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On Homomorphisms of Groups of Integer-valued Functions

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1. Introduction

Let X be a topological space, and let $C(X, \mathbb{Z})$ be the group of all continuous (equivalently, locally constant) integer-valued functions on X. For every function f on X let $\operatorname{Ran} f = f(X)$ be the range of f. Denote by $d(f)$ the diameter of $\operatorname{Ran} f$.

Let $\psi : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ be a homomorphism of additive groups. We say that ψ does not increase range if $\operatorname{Ran} \psi(f) \subset \operatorname{Ran} f$ for every $f \in C(X, \mathbb{Z})$. We say that ψ does not increase diameter if $d(\psi(f)) \leq d(f)$ for every $f \in C(X, \mathbb{Z})$. By a *map* between topological spaces we always mean a continuous map. For every map $h : X \to X$ the dual endomorphism $h^* : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ defined by $h^*(f)(x) = f(h(x))$ ($f \in C(X, \mathbb{Z})$, $x \in X$) does not increase range. If $t : C(X, \mathbb{Z}) \to \mathbb{Z}$ is any homomorphism and $\tau = \pm 1$, the endomorphism $f \mapsto \tau h^*(f) + t(f) \cdot 1$ of $C(X, \mathbb{Z})$ does not increase diameter. The aim of the paper is to prove that for N-compact spaces X the converse is true: every homomorphism $\psi : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ which does not increase range or diameter is of the form described above. Recall that a space is N-compact if it is homeomorphic to a closed subspace of a power of a countable discrete space.

Let A be a finite set of integers. Call the interval $[\min A, \max A]$ the *convex hull* of A and denote it by $\operatorname{Conv} A$. A homomorphism $\psi : C(X, \mathbb{Z}) \to C(Y, \mathbb{Z})$ is *disjointness preserving* if $f \cdot g \equiv 0$ implies $\psi(f) \cdot \psi(g) \equiv 0$ (see [1] for

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a discussion of disjointness preserving homomorphisms of groups of integer-valued functions). A Hausdorff topological space X is 0-*dimensional* if closed-and-open sets form a base of X. H.Ohta posed the following question (we are grateful to M.Sanchis for communicating this question to us). Let X and Y be zero-dimensional spaces, and let $\psi : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ be a homomorphism of additive groups such that $\operatorname{Ran}\psi(f)$ is contained in $\operatorname{Conv}\operatorname{Ran} f$ for every bounded $f \in C(X,\mathbb{Z})$. Does it follow that ψ is disjointness preserving? It follows from Theorem 1.1 that the answer is positive (Corollary 1.2).

Let $C_b(X,\mathbb{Z})$ be the subgroup of $C(X,\mathbb{Z})$ consisting of all bounded functions.

THEOREM 1.1. *Let X and Y be topological spaces, and let $\psi : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ be a homomorphism of additive groups. Suppose that X is N-compact.*

(1) *If $\operatorname{Ran}\psi(f) \subset \operatorname{Conv}\operatorname{Ran} f$ for every $f \in C_b(X,\mathbb{Z})$, then there exists a map $h : Y \to X$ such that $\psi(f) = f \circ h$ for every $f \in C(X,\mathbb{Z})$.*
(2) *If $\operatorname{diam}\operatorname{Ran}\psi(f) \le \operatorname{diam}\operatorname{Ran} f$ for every $f \in C_b(X,\mathbb{Z})$, then there exist a map $h : Y \to X$, a homomorphism $t : C(X,\mathbb{Z}) \to \mathbb{Z}$ and $\tau = \pm 1$ such that $\psi(f)(y) = \tau f(h(y)) + t(f)$ for every $f \in C(X,\mathbb{Z})$ and $y \in Y$.*

COROLLARY 1.2. *Let X and Y be zero-dimensional spaces, and let $\psi : C(X,\mathbb{Z}) \to C(Y,\mathbb{Z})$ be a homomorphism of additive groups. If $\operatorname{Ran}\psi(f) \subset \operatorname{Conv}\operatorname{Ran} f$ for every $f \in C_b(X,\mathbb{Z})$, then ψ is disjointness-preserving.*

Theorem 1.1 follows from Theorem 1.3 which describes homomorphisms $C(X,\mathbb{Z}) \to \mathbb{Z}$ satisfying some boundedness conditions.

