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Composition Operators on Vector-Valued Hardy Spaces

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1. Introduction

If ϕ is an analytic self-map of the unit disc D, then as a consequence of Littlewood's subordination theorem the composition transformation C_ϕ defined by $C_\phi f = f \circ \phi$ for f holomorphic in D turns out to be a bounded operator on the classical Hardy space $H^p(D)$, $(1 \leq p < \infty)$ and is called composition operator induced by ϕ (see Schwartz [5], Nordgren [3] for estimates of the norms of composition operators and Shapiro and Taylor [6] and Cowen and MacCluer [1] for other properties including compactness of these operators on Hardy classes of complex-valued functions). In this paper we attempt to initiate the study of composition operators on a vector-valued Hardy space.

The plan of the rest of the paper is as follows: Next section is preparatory in nature. In this section we collect some known as well as unknown facts about vector-valued Hardy spaces. We also determine generalized reproducing kernels for these spaces and use these kernel functions in the next section as effective tools to study composition operators on vector-valued Hardy spaces. In section 3 we prove that if $\phi\colon D \to D$ is analytic, then C_ϕ is a bounded operator on $H^2_X(D)$. A necessary and sufficient condition for a bounded operator on $H^2_X(D)$ to be a composition operator is given The condition on ϕ for which C^*_ϕ is also a composition operator is presented in section 4. In this section we also present characterizations of normal, unitary and co-isometric composition operators on vector-valued Hardy space $H^2_X(D)$.

2. Background

Let $D = \{z \in \mathbb{C}\colon |z| < 1\}$ and $(X, \|\cdot\|_X)$ be a complex Banach space. For $1 \leq p < \infty$, the vector-valued Hardy space $H^p_X(D)$ consists of all $f\colon D \to X$

such that $e^* \circ f$ is analytic in D for every $e^* \in X^*$ and

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p \, d\theta < \infty.$$

$H_X^p(D)$ is a Banach space with

$$\|f\|_p^p = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|_X^p \, d\theta.$$

Throughout this paper we will assume that $(X, < \cdot >)$ is a separable Hilbert space and so the radial limit $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists a.e. [4, Theorem A, page 84]. In this case $H_X^2(D)$ becomes a Hilbert space under the inner product $\ll \cdot \gg$, defined as

$$\ll f, g \gg = \frac{1}{2\pi} \int_0^{2\pi} < f^*(e^{i\theta}), g^*(e^{i\theta}) > d\theta.$$

For the sake of convenience, we shall denote $f^*(e^{i\theta})$ simply by $f(e^{i\theta})$. For more details about scalar-valued Hardy spaces we refer to Duren [2], and for vector-valued Hardy spaces consult Rosenblum and Rovnyak [4].

The very first result, which we are listing in the form of a lemma, will be used to find kernel functions for $H_X^2(D)$.

LEMMA 2.1. *If $f \in H_X^2(D)$, then*

$$\|f(z)\|_X \leq \frac{\|f\|_2}{(1 - |z|^2)^{1/2}}$$

Proof follows from the Hölder's inequality and the fact that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in H_X^2(D),$$

then $\|f\|_2^2 = \sum_{n=0}^{\infty} \|a_n\|_X^2$.

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for X. For $m, n \in \mathbb{N}$, we define $e_{m,n} : D \to X$ as

$$e_{m,n}(z) = z^m e_n, \quad \forall z \in D.$$

Then clearly $\{e_{m,n} : m, n, \in \mathbb{N}\}$ is an orthnormal subset of $H_X^2(D)$. Further, if $f \in H_X^2(D)$, then

$$\begin{aligned} \ll f, e_{m,n} \gg = 0 \quad &\Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} < f(e^{i\theta}), e_{m,n}(e^{i\theta}) > d\theta = 0 \\ &\Rightarrow \quad \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} < f(e^{i\theta}), e_n > d\theta = 0 \end{aligned}$$

for all $m, n \in \mathbb{N}$. Therefore $< f(e^{i\theta}), e_n >= 0$ a.e. for all $n \in \mathbb{N}$, hence $f(e^{i\theta}) = 0$ a.e. Taking into account the properties of the integral de Poisson, we conclude that $f \equiv 0$. Hence $\{e_{m,n} : m, n, \in \mathbb{N}\}$ is a basis for $H^2_X(D)$.

