X \end{gathered} \tag{1}$$

(called the *associativity, left unit, right unit constraints*, respectively), such that, for all objects $X, Y, Z, T \in \mathbb{G}$ the following two *coherence* conditions hold:

$$a_{X,Y,Z\otimes T} \circ a_{X\otimes Y,Z,T} = (1_X \otimes a_{X,Y,Z}) \circ a_{X,Y\otimes Z,T} \circ (a_{Y,Z,T} \otimes 1_Z) \quad \text{and}$$

$$(1_X \otimes l_Y) \circ a_{X,I,Y} = r_X \otimes 1_Y .$$

A *categorical group* is a monoidal category in which every arrow is invertible and, for each object X, there is an object X^* with an arrow $X^* \otimes X \to I$.

Any group G operating on a commutative ring R by ring automorphisms gives rise to a categorical group $\mathcal{P}ic(S, G)$, which we call the *Picard categorical group* of (S, G), defined as follows. The objects are pairs (P, σ), where $\sigma \in G$ and P is an invertible (S, S)-bimodule such that $\sigma(s)x = xs$ for $x \in P$, $s \in S$. The set of morphisms from (P, σ) to (Q, τ) is empty is $\sigma \neq \tau$ and, in the other case, a morphism $f : (P, \sigma) \longrightarrow (Q, \sigma)$ is an (S, S)-isomorphism $f : P \to Q$. The tensor product is given the tensor product of (S, S)-bimodules

$$\big((P,\sigma) \xrightarrow{f} (Q,\sigma)\big) \otimes \big((P',\tau) \xrightarrow{f'} (Q',\tau)\big) = \big((P \otimes_S P', \sigma\tau) \xrightarrow{f \otimes f'} (Q \otimes_S Q', \sigma\tau)\big) ,$$

the unit object is the pair $(S, 1)$, and the associativity and unit constraints are as usual for the tensor product of bimodules.

Let us recall that that the Picard group $Pic(S)$ of isomorphism classes of invertible S-modules is a G-module, by ${}^{\sigma}[A] = [{}^{\sigma}A]$, where for an invertible

S-module A and each $\sigma \in G$, ${}^{\sigma}A$ means A as abelian group with the action of S given by $s \circ x = \sigma^{-1}(s)x$. Let $Pic(S) \rtimes G$ be the corresponding semidirect product group.

Any object of $\mathcal{P}ic(S, G)$ can be uniquely written in the form (A_σ, σ), where $\sigma \in G$ and A_σ means the (S, S)-bimodule defined by an invertible S-module A, with the given left action of S and the right action by $xs = \sigma(s)x$. Since $A_\sigma \otimes_S B_\tau = (A \otimes_S {}^{\sigma}B)_{\sigma\tau}$, for any invertible S-modules A and B, we have that $Pic(S) \rtimes G$ is identified as the group of connected componets of the categorical group $\mathcal{P}ic(S, G)$.

Furthermore, the group of automorphisms in the unit object $(S, 1)$ is just the group of S-linear isomorphisms of S with itself, or equivalently, the group $U(S)$ of the units of S.

3. The Picard complex

In this paper we use a few well-known basic facts about simplicial sets (see references [7, 14]).

Let $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ be a categorical group. We define the simplicial set *nerve* of $\mathbb{G}$, denoted by $Ner(\mathbb{G}, \otimes)$, as follows. An n-simplex $(f, X) \in Ner_n(\mathbb{G}, \otimes)$ consists of a family of objects X_{ij}, $0 \leq i \leq j \leq n$, and morphisms $f_{ijk} : X_{ij} \otimes X_{jk} \to X_{ik}$, $0 \leq i \leq j \leq k \leq n$, satisfying the following conditions:

1) $X_{ii} = I$,

2) $f_{iik} = l : I \otimes X_{ik} \to X_{ik}$ and $f_{ijj} = r : X_{ij} \otimes I \to X_{ij}$,

3) for any $i \leq j \leq k \leq m$,

$$f_{ijm} \circ (1_{X_{ij}} \otimes f_{jkm}) = f_{ikm} \circ (f_{ijk} \otimes 1_{X_{km}}) \circ a_{X_{ij}, X_{jk}, X_{km}}. \qquad (2)$$