THEOREM 1.3. *Let X be an N-compact space, and let $t : C(X,\mathbb{Z}) \to \mathbb{Z}$ be a non-zero homomorphism of additive groups.*

(1) *If $|t(f)| \le \|f\| = \max\{|f(x)| : x \in X\}$ for every $f \in C_b(X,\mathbb{Z})$, then there exist $x \in X$ and $\tau = \pm 1$ such that $t(f) = \tau f(x)$ for every $f \in C(X,\mathbb{Z})$.*
(2) *If $|t(f)| \le \operatorname{diam}\operatorname{Ran} f$ for every $f \in C_b(X,\mathbb{Z})$, then there exist $x, y \in X$ such that $t(f) = f(x) - f(y)$ for every $f \in C(X,\mathbb{Z})$.*

For a compact space X denote by $C(X)$ the complex Banach space of all complex continuous functions on X. The following theorem was proved in [3]: if X is a first-countable compact space and $\psi : C(X) \to C(X)$ is a linear bijection such that $\operatorname{diam}\operatorname{Ran}\psi(f) = \operatorname{diam}\operatorname{Ran} f$ for every $f \in C(X)$, then there exist a self-homeomorphism $h : X \to X$, a complex number τ with $|\tau| = 1$ and a linear functional $t : C(X) \to \mathbb{C}$ such that $\psi(f)(x) = \tau f(h(x)) + t(f)$ for every $f \in C(X)$ and $x \in X$. We prove that the assumption that X be first-countable is superfluous (Theorem 5.1). In Section 2 we discuss the notion of N-compactness and explain how to deduce Corollary 1.2 from Theorem 1.1. Theorems 1.1 and 1.3 are proved in Sections 4 and 3, respectively.

2. N-compact spaces

There are several equivalent characterizations of N-compact spaces, (see [4]). For the reader's convenience we prove some of them in this section.

If X is a 0-dimensional space, we denote by $\beta_D X$ the 0-dimensional compactification of X characterized by the property that every bounded continuous function $f : X \to \mathbb{Z}$ extends uniquely over $\beta_D X$. For every 0-dimensional compactification bX of X there exist a map $\beta_D X \to bX$ which is identical on X. We call $\beta_D X$ the maximal 0-dimensional compactification of X. If Y is another 0-dimensional space, every map $f : X \to Y$ extends uniquely to a map $\beta_D f : \beta_D X \to \beta_D Y$. If Y is zero-dimensional in the sense of the covering dimension dim, then βY is zero-dimensional, whence $\beta Y = \beta_D Y$. In particular, if Y is discrete, then $\beta_D Y = \beta Y$.

If $X \subset Y$ and $x \in Y \setminus X$, we say that x is *G_δ-separated* from X if there is a G_δ subset A of Y such that $x \in A \subset Y \setminus X$.

THEOREM 2.1. *Let X be a 0-dimensional space. The following conditions are equivalent:*

1. *X is N-compact, that is homeomorphic to a closed subspace of a power of a countable discrete space;*
2. *there exists a 0-dimensional compactification bX of X such that every point $x \in bX \setminus X$ is G_δ-separated from X;*
3. *every point $x \in \beta_D X \setminus X$ is G_δ-separated from X in the space $\beta_D X$;*
4. *for every $x \in \beta_D X \setminus X$ there exists a map $f : X \to \mathbb{Z}$ such that $\beta_D f(x) \in \beta\mathbb{Z} \setminus \mathbb{Z}$.*

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{P}$ be the class of all spaces X for which the property (2) holds. It is clear that: (a) countable discrete spaces are in $\mathcal{P}$; (b) $\mathcal{P}$ is closed under arbitrary (infinite) products; (c) if $X \in \mathcal{P}$ and Y is a closed subspace of X, then $Y \in \mathcal{P}$. It follows that all N-compact spaces are in $\mathcal{P}$.

(2) $\Rightarrow$ (3). Note that the natural map $\beta_D X \to bX$ sends the remainder $\beta_D X \setminus X$ onto the remainder $bX \setminus X$.

(3) $\Rightarrow$ (4). Let $x \in \beta_D X \setminus X$. There exists a decreasing sequence $V_1, V_2, \dots$ of clopen neighbourhoods of x such that $\bigcap V_i \subset \beta_D X \setminus X$. Let $\mathbb{Z} \cup \{\omega\}$ be a one-point compactification of $\mathbb{Z}$. Set $V_0 = \beta_D X$ and define the map $g : \beta_D X \to \mathbb{Z} \cup \{\omega\}$ by: $g(y) = n$ if $y \in V_n \setminus V_{n+1}$ and $g(y) = \omega$ if $y \in \bigcap V_i$. We have $g(X) \subset \mathbb{Z}$, let f be the restriction of g onto X. Let $h : \beta\mathbb{Z} \to \mathbb{Z} \cup \{\omega\}$ be the natural map. We have $g = h \circ \beta_D f$. Since $g(x) = \omega$, it follows that $\beta_D f(x) \in \beta\mathbb{Z} \setminus \mathbb{Z}$.