For each $z \in D$ and $j \in \mathbb{N}$, we define $E^j_z : H^2_X(D) \to \mathbb{C}$ as follows: $E^j_z(f) = < f(z), e_j >$ for every $f \in H^2_X(D)$. Then $E^j_z \in (H^2_X(D))^*$ and so by Riesz representation theorem, there exists $k^j_z \in H^2_X(D)$ such that

$$E^j_z f = \ll f, k^j_z \gg, \quad \forall f \in H^2_X(D).$$

We designate k^j_z's as generalized reproducing kernels or simply kernel functions whenever there is no confusion. The span of the set $\{k^j_z : (z,j) \in D \times \mathbb{N}\}$ will be denoted by $[k^j_z : (z,j) \in D \times \mathbb{N}]$. We now evaluate these kernel functions.

By Parseval's identity,

$$\begin{aligned} k^j_z(w) &= \sum_{m,n \in \mathbb{N}} \ll k^j_z, e_{m,n} \gg \, e_{m,n}(w) \\ &= \sum_{m,n \in \mathbb{N}} \overline{< z^m e_n, e_j >} \, e_{m,n}(w) \\ &= \frac{e_j}{1 - \bar{z}w} \end{aligned}$$

and $\|k^j_z\|^2_2 = \frac{1}{1-|z|^2}$.

LEMMA 2.2. *$[k^j_z : (z,j) \in D \times \mathbb{N}]$ is dense in $H^2_X(D)$.*

Proof. Let $f \in [k^j_z : (z,j) \in D \times \mathbb{N}]^{\perp}$, the orthogonal complement of $[k^j_z : (z,j) \in D \times \mathbb{N}]$. Then $\ll f, g \gg = 0$ for all $g \in [k^j_z : (z,j) \in D \times \mathbb{N}]$. In particular, $\ll f, k^j_z \gg = 0$ for every $(z,j) \in D \times \mathbb{N}$ and so $f \equiv 0$. This completes the proof. ∎

3. COMPOSITION OPERATORS ON $H^2_X(D)$

We begin this section by proving that every analytic self-map of the unit disc induces a composition operator on $H^2_X(D)$.

THEOREM 3.1. *Let $\phi : D \to D$ be analytic. Then C_ϕ is a composition operator on $H^2_X(D)$ and*

$$\|C_\phi\|^2 \le \frac{1 + |\phi(0)|}{1 - |\phi(0)|}.$$

Proof. Since $\phi\colon D \to D$ is analytic, for any $r_1 < 1$ there exists $r_2 < 1$ such that $\phi\colon |z| \le r_1 \to |z| \le r_2$. By [4, Theorem C, p. 89],

$$<f(\phi(r_1e^{i\theta}), e^{i\theta})> = \frac{1}{2\pi}\int_0^{2\pi} P(\phi(r_1e^{i\theta}), r_2e^{it}) <f(r_2e^{it}), e_j> dt,$$

where

$$P(\phi(r_1e^{i\theta}), r_2e^{it}) = \mathrm{Re}\left[\frac{r_2e^{it} + \phi(r_1e^{i\theta})}{r_2e^{it} - \phi(r_1e^{i\theta})}\right]$$

is the Poisson kernel. Since x^2 is a convex function, by Jensen's inequality, we have

$$| <f(\phi(r_1e^{i\theta})), e_j> |^2 \le \frac{1}{2\pi}\int_0^{2\pi} P(\phi(r_1e^{i\theta}), r_2e^{it})| <f(r_2e^{it}), e_j> |^2 dt.$$

Using Parseval's identity, we obtain

$$\begin{aligned}\|f(\phi(r_1e^{i\theta}))\|_X^2 &= \sum_{j\in\mathbb{N}} | <f(\phi(r_1e^{i\theta})), e_j> |^2\\ &\le \frac{1}{2\pi}\int_0^{2\pi} P(\phi(r_1e^{i\theta}), r_2e^{it}) \sum_{j\in\mathbb{N}} | <f(r_2e^{it}), e_j> |^2 dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} P(\phi(r_1e^{i\theta}), r_2e^{it}) \|f(r_2e^{it})\|_X^2\, dt\end{aligned}$$

