EXTRACTA MATHEMATICAE Vol. **14**, Núm. 1, 9 – 14 (1999)

Tight Closure of an Ideal Generated by an *R*-Sequence

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(Research paper presented by Juan Antonio Navarro)

AMS *Subject Class.* (1991): 13B, 13C

Received May 30, 1998

0. Introduction

The theory of tight closure was introduced by M. Hochster and C. Huneke in [2] and [3]. This new theory gives others proofs of many results, in particular the theorem of Briançon-Skoda (see [3] and [4]).

Throughout this paper, R will denote a ring that is Nœtherian commutative with identity and of characteristic $p > 0$.

The phrase "characteristic p" always means positive and prime characteristic. We will use "q" to denote a power of the characteristic p.

We set R° to be the set of elements of R not in any minimal prime of R.

If I is an ideal of R then $I^{[q]}$ denotes the ideal generated by the q^{th} powers of all elements of I. If S generates I then $\{i^q; i \in S\}$ generates $I^{[q]}$.

Definition. Let I be an ideal of R. An element $x \in R$ is said to be in $I^\star$, the tight closure of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all large q. An ideal I with $I^\star = I$ is said to be tightly closed.

It is clear that I^* is an ideal. M. Hochster and C. Huneke showed that $I^\star$ satisfies the closure operations: $I \subset I^\star$, $(I^\star)^\star = I^\star$ and if $I \subset J$ then $I^\star \subset J^\star$ (see [2], [3] and [4]).

Definition. A ring R is called weakly F-regular if every ideal of R is tightly closed.

Definition. We say that the ordered sequence $(a_1, a_2, ..., a_n)$ is an R-sequence in R (see [5]) if:

1) $\langle a_1, a_2, ..., a_n \neq R$.

2) a_1 is regular.
3) a_i is regular mod $\langle a_1, a_2, ..., a_{i-1}\rangle$, $i = 2, ..., n$.

We set $Z(R)$ to be the set of the zero-divisors of R.

The commun length of all maximal R-sequences in I is called the grade of I and written $G(I)$ (see [5]).

In section 1, we show the tight closure $I^\star$ of an ideal I, generated by a regular element, is principal if and only if $I^\star = I$.

In section 2, we show that $I^\star \neq R$ and $G(I^\star) = G(I)$ for all ideals I generated by an R-sequence.

1. Tight closure of an ideal generated by a regular element

Proposition 1.1. *Let $I \neq R$ be a principal ideal of a local ring R. If $G(I) = 1$ then $I^\star \neq R$.*

Proof. Since I is principal and $G(I) = 1$ then there exists $a \notin Z(R)$ such that $I = \langle a\rangle$ (see [5]).

We suppose that $I^\star = R$. There exists $c \in R^\circ$ such that $c \in \langle a^q\rangle$ for all $q = p^e \geq p^{e'}$.

Since R is a local ring and $I = \langle a\rangle \neq R$ all elements of $1 + \langle a\rangle$ are invertible in R. Hence after Krull's theorem (see [1]) $c = 0$ thus a contradiction. ∎

Corollary 1.2. *Let $I \neq R$ be a principal ideal of R. If $G(I) = 1$ then $I^\star \neq R$.*

Proof. There exists $a \notin Z(R)$ such that $I = \langle a\rangle \neq R$ (see [5]). Since $I \neq R$, there exists a maximal ideal $\mathfrak{m}$ of R such that $I \subset \mathfrak{m}$. We have $I_{\mathfrak{m}} = \langle \frac{a}{1}\rangle \neq R_{\mathfrak{m}}$ and $\frac{a}{1} \notin Z(R_{\mathfrak{m}})$.

Since $R_{\mathfrak{m}}$ is local, it follows from Proposition 1.1 that $(I_{\mathfrak{m}})^\star \neq R_{\mathfrak{m}}$. Since $(I_{\mathfrak{m}})^\star = (I^\star)_{\mathfrak{m}}$ (see [3] and [4]), $I^\star \neq R$. ∎

Proposition 1.3. *Let $I \neq R$ be a principal ideal of R. If $G(I) = 1$ then $G(I^\star) = 1$.*

Proof. There exists $a \notin Z(R)$ such that $I = \langle a\rangle \neq R$ (see [5]). It follows from Corollary 1.2 that $I^\star \neq R$. Since $I \subset I^\star$, $G(I^\star) \geq 1$.