(4) $\Rightarrow$ (1). Let $\mathcal{F} = \mathcal{C}(\mathcal{X}, \mathbb{Z})$. The diagonal product of the family $\{\beta_D f : f \in \mathcal{F}\}$ is an imbedding $g : \beta_D X \to (\beta\mathbb{Z})^{\mathcal{F}}$. The condition (4) implies that $g(X) = g(\beta_D X) \cap \mathbb{Z}^{\mathcal{F}}$. Thus X is homeomorphic to a closed subspace of a power of $\mathbb{Z}$. ∎

Given a zero-dimensional space X, let νX be the *N-compactification* of X. This space is characterized by the following properties: νX is N-compact, X is dense in νX, and the restriction homomorphism $r_X : C(\nu X, \mathbb{Z}) \to C(X, \mathbb{Z})$ is bijective. We can take for νX the space of all $y \in \beta_D X$ which are not G_δ-separated from X. The notion of N-compactification can be used to deduce Corollary 1.2 from Theorem 1.1. Let $\psi : C(X, \mathbb{Z}) \to C(Y, \mathbb{Z})$ be a homomorphism satisfying the condition of Corollary 1.2. In virtue of Theorem 1.1, the homomorphism $\psi r_X : C(\nu X, \mathbb{Z}) \to C(Y, \mathbb{Z})$ has the form $f \mapsto f \circ h$ for some $h : Y \to \nu X$. It follows that ψr_X is disjointness preserving, and hence so is ψ.

3. Proof of Theorem 1.3

We first prove Theorem 1.3 for compact spaces. Let X be compact and 0-dimensional, and let $t : C(X, \mathbb{Z}) \to \mathbb{Z}$ be a homomorphism such that $|t(f)| \le \|f\|$ (respectively, $|t(f)| \le d(f)$) for every $f \in C(X, \mathbb{Z})$. Let E be the linear space over $\mathbb{Q}$ of all locally constant functions on X with rational values. We have $E = \mathbb{Q} \otimes C(X, \mathbb{Z})$, hence the homomorpism t can be extended to a $\mathbb{Q}$-linear functional $t_1 : E \to \mathbb{Q}$. Clearly the inequality $|t_1(f)| \le \|f\|$ (respectively, $|t_1(f)| \le d(f) \le 2\|f\|$) holds true for every $f \in E$. Since $\dim X = 0$, E is dense in the Banach space $C(X, \mathbb{R})$, and the functional t_1 extends to an $\mathbb{R}$-linear functional t_2 on $C(X, \mathbb{R})$. The norm of t_2 is ≤ 1 (respectively, ≤ 2), so we can identify t_2 with a regular Borel measure on X. Denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between functions and measures on X. For every integer-valued continuous function f on X we have $\langle f, t_2 \rangle = t(f) \in \mathbb{Z}$.

According to Lemma 3.1 below, the measure t_2 can be written as a linear combination $\sum n_i x_i$ of points of X with integer coefficients. Here we identify each point $x \in X$ with a measure of mass 1 concentrated at x. Since $\sum |n_i| = \| \sum n_i x_i \| = \|t_2\| \leq 2$, it follows that $t_2 = \pm x$ or $t_2 = \pm x \pm y$ for some $x, y \in X$. In the second case the signs of x and y must be different, since from the inequality $|t(f)| \leq d(f)$ it follows that $t(f) = 0$ whenever f is constant.

To complete the proof of Theorem 1.3 for compact spaces, it remains to establish the following lemma:

LEMMA 3.1. *Let X be a 0-dimensional compact space, and let μ be a regular Borel measure on X such that $\langle f, \mu \rangle \in \mathbb{Z}$ for every $f \in C(X, \mathbb{Z})$. Then μ can be written as a finite linear combination of points of X with integer coefficients.*

Proof. If V is clopen in X, then $\mu(V) = \langle \chi_V, \mu \rangle \in \mathbb{Z}$, where χ_V is the characteristic function of V. It follows that for every $x \in X$ we have $\mu(\{x\}) \in \mathbb{Z}$, since $\mu(\{x\}) = \lim \mu(V)$, where V runs over the collection of clopen neighbourhoods of x. Call a point $x \in X$ an atom if $\mu(\{x\}) \neq 0$. There are finitely many atoms, and if we subtract from μ a linear combination of atoms with integer coefficients, we obtain an atomless measure satisfying the condition of the lemma. Thus we may assume that μ is atomless. Then for every $x \in X$ there exists an open neighbourhood U of x such that for every clopen neighbourhood V of x contained in U we have $|\mu(V)| < 1$ and hence $\mu(V) = 0$. It follows that the restriction of μ onto U is zero. Thus μ is locally zero and hence zero. ∎