Integrating with respect to θ, using Fubini's theorem to interchange the order of integration in the double integral and well known property of Poisson kernel that

$$\frac{1}{2\pi}\int_0^{2\pi} P(\phi(r_1e^{i\theta}), r_2e^{it})d\theta = P(\phi(0), r_2e^{it}) \le \frac{r_2 + |\phi(0)|}{r_2 - |\phi(0)|},$$

we get

$$\frac{1}{2\pi}\int_0^{2\pi} \|f(\phi(r_1e^{i\theta}))\|_X^2\, d\theta \le \frac{r_2 + |\phi(0)|}{r_2 - |\phi(0)|}\frac{1}{2\pi}\int_0^{2\pi} \|f(r_2e^{it})\|_X^2\, dt.$$

If $r_1 \to 1$, then $r_2 \to 1$, so that

$$\|C_\phi f\|_2^2 \le \frac{1 + |\phi(0)|}{1 - |\phi(0)|}\|f\|_2^2.$$

This implies that

$$\|C_\phi\|^2 \le \frac{1 + |\phi(0)|}{1 - |\phi(0)|},$$

hence the theorem. ∎

We next present a necessary and sufficient condition for an operator A on $H^2_X(D)$ to be a composition operator.

THEOREM 3.2. *Let A be an operator on $H^2_X(D)$. Then A is a composition operator if and only if for each $z \in D$ there exists unique $w \in D$ such that $A^* k_z^j = k_w^j$ for every $j \in \mathbb{N}$.*

Proof. If $A = C_\phi$, a composition operator, then

$$\begin{aligned} \ll f, A^* k_z^j \gg &= \ll C_\phi f, k_z^j \gg = E_z^j C_\phi f \\ &= E^j_{\phi(z)} C_\phi f = \ll f, k^j_{\phi(z)} \gg, \end{aligned}$$

for every $j \in \mathbb{N}$.

Conversely, suppose that for each $z \in D$ there exists unique $w \in D$ such that $A^* k_z^j = k_w^j$ for every $j \in \mathbb{N}$. Thus, if we define ϕ as $\phi(z) = w$, then

$$\begin{aligned} \phi(z) &= <\phi(z) e_1, e_1> = E_w^1(e_{11}) \\ &= \ll e_{11}, A^* k_z^1 \gg = \ll A e_{11}, k_z^1 \gg \\ &= E_z^1(A e_{11}) = <(A e_{11})(z), e_1> . \end{aligned}$$

This proves that ϕ is analytic and so, by [4, Theorem C, p. 76], $C_\phi f \in H^2_X(D)$, for every $f \in H^2_X(D)$.

Now

$$\begin{aligned} \ll Af, k_z^j \gg &= \ll f, A^* k_z^j \gg = \ll f, k^j_{\phi(z)} \gg \\ &= E^j_{\phi(z)}(f) = E_z^j(C_\phi f) = \ll C_\phi f, k_z^j \gg \end{aligned}$$

for every $(z, j) \in D \times \mathbb{N}$ and every $f \in H^2_X(D)$. Since $[k_z^j \colon (z, j) \in D \times \mathbb{N}]$ is dense in $H^2_X(D)$, we conclude that $A = C_\phi$. ∎

As an application of the above theorem, we obtain a lower bound for the norm of a composition operator.

COROLLARY 3.3. *If ϕ is an analytic self-map of the unit disc, then*

$$\sup_{z \in D} \frac{1 - |z|^2}{1 - |\phi(z)|^2} \leq \|C_\phi\|^2.$$

Proof. By Theorem 3.2

$$\frac{1-|z|^2}{1-|\phi(z)|^2} = \frac{\|k^j_{\phi(z)}\|_2^2}{\|k^j_z\|_2^2} = \frac{\|C^*_\phi k^j_z\|_2^2}{\|k^j_z\|_2^2} \leq \|C^*_\phi\|^2 = \|C_\phi\|^2$$

for every $(z,j) \in D \times \mathbb{N}$. This implies that

$$\sup_{z\in D} \frac{1-|z|^2}{1-|\phi(z)|^2} \leq \|C_\phi\|^2.$$

∎

4. Normal, unitary and co-isometric composition operators

In general, the adjoint of a composition operator may or may not be a composition operator. We give a necessary and sufficient condition on ϕ for which C^*_ϕ is also composition operator. The scalar-valued version of this result was proved by H.J. Schwartz [5] by using the technique of Fourier coefficients. Our method of proof is based on the generalized reproducing kernels.