We suppose that $G(I^\star) > 1$. There exists $b \in I^\star$ such that (a, b) is an R-sequence in $I^\star$.

Since $b \in I^\star$, there exists $c \in R^\circ$ such that $cb^q \in \langle a^q\rangle$ for all large q.

Since (a, b) is an R-sequence in $I^\star$, (a^q, b^q) is an R-sequence in $I^\star$ (see [6]), then $c \in \langle a^q \rangle$ for all large q, a contradiction. ∎

Remark. Let $I \neq R$ be a principal ideal of R. If R is reduced then $I^\star \neq R$ and $G(I^\star) = G(I)$: In fact, if $G(I) = 0$ then $I \subset \mathfrak{p}$ where $\mathfrak{p}$ is a minimal prime ideal of R, so $I^\star \subset \mathfrak{p}$. Hence $I^\star \neq R$ and $G(I) = 0$.

THEOREM 1.4. *Let $I = \langle a \rangle \neq R$ be an ideal of R such that $a \notin Z(R)$. $I^\star = I$ if and only if $I^\star$ is a principal ideal.*

Proof. Since $G(I) = 1$, it follows from Proposition 1.3 that $G(I^\star) = 1$. There exists $b \notin Z(R)$ such that $I^\star = \langle b \rangle$ (see [5]).

There exists $\alpha \in R$ such that: $a = \alpha b$ because $I \subset I^\star$. Since $b \in I^\star$, there exists $c \in R^\circ$ such that $cb^q \in \langle a^q \rangle$ for all $q = p^e \geq p^{e'}$. Hence for all $q = p^e \geq p^{e'}$, there exists $r_e \in R$ such that: $cb^q = r_e a^q = r_e \alpha^q b^q$, then $c \in \langle \alpha^q \rangle$ for all $q = p^e \geq p^{e'}$ because $b \notin Z(R)$. Hence $\langle \alpha \rangle^\star = R$.

Since $\alpha \notin Z(R)$, it follows from Corollary 1.2 that $\langle \alpha \rangle = R$, so $b \in \langle a \rangle$.

The converse is obvious. ∎

The counter example below, shows that we can have $I^\star \neq I$ even if $I = \langle a \rangle$ is an ideal of R and (a) is an R-sequence.

COUNTEREXAMPLE. Let $R = K[X, Y]/\langle X^3 + Y^3 \rangle$, where k is a field of characteristic $p \neq 3$. Let x, y denote the image of X, Y in R.

It is easy to show that (x) is an R-sequence and $y \notin \langle x \rangle$.

We show that $y \in \langle x \rangle^\star$:

Let $q = p^e$. Write $2q = 3k + i$, where i is either 1 or 2, then: $y^{3-i} y^q = y^{3(k+1)} = (y^3)^{k+1} = (-1)^{k+1} (x^3)^{k+1}$.

$3(k+1) = 3k+3 > 3k+i = q$ so: $y^{3-i} y^q \in \langle x \rangle^q$. It follows that $y^3 y^q \in \langle x \rangle^q$. So $y \in \langle x \rangle^\star$ because $y^3 \notin Z(R)$.

Remarks. 1) It follows from Theorem 1.4 that the ideal $\langle x \rangle^\star$ of the above counterexample is not principal.

2) Let R be an Artinian ring. If I is an ideal of R then $I^\star = \sqrt{I}$.

2. THE GRADE OF $I^\star$ WHEN I IS AN IDEAL GENERATED BY A REGULAR SEQUENCE

PROPOSITION 2.1. *$G(I) = G(I^\star)$ for all ideals I of R if and only if $G(\mathfrak{p}) = G(\mathfrak{p}^\star)$ for all prime ideals $\mathfrak{p}$ of R.*

Proof. We suppose that $G(\mathfrak{p}) = G(\mathfrak{p}^\star)$ for all prime ideals $\mathfrak{p}$ of R. Let $I \neq R$ an ideal of R. There exists a prime ideal $\mathfrak{p}$ of R such that $I \subset \mathfrak{p}$ and $G(I) = G(\mathfrak{p})$ (see [5]).