If $\alpha : [n] \to [m]$ is a non-decreasing map, the induced map

$$Ner_m(\mathbb{G}, \otimes) \to Ner_n(\mathbb{G}, \otimes),$$

is given by $\alpha^*(f, X) = (\alpha^* f, \alpha^* X)$, where

$$(\alpha^* f)_{ijk} = f_{\alpha(i)\alpha(j)\alpha(k)} \quad \text{and} \quad (\alpha^* X)_{ij} = X_{\alpha(i)\alpha(j)}.$$

Let us note that if $x = (f, X)$ is an n-simplex of $Ner(\mathbb{G}, \otimes)$, $n \geq 3$, then x is completely determined by any three of its faces $d_m(x) = (\delta_m^* f, \delta_m^* X)$, where

$\delta_m : [n] \to [n+1]$ is the injective non-decreasing map that does not take the value $m \in [n]$, since a face $d_m(x)$ includes all the morphisms f_{ijk} such that $m \notin \{i,j,k\}$. Then, if one knows $d_m(x)$, $d_r(x)$ and $d_s(x)$, $0 \leq m < r < s \leq n$, one also knows all f_{ijk} except x_{mrs}; but taking any $i \notin \{m,r,s\}$ (such an integer exists because $n \geq 3$) and the corresponding diagram (2) for i, m, r and s (in the corresponding order), we can see that f_{mrs} is also determined by the others.

An immediate consequence of the above observation is that the canonical simplicial map $Ner(\mathbb{G},\otimes) \to Cosk^3(Ner(\mathbb{G},\otimes))$ is an isomorphism, where $Cosk^3(-)$ is the third Verdier's coskeleton functor [17], and then the simplicial set $Ner(\mathbb{G},\otimes)$ is isomorphic to the nerve of the categorical group $\mathbb{G}$ as defined in [3] (there denoted by $Ner_2(\mathbb{G})$).

The *classifying space* of a categorical group $\mathbb{G}$, $B(\mathbb{G},\otimes)$, is the geometrical realization of the simplicial set $Ner(\mathbb{G},\otimes)$. The next proposition describes the homotopy type of $B(\mathbb{G},\otimes)$ and was proved in [3].

PROPOSITION 3.1. *Let $\mathbb{G} = (\mathbb{G},\otimes,a,I,l,r)$ be a categorical group. Then:*
i) *The homotopy groups of $B(\mathbb{G},\otimes)$ are*

$$\pi_i(B(\mathbb{G},\otimes)) = \begin{cases} 0 & i \neq 2 \\ [\mathbb{G}] \quad \textit{(the group of connected components of } \mathbb{G}) & i = 1 \\ Aut_{\mathbb{G}}(I) \quad \textit{(the group of automorphisms in } I) & i = 2 \end{cases}$$

ii) *There exists a canonical homotopy equivalence $B(\mathbb{G}) \simeq \Omega B(\mathbb{G},\otimes)$. Thus, the classifying space of category $\mathbb{G}$ (forgetting the tensor structure) is a loop space of the classifying $B(\mathbb{G},\otimes)$.*

iii) *Any path-connected CW-complex with trivial homotopy groups in dimensions other than 1 and 2 is homotopy equivalent to the classifying space of a categorical group. Specifically, if X is such a space and $* \in X$ is a base point, then $X \simeq B(\mathcal{P}_2(X,*))$ where $\mathcal{P}_2(X,*)$ is the categorical group of loops based on $*$, that is, the fundamental groupoid of the loop space $\Omega(X,*)$, with the tensor structure given by concatenation of loops. Furthemore, if X is homotopy equivalent to $B(\mathbb{G},\otimes)$ for some $\mathbb{G}$, then $\mathcal{P}_2(X,*)$ and $\mathbb{G}$ are monoidal equivalent.*

The above results show that categorical groups provide algebraic models for homotopy 2-types of connected spaces. This fact is not new (see [16]), although the better-known references consider the strict case, where Whitehead's crossed modules provide the algebraic models [13].

For any group G we can consider the discrete groupoid $dis(G)$ with only identities. The multiplication in G determines a strict categorical group structure on $dis(G)$, $\sigma\otimes\tau=\sigma\tau$, and $Ner(dis(G),\otimes)$ is just the minimal Eilenberg-Mac Lane complex $K(G,1)$, whose n-simplices σ consist of a family of elements $\sigma_{ij}\in G$, $0\le i\le j\le n$, such that $\sigma_{ij}\sigma_{jk}=\sigma_{ik}$ for $i\le j\le k$.

Analogously, if A is an abelian group regarded as a category with one object, then the multiplication in A defines a structure of categorical group whose nerve is isomorphic to the minimal Eilenberg-Mac Lane complex $K(A,2)$.