Theorem 4.1. *C^*_ϕ, the adjoint of C_ϕ is a composition operator if and only if $\phi(z) = \alpha z$, $|\alpha| \leq 1$.*

Proof. We first suppose that $\phi(z) = \alpha z$, $|\alpha| \leq 1$. Let $\psi(z) = \bar{\alpha} z$. Then clearly ψ is an analytic mapping from D into itself. We shall show that $C^*_\phi = C_\psi$. Let $(z,j) \in D \times \mathbb{N}$. Then

$$(C^*_\phi)^* k^j_z(w) = C_\phi k^j_z(w) = k^j_z(\phi(w)) = k^j_{\bar{\alpha} z}(w) = k^j_{\psi(z)}(w)$$

for every $w \in D$, i.e., $(C^*_\phi)^* k^j_z = k^j_{\psi(z)}$. Hence, by Theorem 3.2, C^*_ϕ is a composition operator and $C^*_\phi = C_\psi$.

Conversely, suppose that $C^*_\phi = C_\psi$ for some ψ. Let $\phi(z) = \sum_{n=0}^\infty a_n z^n$ and $\psi(z) = \sum_{n=0}^\infty b_n z^n$. Since $C^*_\phi = C_\psi$, we have for any j,

$$\|C^*_\phi k^j_0\|_2^2 = \|C^*_\psi k^j_0\|_2^2 \Rightarrow \|k^j_{\phi(0)}\|_2^2 = 1 \Rightarrow \frac{1}{1-|\phi(0)|^2} = 1 \Rightarrow \phi(0) = 0 \Rightarrow a_0 = 0$$

Similarly, we can show that $b_0 = 0$. Hence, for any integer k, the first k Fourier coefficients of ϕ^k and ψ^k are zero.

Now for $n \geq 1$

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_0^{2\pi} \psi(e^{i\theta})e^{-in\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} <\psi(e^{i\theta})e_1, e^{in\theta}e_1> d\theta \\ &= \ll C_\psi e_{11}, e_{n1} \gg = \ll e_{11}, C_\phi e_{n1} \gg \\ &= \frac{1}{2\pi}\int_0^{2\pi} <e^{i\theta}e_1, \phi^n e_1> d\theta = \frac{1}{2\pi}\int_0^{2\pi} \overline{\phi^n e^{-i\theta}} d\theta = \bar{a}_1\delta_{n1}, \end{aligned}$$

where δ_{n1} is the Kronecker delta. Therefore $\psi(z) = \bar{a}_1 z$.

But $C_\psi^* = C_\phi$, so by first part of the theorem, we have $\phi(z) = a_1 z$. ∎

In the next theorem we present a criterion for the normality of a composition operator on $H_X^2(D)$.

THEOREM 4.2. *C_ϕ is normal if and only if $\phi(z) = \alpha z$, $|\alpha| \leq 1$.*

Proof. Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$. We first suppose that C_ϕ is normal. Then $\|C_\phi^* f\|_2 = \|C_\phi f\|_2$, for every $f \in H_X^2(D)$.

In particular, taking $f = k_0^j$, we get $\phi(0) = 0$. Hence for any integer k, the first k Fourier coefficients of ϕ^k are zero. Since $\{e_{m,n} \colon m, n \in \mathbb{N}\}$ is an orthonormal basis for $H_X^2(D)$, by Parseval's identity, we have

$$\begin{aligned} \|C_\phi^* e_{11}\|_2^2 &= \sum_{m,n} | \ll C_\phi^* e_{11}, e_{m,n} \gg |^2 = \sum_{m,n} | \ll e_{11}, C_\phi e_{m,n} \gg |^2 \\ &= \sum_{m,n} \left| \frac{1}{2\pi}\int_0^{2\pi} <e^{i\theta}e_1, \phi^m e_n> d\theta \right|^2 \\ &= \sum_{m,n} \left| \frac{1}{2\pi}\int_0^{2\pi} \overline{e^{-i\theta}\phi^m} <e_1, e_n> d\theta \right|^2 \\ &= \sum_{m} \left| \frac{1}{2\pi}\int_0^{2\pi} \overline{e^{-i\theta}\phi^m} d\theta \right|^2 = \sum_m |\bar{a}_1\delta_{m1}|^2 = |a_1|^2. \end{aligned}$$

Also

$$\begin{aligned} \|C_\phi^* e_{11}\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi} \|e_{11}(\phi(e^{i\theta}))\|_X^2 d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} |\phi(e^{i\theta}))|^2 d\theta. \end{aligned}$$

Therefore we have

$$|a_1|^2 = \sum_{n=1}^{\infty} |a_n|^2 \Rightarrow a_n = 0 \text{ for } n \geq 2.$$

Hence $\phi(z) = a_1 z$.