We have $I^\star \subset \mathfrak{p}^\star$ so: $G(I^\star) \leq G(\mathfrak{p}^\star)$. Then $G(I^\star) = G(I)$.

The converse is obvious. ∎

Proposition 2.2. *Let I be an ideal of a local ring R. If I is generated by an R-sequence then $I^\star \neq R$.*

Proof. We will use an induction on the number n of the elements of the R-sequence.

For $n = 1$: See Proposition 1.1.

Since the passage from n to $n+1$, using the induction hypothesis, is exactly similar to the passage from $n = 1$ to $n = 2$, it is then enough to show this for $n = 2$:

Let $I = \langle a, b\rangle$ be an ideal of R, where (a, b) is an R-sequence. We suppose that $I^\star = R$, so there exists $c \in R^\circ$ such that $c \in \langle a^q, b^q\rangle$ for all $q = p^e \geq p^{e'}$. So for all $i \geq e' : c = \alpha_i a^{p^i} + \beta_i b^{p^i}$ where $\alpha_i, \beta_i \in R$.

Hence for $m \geq e' : \alpha_m a^{p^m} + \beta_m b^{p^m} = \alpha_{m+1} a^{p^{m+1}} + \beta_{m+1} b^{p^{m+1}}$.

So $b^{p^m}(\beta_{m+1} b^{p^{m+1}-p^m} - \beta_m) = a^{p^m}(\alpha_m - \alpha_{m+1} a^{p^{m+1}-p^m})$.

Since (a, b) is an R-sequence, (a^{p^m}, b^{p^m}) is an R-sequence (see [6]), so $\beta_m = r_m a^{p^m} + \beta_{m+1} b^{p^{m+1}-p^m}$, where $r_m \in R$. Hence $\beta_m \in \langle a, \beta_{m+1}\rangle$. Thus, we obtain an increasing sequence of ideals of R:

$$\langle \beta_{e'}, a\rangle \subset \langle \beta_{e'+1}, a\rangle \subset ... \subset \langle \beta_m, a\rangle \subset \langle \beta_{m+1}, a\rangle \subset \dots$$

Since R is Notherian, there exists $k \geq e'$ such that: $\langle \beta_k, a\rangle = \langle \beta_{k+1}, a\rangle$. So $\beta_{k+1} = \gamma_k \beta_k + \mu_k a$, where $\gamma_k, \mu_k \in R$.

So $\beta_{k+1} = \gamma_k(r_k a^{p^k} + \beta_{k+1} b^{p^{k+1}-p^k}) + \mu_k a$. Hence: $\beta_{k+1}(1 - b^{p^{k+1}-p^k}) \in \langle a\rangle$.

Since R is local, $\beta_{k+1} \in \langle a\rangle$. Hence: $c \in \langle a\rangle$.

To show that $c \in \langle a\rangle^{p^s}$ for all s, we set $x = a^{p^s}, y = b^{p^s}$. We have $c \in \langle x^q, y^q\rangle$ for all $q = p^e \geq p^{e'}$. Since (x, y) is an R-sequence (see [6]), $c \in \langle x\rangle = \langle a\rangle^{p^s}$. Hence $\langle a\rangle^\star = R$, a contradiction. ∎

Theorem 2.3. *If I is an ideal of R generated by an R-sequence then $I^\star \neq R$.*

Proof. Since $I \neq R$, there exists a maximal ideal $\mathfrak{m}$ of R such that $I \subset \mathfrak{m}$. We set $I = \langle a_1, ..., a_n\rangle$ where $(a_1, ..., a_n)$ is an R-sequence.