For every abelian group A let A^* denote the group $\mathrm{Hom}(A, \mathbb{Z})$. If $f : A \to B$ is a homomorphism, the dual homomorphism $f^* : B^* \to A^*$ is defined by $f^*(g) = g \circ f$. Let $A_{\mathbb{Z}}(X)$ be the subgroup of $C(X, \mathbb{Z})^*$ generated by the image of the natural embedding $X \to C(X, \mathbb{Z})^*$. It is easy to see that $A_{\mathbb{Z}}(X)$ can be identified with the free abelian group on X.

PROPOSITION 3.2. *Let X be a $0-$dimensional space, and let $i : C(\beta_D X, \mathbb{Z}) \to C(X, \mathbb{Z})$ be the restriction homomorphism. Then the homomorphism $i^* : C(X, \mathbb{Z})^* \to C(\beta_D X, \mathbb{Z})^*$ is injective. If X is N-compact, then $(i^*)^{-1}(A_{\mathbb{Z}}(\beta_D X)) = A_{\mathbb{Z}}(X)$.*

Proof. Consider first the case when X is a discrete countable space. Then $C(X, \mathbb{Z})$ is a countable power of the group Z, and it is known that in this case $C(X, \mathbb{Z})^* = A_{\mathbb{Z}}(X)$ [2, Corollary 94.6]. It follows that the proposition holds true in this case. Let us reduce the general case to the case of a countable discrete space. Let $f : X \to Y$ be a map of X to a countable discrete space Y. Consider the commutative diagram

$$\begin{array}{ccc} C(X,\mathbb{Z})^* & \xrightarrow{i^*} & C(\beta_D X,\mathbb{Z})^* \\ {\scriptstyle f^{**}}\big\downarrow & & \big\downarrow{\scriptstyle (\beta_D f)^{**}} \\ C(Y,\mathbb{Z})^* & \xrightarrow[i_Y^*]{} & C(\beta Y,\mathbb{Z})^*, \end{array}$$

where $i_Y : C(\beta Y,\mathbb{Z}) \to C(Y,\mathbb{Z})$ is the restriction homomorphism. We saw that the lower horizontal arrow is injective. A routine verification shows that in order to deduce that the upper horizontal arrow is also injective, it suffices to prove that for every non-zero element $t \in C(X,\mathbb{Z})^*$ there exists a map f of X to a countable discrete space Y such that $f^{**}(t) \neq 0$. Put $Y = \mathbb{Z}$, and pick $f \in C(X,\mathbb{Z})$ so that $t(f) \neq 0$. Then $f^{**}(t)$ is a non-zero element of $C(\mathbb{Z},\mathbb{Z})^*$, since $f^{**}(t)(\mathrm{id}_{\mathbb{Z}}) = t(f^*(\mathrm{id}_{\mathbb{Z}})) = t(f) \neq 0$. Assume that X is N-compact. Since $i^*(A_{\mathbb{Z}}(X)) \subset A_{\mathbb{Z}}(\beta_D X)$, we have $A_{\mathbb{Z}}(X) \subset (i^*)^{-1}(A_{\mathbb{Z}}(\beta_D X))$. Suppose that the last inclusion is proper. Then there exists $s \in C(X,\mathbb{Z})^*$ such that $s \notin A_{\mathbb{Z}}(X)$ and $i^*(s) \in A_{\mathbb{Z}}(\beta_D X)$. We can write $i^*(s)$ as a sum $t+t'$, where t is a linear combination of points of $\beta_D X \setminus X$ and t' is a linear combination of points of X. Since i^* is injective, as we have proved in the preceding paragraph, and $s \notin A_{\mathbb{Z}}(X)$, it follows that $t \neq 0$. Since t' is in the range of i^*, so is $t = i^*(s) - t'$. Thus in order to prove that $(i^*)^{-1}(A_{\mathbb{Z}}(\beta_D X)) = A_{\mathbb{Z}}(X)$, it suffices to show that the range of i^* contains no non-zero linear combination $t = \sum_{i=1}^n k_i p_i$ ($k_i \in \mathbb{Z}$, $p_i \in \beta_D X \setminus X$) of points of the remainder $\beta_D X \setminus X$. According to Theorem 2.1, there exist maps $f_i : X \to \mathbb{Z}$ such that $\beta_D f_i(p_i) \in \beta\mathbb{Z} \setminus \mathbb{Z}$, $1 \le i \le n$. For every pair of distinct indices i and j pick a function $g_{ij} : X \to \mathbb{Z}$ such that $\beta_D g_{ij}(p_i) \neq \beta_D g_{ij}(p_j)$. Let $\mathcal{F}$ be the finite set of all the functions f_i and g_{ij}. The diagonal product f of $\mathcal{F}$ is a map of X to a countable discrete space Y such that the points $q_i = \beta_D f(p_i)$, $1 \le i \le n$, are pairwise distinct and belong to $\beta Y \setminus Y$. It follows that $(\beta_D f)^{**}(t) = \sum_{i=1}^n k_i q_i$ is a non-zero linear combination of points of $\beta Y \setminus Y$. We have observed that the proposition is true for Y. Considering the same commutative diagram as above, we see that $(\beta_D f)^{**}(t)$ does not belong to the range of the lower arrow. It follows that t does not belong to the range of the upper arrow. ∎