Now, let G be a group operating on a commutative ring S by ring automorphisms.

We define the *Picard complex*, also denoted by $\mathcal{P}ic(S,G)$, as the nerve of the Picard categorical group $\mathcal{P}ic(S,G)$.

Then, an n-simplex $(f,P,\sigma)\in\mathcal{P}ic_n(S,G)$ consists of a family of elements $\sigma_{ij}\in G$, $0\le i\le j\le n$, invertible (S,S)-bimodules P_{ij}, $0\le i\le j\le n$, such that $\sigma_{ij}(s)x=xs$ for $x\in P_{ij}$, $s\in S$ and (S,S)-isomorphisms $f_{ijk}:P_{ij}\otimes_S P_{jk}\xrightarrow{\sim}P_{ik}$, $0\le i\le j\le k\le n$, satisfying the following conditions:

1) $\sigma_{ij}\sigma_{jk}=\sigma_{ik}$ for $i\le j\le k$,

2) $P_{ii}=S$

3) $f_{iij}(s\otimes x)=sx,\ f_{ijj}(x\otimes s)=xs$ for $i\le j$, $s\in S$ and $x\in P_{ij}$,

4) for any $i\le j\le k\le m$ the diagram

$$\begin{array}{ccc} P_{ij}\otimes_S P_{jk}\otimes_S P_{km} & \xrightarrow{1\otimes f_{jkm}} & P_{ij}\otimes_S P_{jm} \\ {\scriptstyle f_{ijk}\otimes 1}\downarrow & & \downarrow{\scriptstyle f_{ijm}} \\ P_{ik}\otimes_S P_{km} & \xrightarrow{f_{ikm}} & P_{im} \end{array} \tag{3}$$

is commutative.

If $\alpha:[n]\to[m]$ is a non-decreasing map, the induced map $\mathcal{P}ic_m(S,G)\to\mathcal{P}ic_n(S,G)$ is given by $\alpha^*(f,P,\sigma)=(\alpha^*P,\alpha^*P,\alpha^*\sigma)$, where

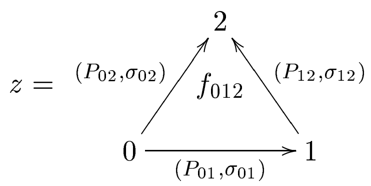
$$(\alpha^*f)_{ijk}=f_{\alpha(i)\alpha(j)\alpha(k)},\quad (\alpha^*P)_{ij}=P_{\alpha(i)\alpha(j)}$$

and analogously

$$(\alpha^*\sigma)_{ij}=\sigma_{\alpha(i)\alpha(j)}.$$

In particular, let us note that a 1-simplex of $\mathcal{P}ic(S,G)$ is just a pair (P,σ), where $\sigma\in G$ and P is an invertible (S,S)-bimodule such that $\sigma(s)x=xs$

for $x \in P$, $s \in S$, with 0-face and 1-face the ring S. Furthermore, giving a 2-simplex $z \in \mathcal{P}ic(S,G)$ with faces $d_0(z) = (P_{12}, \sigma_{12})$, $d_2(z) = (P_{01}, \sigma_{01})$ and $d_1(z) = (P_{02}, \sigma_{02} = \sigma_{01}\sigma_{12})$ is equivalent to giving an (S,S)-isomorphism $f_{012} : P_{01} \otimes_S P_{12} \to P_{02}$



(4)

For our purpose, what is most interesting about this complex is sumarized below.

Proposition 3.2. *For any group G operating on a commutative ring S by ring automorphisms, the simplicial complex $\mathcal{P}ic(S,G)$ has the following properties:*

a) *$\mathcal{P}ic(S,G)$ is a reduced (and hence connected) Kan-complex;*

b) *it has at most two non-trivial homotopy groups,viz.:*

$$\pi_1(\mathcal{P}ic(S,G)) = Pic(S) \rtimes G, \quad \text{and}$$
$$\pi_2(\mathcal{P}ic(S,G)) = U(S) \ ;$$

c) *it is a 2-dimensional hypergroupoid, that is, for $n \geq 3$ to give an n-simplex of $\mathcal{P}ic(S,G)$ is equivalent to knowing any three of its faces.*

4. The representation theorem

Let G be a group operating on a commutative ring S by ring automorphisms. From the definition of the complex $\mathcal{P}ic(S,G)$, it is immediate that there is a (Kan) fibration

$$\wp : \mathcal{P}ic(S,G) \longrightarrow K(G,1) \qquad (5)$$

given by $\wp(f, J, \sigma) = \sigma$ for any simplex $(f, J, \sigma) \in \mathcal{P}ic(S,G)$.