Conversely, suppose that $\phi(z) = \alpha z$, $|\alpha| \leq 1$. Then by Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z$. Therefore

$$\begin{aligned} C_\phi C_\psi f(z) &= C_\phi(f(\psi(z))) = C_\phi(f(\bar{\alpha} z)) \\ &= f(\alpha\bar{\alpha} z) = f(\psi(\alpha z)) \\ &= C_\psi f(\phi(z)) = C_\psi C_\phi f(z), \end{aligned}$$

for every $f \in H_X^2(D)$ and every $z \in D$. Hence $C_\phi C_\psi = C_\psi C_\phi$ and C_ϕ is normal. ∎

THEOREM 4.3. *C_ϕ is hermitian if and only if $\phi(z) = \alpha z$, where $\alpha \in \mathbb{R}$ and $|\alpha| \leq 1$.*

Proof. If C_ϕ is hermitian, then it is normal and hence, by Theorem 4.2 $\phi(z) = \alpha z$, $|\alpha| \leq 1$. Also, by Theorem 4.1, $C_\phi^* = C_\psi$, where $\phi(z) = \bar{\alpha} z$. But $C_\phi^* = C_\phi$, so $\phi = \psi$, which implies that $\alpha = \bar{\alpha}$, i.e., α is real.

Conversely, we suppose that $\phi(z) = \alpha z$, $\alpha \in \mathbb{R}$ and $|\alpha| \leq 1$. Then, by Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z = \alpha z = \phi(z)$. Thus $C_\phi^* = C_\phi$, i.e., C_ϕ is hermitian. ∎

THEOREM 4.4. *C_ϕ is a unitary operator if and only if $\phi(z) = \alpha z$, $|\alpha| = 1$.*

Proof. We first suppose that $\phi(z) = \alpha z$, $|\alpha| = 1$. Then, by Theorems 4.2 and 4.1, C_ϕ is normal and $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z$. Therefore,

$$C_\phi C_\phi^* f(z) = f(\psi(\phi(z))) = f(z),$$

for every $f \in H_X^2(D)$ and every $z \in D$, which implies $C_\phi C_\phi^* = I$, the identity operator. Hence, by normality of C_ϕ, we conclude that C_ϕ is unitary.

Conversely, suppose that C_ϕ is unitary. Then C_ϕ is normal. Hence by Theorem 4.2, $\phi(z) = \alpha z$, $|\alpha| \leq 1$. By Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z$, and $C_\phi C_\phi^* = I$ implies that

$$\begin{aligned} & C_\phi C_\phi^* k_z^j = k_z^j, \text{ for every } (z, j) \in D \times \mathbb{N} \\ & \Rightarrow k_z^j(\bar{\alpha}\alpha w) = k_z^j(w), \text{ for every } w \in D \\ & \Rightarrow \bar{\alpha}\alpha w = w, \text{ for every } w \in D \\ & \Rightarrow |\alpha| = 1 \end{aligned}$$

Hence $\phi(z) = \alpha z$, $|\alpha| = 1$. ∎

THEOREM 4.5. *C_ϕ^* is an isometry if and only if $\phi(z) = \alpha z$, $|\alpha| = 1$.*

Proof. If C_ϕ^* is an isometry, then $\|C_\phi^* f\|_2 = \|f\|_2$, for every $f \in H_X^2(D)$. In particular, taking $f = k_z^j$ and using Theorem 3.2, we obtain $\|k_{\phi(z)}^j\|_2^2 = \|k_z^j\|_2^2$. So

$$\frac{1}{1 - |\phi(z)|^2} = \frac{1}{1 - |z|^2}.$$

This implies that $\phi(z) = \alpha z$, $|\alpha| = 1$.

Conversely, if $\phi(z) = \alpha z$, $|\alpha| = 1$, then, by Theorem 4.1, $C_\phi^* = C_\psi$, where $\psi(z) = \bar{\alpha} z$. Thus, if $f \in H_X^2(D)$ and $f(z) = \sum_{n=0}^\infty a_n z^n$, then

$$\begin{aligned} \|C_\phi^* f\|_2^2 &= \|C_\psi f\|_2^2 \quad \text{for every } f \in H_X^2(D) \\ &= \sum_{n=0}^\infty \|a_n\|_X^2 = \|f\|_2^2. \end{aligned}$$

Hence C_ϕ^* is an isometry. ∎

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