We now prove Theorem 1.3 in full generality, reducing the general case to the compact case considered above. Let X be an N-compact space, and let $t : C(X,\mathbb{Z}) \to \mathbb{Z}$ be a homomorphism satisfying one of the conditions of Theorem 1.3. Let $i : C(\beta_D X, \mathbb{Z}) \to C(X, \mathbb{Z})$ be, as above, the restriction homomorphism, and let $t' = t \circ i \in C(\beta_D X, \mathbb{Z})^*$. Applying the compact case of Theorem 1.3 to t', we see that the functional t' on $C(\beta_D X, \mathbb{Z})$ is represented, up to a sign, by a point $x \in \beta_D X$ or, respectively, by a formal difference $x - y$, where $x, y \in \beta_D X$. In any case we have $i^*(t) = t' \in A_{\mathbb{Z}}(\beta_D X)$, and Proposition 3.2 implies that $t \in A_{\mathbb{Z}}(X)$. It follows that $x \in X$ or, respectively, that $x, y \in X$. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.1

Let X and Y be topological spaces such that X is N-compact, and let $\psi : C(X, \mathbb{Z}) \to C(Y, \mathbb{Z})$ be a homomorphism such that $\operatorname{Ran} \psi(f) \subset \operatorname{Conv} \operatorname{Ran} f$ for every $f \in C(X, \mathbb{Z})$. We must show that there exists a map $h : Y \to X$ such that $\psi(f) = f \circ h$ for every $f \in C(X, \mathbb{Z})$. Fix $y \in Y$, and consider the functional $t : C(X, \mathbb{Z}) \to \mathbb{Z}$ defined by $t(f) = \psi(f)(y)$. Since $\psi(f)(y) \in \operatorname{Ran} \psi(f) \subset \operatorname{Conv} \operatorname{Ran} f$, we have $|t(f)| \le \|f\|$ for every $f \in C(X, \mathbb{Z})$. Theorem 1.3 implies that there exist $x \in X$ and $\tau = \pm 1$ such that $t(f) = \tau f(x)$ for all $f \in C(X, \mathbb{Z})$. Since ψ maps the constant 1 to the constant 1, it follows that $\tau = 1$. Put $h(y) = x$. We have $\psi(f)(y) = f(h(y))$ for every $f \in C(X, \mathbb{Z})$ and $y \in Y$. Since for every $f \in C(X, \mathbb{Z})$ the function $y \to f(h(y))$, being equal to $\psi(f)$, is continuous on Y, it follows that $h : Y \to X$ is continuous. This proves the first part of Theorem 1.1.

To prove the second part of Theorem 1.1, we use the following lemma:

Lemma 4.1. *Let G_1 and G_2 be abelian groups, X_i be a free subset of G_i, and $T_i = \{x - y : x, y \in X_i\} \subset G_i$, $i = 1, 2$. Let $f : G_1 \to G_2$ be a homomorphism such that $f(T_1) \subset T_2$. Then there exists a map $h : X_1 \to X_2$ and $\tau = \pm 1$ such that $f(x - y) = \tau(h(x) - h(y))$ for all $x, y \in X$.*

We say that a subset X of an abelian group G is free if no non-trivial linear combination of elements of X with integer coefficients is zero.