Our main result here is

THEOREM 4.1. *Let $S \supseteq R$ be a Galois extension of commutative rings, with finite Galois group $G \subseteq Aut(S)$. Then there is a bijection (really a group isomorphism)*

$$Br(S/R) \cong \Gamma\left({}^{\mathcal{P}ic(S,G)}/_{K(G,1)}\right) \tag{6}$$

between the Brauer group of Azumaya R-algebras split by S and the set of fiber homotopy classes of cross-sections of $\wp$, that is, of fiber homotopy classes of simplicial maps $\ell : K(G,1) \longrightarrow \mathcal{P}ic(S,G)$ such that $\wp\ell$ is the identity map on $K(G,1)$.

Proof. We begin with a brief review of the fundamental facts concerning generalized crossed products. A *Kanzaki factor set* [11, 8] (f,P) consists of a family of invertible (S,S)-bimodules P_σ, $\sigma \in G$, such that $\sigma(s)x = xs$ for $x \in P_\sigma$, $s \in S$ and (S,S)-isomorphisms $f_{\sigma,\tau} : P_\sigma \otimes_S P_\tau \xrightarrow{\sim} P_{\sigma\tau}$, $\sigma, \tau \in G$, satisfying the conditions: 1) $P_1 = S$; 2) $f_{1,\sigma}(s \otimes x) = sx$ and $f_{\sigma,1}(x \otimes s) = xs$ for $\sigma \in G$, $x \in P_\sigma$, $s \in S$; 3) for every $\sigma, \tau, \gamma \in G$ the following diagrams are commutative

$$\begin{array}{ccc} P_\sigma \otimes_S P_\tau \otimes_S P_\gamma & \xrightarrow{1\otimes f_{\tau,\gamma}} & P_\sigma \otimes_S P_{\tau\gamma} \\ {\scriptstyle f_{\sigma,\tau}\otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle f_{ijr}} \\ P_{\sigma\tau} \otimes_S P_\gamma & \xrightarrow{f_{\sigma\tau,\gamma}} & P_{\sigma\tau\gamma} \end{array} \tag{7}$$

A factor set (f,P) gives rise to the *generalized crossed product algebra*, of S and G, $\Delta(f,P) = \bigoplus_{\sigma\in G} J_\sigma$, where the multiplication is defined by $x \circ y = f_{\sigma,\tau}(x \otimes y)$ for $x \in P_\sigma$, $y \in P_\tau$, which is an Azumaya R-algebra (i.e., central separable) with a maximal commutative subalgebra S [11, Prop.2]. Any Azumaya R-algebra containing S as a maximal commutative subalgebra is S-isomorphic to a generalized crossed product $\Delta(f,P)$ for some factor set (f,P) [8, 9, Prop. 3, Th. E].

Since an Azumaya R-algebra is split by S if, and only if, there exists an Azumaya R-algebra equivalent to it which contains a maximal commutative subalgebra isomorphic to S ([5, Th. 5.5], any element of $Br(S/R)$ can be represented by a generalized crossed product $\Delta(f,P)$. Thus, the map $(f,P) \longmapsto \Delta(f,P)$ is a surjection from the set of Kanzaki factor sets to $Br(S/R)$. Moreover [8, 9, Th. I, Th. 3], two factor sets, (f,P) and (f',P'), define the same element of $Br(S/R)$ if, and only if, there exist an invertible S-module A and a family of (S,S)-isomorphisms $g_\sigma : A \otimes_S P_\sigma \xrightarrow{\sim} P'_\sigma \otimes_S A$,

$\sigma \in G$, with $g_1(x \otimes s) = s \otimes x$ for $x \in A,\ s \in S$ and making the following diagrams commutative

$$\begin{array}{ccccc} A \otimes_S P_\sigma \otimes_S P_\tau & \xrightarrow{1\otimes f_{\sigma,\tau}} & A \otimes_S P_{\sigma\tau} & \xrightarrow{g_{\sigma\tau}} & P'_{\sigma\tau} \otimes_S A \\ \big\downarrow{\scriptstyle g_\sigma \otimes 1} & & & & \big\uparrow{\scriptstyle f'_{\sigma,\tau}\otimes 1} \\ P'_\sigma \otimes_S A \otimes_S P_\tau & & \xrightarrow{1\otimes g_\tau} & & P'_\sigma \otimes_S P'_\tau \otimes_S A \end{array} \tag{8}$$

for every $\sigma,\ \tau \in G$.