After localizing at $\mathfrak{m}$, we have: $I_{\mathfrak{m}} = \langle \frac{a_1}{1}, ..., \frac{a_n}{1} \rangle$ and $(\frac{a_1}{1}, ..., \frac{a_n}{1})$ is an R-sequence in $R_{\mathfrak{m}}$ (see [5]). Since $R_{\mathfrak{m}}$ is local, it follows from Proposition 2.2 that $(I_{\mathfrak{m}})^\star \neq R_{\mathfrak{m}}$. Since $(I_{\mathfrak{m}})^\star = (I^\star)_{\mathfrak{m}}$ (see [4]), $I^\star \neq R$. ∎

PROPOSITION 2.4. *If I is an ideal of R generated by an R-sequence then $G(I^\star) = G(I)$.*

Proof. Let $I = \langle a_1, ..., a_n \rangle$ where $(a_1, ..., a_n)$ is an R-sequence. It follows from Theorem 2.3 that $I^\star \neq R$.

We Suppose that $G(I^\star) > n$: There exists $b \in I^\star$ such that $(a_1, ..., a_n, b)$ is an R-sequence.

Since $b \in I^\star$, there exists $c \in R^\circ$ such that $cb^q \in (a_1^q, ..., a_n^q)$ for all large q. Since $(a_1, ..., a_n, b)$ is an R-sequence, $(a_1^q, ..., a_n^q, b^q)$ is an R-sequence (see [6]). Hence $c \in (a_1^q, ..., a_n^q)$ for all large q, then $I^\star = R$, a contradiction. ∎

Remarks. 1) If $\mathfrak{p}$ is a prime ideal of R generated by an R-sequence then $\mathfrak{p}^\star = \mathfrak{p}$.

2) If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of R generated by an R-sequence then $\sqrt{\mathfrak{q}^\star} = \mathfrak{p}$.

3) By Theorem 2.3 we can show that a local regular ring is weakly F-regular without using the fact $\cap_q \mathfrak{m}^q = 0$. In fact Since $\mathfrak{m}$ is generated by an R-sequence, $\mathfrak{m}^\star = \mathfrak{m}$. Let I a $\mathfrak{m}$-primary ideal and $x \in I^\star$, there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all q. Since R is regular, $c \in (I : x)^{[q]}$. If we suppose that $(I : x) \neq R$ then $c \in \mathfrak{m}^{[q]}$. Hence $\mathfrak{m}^\star = R$, a contradiction.

LEMMA. *If J is an ideal and x an element of R then:*
$[J : x]^{[q]} \subset [J^{[q]} : x^q]$ for all $q = p^e$.

Proof. We set $[J : x] = \langle r_1, ..., r_n \rangle$ where $r_i \in R$.

We have $[J : x]^{[q]} = \langle r_1^q, ..., r_n^q \rangle$. For all $i = 1, ..., n : r_i x \in J$. So $r_i^q x^q = (r_i x)^q \in J^{[q]}$. Hence $r_i^q \in [J^{[q]} : x^q]$. ∎

PROPOSITION 2.5. *Let I be an ideal of R containing a maximal R-sequence $(a_1, ..., a_n)$ in I such that $\langle a_1, ..., a_n \rangle^\star = \langle a_1, ..., a_n \rangle$.*

If $I^\star \neq R$ then $G(I^\star) = G(I)$.

Proof. We set $J = \langle a_1, ..., a_n \rangle$. We have: $J^\star = J \subset I$. Since $(a_1, ..., a_n)$ is a maximal R-sequence in I, $I \subset Z(R/J)$. So there exists a prime ideal $\mathfrak{p}$ such that $I \subset \mathfrak{p} = (J : x)$, where $x \notin J$. Since $I^\star \neq R$ and $I \subset I^\star$, $G(I^\star) \geq n$.

We suppose that $G(I^\star) > n$. There exists $b \in I^\star$ such that $b \notin Z(R/J)$. Since $b \in I^\star$, there exists $c \in R^\circ$ such that $cb^q \in I^{[q]} \subset [J : x]^{[q]}$ for all large

q. It follows from the Lemma that: $cb^q x^q \in J^{[q]} = \langle a_1^q, ..., a_n^q \rangle$ for all large q. Since $(a_1, ..., a_n, b)$ is an R-sequence, $(a_1^q, ..., a_n^q, b^q)$ is an R-sequence (see [6]). So $cx^q \in J^{[q]}$ for all large q. Hence $x \in J^\star = J$, a contradiction. ∎

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