Proof. Fix $a \in X_1$ and put $g(x) = f(x - a)$ for every $x \in X_1$. Then g is a map from X_1 to G_2 such that $f(x - y) = g(x) - g(y)$ for all $x, y \in X_1$. Put $Y = g(X_1)$. We have $0 \in Y \subset Y - Y = f(T_1) \subset T_2$. For every $c \in X_2$ let $S_c = \{x - c : x \in X_2\} \subset T_2$. It is easy to see that every subset A of T_2 such that $A - A \subset T_2$ is contained in S_c or $-S_c$ for some $c \in X_2$. In particular, there exists $c \in X_2$ such that $Y \subset S_c$ or $Y \subset -S_c$. If $Y \subset S_c$, put $\tau = 1$ and $h(x) = g(x) + c$. If $Y \subset -S_c$, put $\tau = -1$ and $h(x) = c - g(x)$. In any case we have $h(X_1) \subset c + S_c \subset X_2$ and $f(x-y) = g(x) - g(y) = \tau(h(x) - h(y))$ for all $x, y \in X$. ∎

We now prove the second part of Theorem 1.1. Let X and Y be topological spaces such that X is N-compact, and let $\psi : C(X, \mathbb{Z}) \to C(Y, \mathbb{Z})$ be a homomorphism of additive groups such that $\operatorname{diam} \operatorname{Ran} \psi(f) \le \operatorname{diam} \operatorname{Ran} f$ for every $f \in C(X, \mathbb{Z})$. We must show that there exist a map $h : Y \to X$, a homomorphism $t : C(X, \mathbb{Z}) \to \mathbb{Z}$ and $\tau = \pm 1$ such that $\psi(f)(y) = \tau f(h(y)) + t(f)$ for every $f \in C(X, \mathbb{Z})$ and $y \in Y$. Let $G_1 = C(Y, \mathbb{Z})^*$ and $G_2 = C(X, \mathbb{Z})^*$. Identify X with its image under the natural embedding $i : X \to G_2$, and similarly for Y and G_1. As above, let $\psi^* : G_1 \to G_2$ be the homomorphism defined by $\psi^*(t) = t \circ \psi$. Let $T_1 = \{x - y : x, y \in Y\} \subset G_1$ and $T_2 = \{x - y : x, y \in X\} \subset G_2$. We claim that $\psi^*(T_1) \subset T_2$. To see this, fix $x, y \in Y$, and let $t_{x,y} = \psi^*(x - y)$. The homomorphism $t_{x,y} : C(X, \mathbb{Z}) \to \mathbb{Z}$ is defined by $t_{x,y}(f) = \psi(f)(x) - \psi(f)(y)$. We have $|t_{x,y}(f)| \le \operatorname{diam} \operatorname{Ran} \psi(f) \le \operatorname{diam} \operatorname{Ran} f$ for every $f \in C(X, \mathbb{Z})$. In virtue of Theorem 1.3 there exist $a, b \in X$ such that $t_{x,y}(f) = f(a) - f(b)$ for every f. This means that $t_{x,y} = a - b \in T_2$. We have verified that $\psi^*(T_1) \subset T_2$.

Lemma 4.1 implies that there exist a map $h : Y \to X$ and $\tau = \pm 1$ such that $t_{x,y} = \tau(h(x) - h(y))$ for every $x, y \in Y$. This means that $\psi(f)(x) - \psi(f)(y) = \tau(f(h(x)) - f(h(y)))$ for every $f \in C(X, \mathbb{Z})$ and $x, y \in Y$. It follows that the number $\psi(f)(x) - \tau f(h(x))$ does not depend on $x \in Y$. Denote this number by $t(f)$. It is clear that the map $t : C(X, \mathbb{Z}) \to \mathbb{Z}$ is a homomorphism. We have $\psi(f)(x) = \tau f(h(x)) + t(f)$ ($f \in C(X, \mathbb{Z})$, $x \in Y$), as required.

5. Diameter-preserving linear bijections of $C(X)$

For a compact space X we denote by $M(X)$ the Banach dual of $C(X)$. Elements of $M(X)$ can be identified with regular complex Borel measures on X. Let X and Y be compact, and let $\psi : C(X) \to C(Y)$ be a norm-preserving linear bijection. Then there exists a homeomorphism $h : Y \to X$ and a function $\alpha : Y \to \mathbb{C}$ such that $|\alpha(x)| = 1$ for every $x \in X$ and $\psi(f)(y) = \alpha(y) f(h(y))$. This easily follows from the following two facts: (1) the dual map ψ^* establishes a linear bijection between the unit balls in the spaces $M(X)$ and $M(Y)$; (2) the extreme points of the unit ball in $M(X)$ are of the form αx, where $\alpha \in \mathbb{C}$, $|\alpha| = 1$ and $x \in X$. We prove a similar result for linear bijections which preserve the diameter of the range.