Now, let ℓ be a cross-section of $\wp$. Since both complexes are reduced, ℓ_0 is the constant map. Then, by the proposition 3.2, c) and the simplicial identities, we see that ℓ is determined by the maps $\ell_m : K_m(G,1) \to \mathcal{P}ic_m(S/R)$ for $m = 1, 2$. Now, ℓ_1 has the form $\ell_1(\sigma) = (P_\sigma, \sigma)$, where P_σ is an invertible (S,S)-bimodule such that $\sigma(s)x = xs$ for $x \in P_\sigma,\ s \in S$; and therefore ℓ_2 has the form

$$\ell_2(\sigma,\tau) = \begin{array}{ccc} & S & \\ {\scriptstyle (P_{\sigma\tau},\sigma\tau)}\nearrow & f_{\sigma,\tau} & \nwarrow{\scriptstyle (P_\tau,\tau)} \\ S & \xrightarrow[(P_\sigma,\sigma)]{} & S \end{array}$$

where $f_{\sigma,\tau} : P_\sigma \otimes_S P_\tau \xrightarrow{\sim} P_{\sigma\tau}$ is an (S,S)-isomorphism. Since, for each $\sigma, \tau, \gamma \in G$, the faces of the 3-simplex $\ell_3(\sigma,\tau,\gamma) \in \mathcal{P}ic(S,G)$ are

$$\begin{aligned} d_0\ell_3(\sigma,\tau,\gamma) &= (f_{\tau,\gamma}, P_\tau, P_\gamma, P_{\tau\gamma}, \tau, \gamma, \tau\gamma), \\ d_1\ell_3(\sigma,\tau,\gamma) &= (f_{\sigma\tau,\gamma}, P_{\sigma\tau}, P_\gamma, P_{\sigma\tau\gamma}, \sigma\tau, \gamma, \sigma\tau\gamma), \\ d_2\ell_3(\sigma,\tau,\gamma) &= (f_{\sigma,\tau\gamma}, P_\sigma, P_{\tau\gamma}, P_{\sigma\tau\gamma}, \sigma, \tau\gamma, \sigma\tau\gamma) \end{aligned}$$

and

$$d_3\ell_3(\sigma,\tau,\gamma) = (f_{\sigma,\tau}, P_\sigma, P_\tau, P_{\sigma\tau}, \sigma, \tau, \sigma\tau),$$

the commutativity of the diagrams (3) gives the corresponding commutativity of (7). Thus, we see that (f,P) is actually a Kanzaki factor set, which clearly determines ℓ_1 and ℓ_2, and therefore ℓ.

Moreover, any Kanzaki factor set (f,P) arises from a cross-section ℓ defined as follows: if $\sigma = (\sigma_{ij})_{0\le i\le j\le n}$ is an n-simplex of $K(G,1)$, then $\ell(\sigma) = (f,P,\sigma) \in \mathcal{P}ic_n(S,G)$, where $P_{ij} = P_{\sigma_{ij}}$ for $0 \le i \le j \le n$ and $f_{ijk} = f_{\sigma_{ij},\sigma_{jk}}$ for $0 \le i \le j \le k \le n$.

Thus, $\ell \mapsto (f,P)$ establishes a bijection between the set of cross-sections of $\wp$ and the set of Kanzaki factor sets.

Now, let ℓ, ℓ' be two cross-sections of $\wp$, associated respectively to Kanzaki factor sets (f,P) and (f',P') by the above bijection, and let $h = \big(h_i^n : K_n(G,1) \to \mathcal{P}ic_n(S,G)\big)_{0\leq i\leq n}$ be a homotopy of ℓ to ℓ' which is stationary over $K(G,1)$, that is, such that $\wp h$ is the identity homotopy.