THEOREM 5.1. *Let X and Y be compact spaces, and let $\psi : C(X) \to C(Y)$ be a linear bijection (not necessarily continuous) such that* $\operatorname{diam}\operatorname{Ran}\psi(f) = \operatorname{diam}\operatorname{Ran} f$ *for every $f \in C(X)$. Then there exist a homeomorphism $h : Y \to X$, a complex number α with $|\alpha| = 1$ and a linear functional $t : C(X) \to \mathbb{C}$ such that $\psi(f)(y) = \alpha f(h(y)) + t(f)$ for every $f \in C(X)$ and $y \in Y$. If ψ is continuous, then so is t, hence there exist a complex measure μ on X such that $t(f) = \int_X f\mu$ for every $f \in C(X)$.*

If E is complex locally convex space and $A \subset E$, the *polar* of A is the set $A^\circ = \{y \in E^* : \operatorname{Re}\langle x, y\rangle \le 1 \text{ for all } x \in A\}$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing between E and E^*. The polar $A^\circ \subset E$ of a subset $A \subset E^*$ is defined similarly. If $A \subset E^*$ and $0 \in A$, the closed convex hull of A with respect to the weak* topology on E^* is equal to $A^{\circ\circ}$ [5, Ch. 4,Theorem 1.5].

For a compact space X let $M_0(X) = \{\mu \in M(X) : \mu(X) = 0\}$. We consider X as a subspace of $M(X)$, identifying each point $x \in X$ with the atomic measure of mass 1 concentrated at x.

LEMMA 5.2. *Let X be a compact space containing more than one point, and let $T = \{\alpha(x - y) : x, y \in X,\ \alpha \in \mathbb{C},\ |\alpha| = 1\} \subset M_0(X)$. Let $T^{\circ\circ}$ be the closed convex hull of T with respect to the weak* topology on $M_0(X)$. Then the set of extreme points of $T^{\circ\circ}$ is equal to $T \setminus \{0\}$.*

Proof. The sets T and $T^{\circ\circ}$ are compact. According to the Milman theorem [5, Ch. 2, Theorem 10.5], all extreme points of $T^{\circ\circ}$ belong to T. Plainly 0 is not an extreme point of $T^{\circ\circ}$. We must prove that every non-zero element $\alpha(x-y)$ of T is an extreme point of $T^{\circ\circ}$. Since T and $T^{\circ\circ}$ are invariant under multiplication by complex numbers of absolute value 1, we may assume that $\alpha = 1$. Let $B = \{\mu \in M_0(X) : \|\mu\| = |\mu|(X) \le 2\}$ be the ball of radius 2 in $M_0(X)$; here $|\mu|$ stands for the variation of μ. Since $T^{\circ\circ} \subset B$, it suffices to prove that $x - y$ is an extreme point of B. Suppose that $\lambda \in M_0(X)$ and $x - y \pm \lambda \in B$. We must show that $\lambda = 0$. Write λ in the form $\lambda = \beta x + \gamma y + \nu$, where $\nu \in M(X)$ is such that $\nu(\{x\}) = \nu(\{y\}) = 0$. We have

$$
\begin{aligned}
2 \ge \|x - y + \lambda\| &= \|(1+\beta)x - (1-\gamma)y + \nu\| \\
&= |1+\beta| + |1-\gamma| + \|\nu\|
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
2 \ge \|x - y - \lambda\| &= \|(1-\beta)x - (1+\gamma)y - \nu\| \\
&= |1-\beta| + |1+\gamma| + \|\nu\|.
\end{aligned}
\tag{2}
$$

Adding these inequalities, we get

$$4 \geq |1+\beta| + |1-\beta| + |1-\gamma| + |1+\gamma| + 2\|\nu\|.$$

Since $|1+\beta| + |1-\beta| \geq 2$ and $|1+\gamma| + |1-\gamma| \geq 2$, it follows that $\|\nu\| = 0$. Hence $\nu = 0$ and $\lambda = \beta x + \gamma y$. Since $\lambda \in M_0(X)$, we must have $\beta = -\gamma$. From the inequalities (1) and (2) it follows that $|1+\beta| \leq 1$ and $|1-\beta| \leq 1$. Hence $\beta = 0$, $\gamma = 0$ and $\lambda = 0$. ∎

Proof of Theorem 5.1. Let E (respectively F) be the Banach quotient of $C(X)$ (respectively $C(Y)$) by the one-dimensional subspace of constants. Since ψ maps constants to constants, there exist a map $\Lambda : E \to F$ such that $\Lambda(\tilde{f}) = \widetilde{\psi(f)}$ for every $f \in C(X)$, where the wave denotes the class of a function modulo constants. We claim that Λ is continuous. Let $u \in E$ be such that $\|u\| < 1$. Pick $f \in C(X)$ so that $\tilde{f} = u$ and $\|f\| < 1$. Then $\operatorname{Ran} f$ is contained in the unit disc, hence $\operatorname{diam} \operatorname{Ran} \psi(f) = \operatorname{diam} \operatorname{Ran} f \leq 2$. Pick $c \in \operatorname{Ran} \psi(f)$ and put $g = \psi(f) - c \in C(Y)$. Then $\|g\| \leq 2$ and $\tilde{g} = \widetilde{\psi(f)} = \Lambda(\tilde{f}) = \Lambda(u)$, hence $\|\Lambda(u)\| \leq \|g\| \leq 2$. Thus $\|\Lambda\| \leq 2$. The Banach dual E^* of E can be identified with the hyperplane $M_0(X)$ in $M(X)$ of measures of full mass 0. Let $T_1 = \{\alpha(x-y) : x, y \in X,\ \alpha \in \mathbb{C},\ |\alpha| = 1\} \subset E^* = M_0(X)$, and define $T_2 \subset F^* = M_0(Y)$ similarly, replacing X by Y. The polar $T_1^\circ \subset E$ of T_1 is equal to $\{\tilde{f} \in E : f \in C(X),\ d(f) \leq 1\}$. Similarly, $T_2^\circ = \{\tilde{f} \in F : f \in C(Y),\ d(f) \leq 1\}$. Since ψ is diameter-preserving, Λ establishes a bijection between T_1° and T_2°. It follows that the map $\Lambda^* : M_0(Y) \to M_0(X)$ dual to Λ establishes a bijection between $T_2^{\circ\circ}$ and $T_1^{\circ\circ}$. Lemma 5.2 implies that $\Lambda^*(T_2) = T_1$.

Fix $a \in Y$ and put $g(y) = \Lambda^*(y-a)$ for every $y \in Y$. Then g is a map from Y to T_1 such that $\Lambda^*(x-y) = g(x) - g(y)$ for all $x, y \in Y$. Put $P = g(Y)$. We have $P \subset T_1$ and $P - P \subset \Lambda^*(T_2) = T_1$. For every $c \in X$ and $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ let $\alpha S_c = \{\alpha(x-c) : x \in X\} \subset T_1$. It is easy to see that every subset A of T_1 such that $A - A \subset T_1$ is contained in αS_c for some $c \in X$ and $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Indeed, pick a point $p_1 = \alpha x - \alpha y \in A$. Every other point $p_2 \in A$ must be of the form $\alpha x - \alpha z$ or $\alpha z - \alpha y$, since $p_1 - p_2 \in T_1$. Suppose, for example, that $p_2 = \alpha x - \alpha z$. Then any point $p_3 \in A$ must be of the form $\alpha x - \alpha u$, since both $p_1 - p_3$ and $p_2 - p_3$ are in T_1. It follows that $A \subset -\alpha S_x$.

Thus there exist $c \in X$ and $\alpha \in \mathbb{C}$, $|\alpha| = 1$, such that $P \subset \alpha S_c$. This means that for every $x \in Y$ there exists $h(x) \in X$ such that $g(x) = \alpha h(x) - \alpha c$. The function $h : Y \to X$ is continuous, since $h(x) = \alpha^{-1} g(x) + c$. We have $\Lambda^*(x-y) = g(x) - g(y) = \alpha(h(x) - h(y))$. Since Λ^* is a bijection between T_2 and

T_1, it follows that $h : Y \to X$ also is bijective. Hence h is a homeomorphism between Y and X.

Let $h^* : E \to F$ be the isometry induced by h. Consider the map $h^{**} : F^* \to E^*$. The maps Λ^* and αh^{**} coincide on the set T_2, since $\Lambda^*(\beta(x-y)) = \beta\alpha(h(x) - h(y))$ for all $x, y \in Y$, $\beta \in \mathbb{C}$, $|\beta| = 1$. Since the polar T_2° of T_2 is bounded in F (it is easy to see that T_2° is contained in the ball of radius two in F), the linear subspace spanned by T_2 is weakly* dense in $F^* = M_0(Y)$. It follows that $\Lambda^* = \alpha h^{**}$ and $\Lambda = \alpha h^*$. This means that for every $f \in C(X)$ the functions $\psi(f)$ and $\alpha f \circ h$ represent the same element of F. In other words, the difference $\psi(f) - \alpha f \circ h$ is constant. Denote this constant by $t(f)$. We have $\psi(f)(y) = \alpha f(h(y)) + t(f)$ for every $f \in C(X)$ and $y \in Y$. It is clear that $t : C(X) \to \mathbb{C}$ is linear and that it is continuous whenever ψ is so. ∎

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