From the homotopy simplicial identities, we see that h_0^0, h_0^1 and h_1^1 take the form

$$h_0^0(1) = \ S \xrightarrow{(A,1)} S,$$

$$h_0^1(\sigma) = \begin{array}{ccc} & S & \\ {}^{(Q_\sigma,\sigma)}\nearrow & \varphi_\sigma & \nwarrow^{(P_\sigma,\sigma)} \\ S & \xrightarrow[(A,1)]{} & S, \end{array} \qquad h_1^1(\sigma) = \begin{array}{ccc} & S & \\ {}^{(Q_\sigma,\sigma)}\nearrow & \psi_\sigma & \nwarrow^{(A,1)} \\ S & \xrightarrow[(P'_\sigma,\sigma)]{} & S \end{array}$$

for an invertible S-module A, an invertible (S,S)-bimodule Q_σ such that $\sigma(s)x = xs$, $x \in Q$, $s \in S$, and (S,S)-isomorphisms $\varphi_\sigma : A \otimes_S P_\sigma \to Q$ and $\psi_\sigma : P'_\sigma \otimes_S A \to Q$. By computing the faces of the 3-simplices $h_i^2(\sigma,\tau) \in \mathcal{P}ic_3(S,G)$, $i = 0,1,2$, we obtain, for each $\sigma,\tau \in G$, the following diagram of (S,S)-isomorphisms

$$\begin{array}{ccccc}
A\otimes_S P_\sigma \otimes_S P_\tau & \xrightarrow{1\otimes f_{\sigma,\tau}} & A\otimes_S P_{\sigma\tau} & & \\
{\scriptstyle \varphi_\sigma\otimes 1}\downarrow & (0) & \downarrow{\scriptstyle \varphi_{\sigma\tau}} & & \\
Q_\sigma \otimes_S P_\tau & \longrightarrow & Q_{\sigma\tau} & \xleftarrow{\psi_{\sigma\tau}} & P'_{\sigma\tau}\otimes_S A \\
{\scriptstyle \psi_\sigma\otimes 1}\downarrow & (1) & \uparrow & (2) & \uparrow{\scriptstyle f'_{\sigma,\tau}\otimes 1} \\
P'_\sigma \otimes_S A \otimes_S P_\tau & \xrightarrow{1\otimes\varphi_\tau} & P'_\sigma\otimes_S Q_\tau & \xleftarrow{1\otimes\psi_\tau} & P'_\sigma\otimes_S P'_\tau \otimes_S A
\end{array}$$

where each square (i), $i = 0,1,2$, is commutative since $h_i^2(\sigma,\tau) \in \mathcal{P}ic_3(S,G)$ (note the homotopy simplicial identities $d_1h_0^2 = d_1h_1^2$ and $d_2h_1^2 = d_2h_2^2$).

Then, considering the composition $g_\sigma = \psi_\sigma^{-1}\circ\varphi_\sigma : A\otimes_S P_\sigma \longrightarrow P'_\sigma \otimes_S A$ for each $\sigma \in G$, the diagrams (8) are commutative for every $\sigma,\tau \in G$; which turns out to be that the factor sets (f,P) and (f',P') define equivalent Azumaya R-algebras, that is, the same element in $Br(S/R)$.

Conversely, if there exists an invertible S-module A and a family of (S,S)-isomorphisms $g_\sigma =: A \otimes P_\sigma \to P'_\sigma \otimes A$, $\sigma \in G$, with $g_1(x\otimes s) = s\otimes x$ for

$x \in A$, $s \in S$ and making the diagrams (8) commutative, we find a fiber truncated homotopy (h_0^0, h_0^1, h_1^1) of ℓ to ℓ' defined by

$$h_0^1(\sigma) = \begin{array}{ccc} & S & \\ {}^{(P'_\sigma \otimes_S A,\sigma)}\nearrow & g_\sigma & \nwarrow{}^{(P_\sigma,\sigma)} \\ S & \xrightarrow[(A,1)]{} & S, \end{array} \qquad h_1^1(\sigma) = \begin{array}{ccc} & S & \\ {}^{(P'_\sigma \otimes_S A,\sigma)}\nearrow & id & \nwarrow{}^{(A,1)} \\ S & \xrightarrow[(P'_\sigma,\sigma)]{} & S \end{array}$$

which, by [2, Lemma 2.8], extends to a full fiber homotopy $h : \ell \to \ell'$, and so the theorem is established. ∎

To conclude, we shall remark on the existence of a fibration (of fibered complexes over $K(G,1)$),

$$\mathbf{q} : \mathcal{P}ic(S,G) \longrightarrow K(Pic(S) \rtimes G, 1),$$

where $K(Pic(S) \rtimes G, 1)$ is the Eilenberg-MacLane complex associated to the semidirect product group $Pic(S) \rtimes G$, which is given by $\mathbf{q}(f,P,\sigma) = ([P],\sigma)$. The homotopy exact sequence, linking the groups of fiber homotopy classes of cross-sections, induced by the fibration $\mathbf{q}$ is precisely the Chase-Harrison-Rosenberg six-term exact sequence [4].

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