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A Note on the Singular Spectrum

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1. Introduction

We compare the singular spectrum of a Banach algebra element with the usual spectrum and exponential spectrum.

Throughout A (or B) will denote a complex Banach algebra with unit element $1 \neq 0$. The invertible group of A will be denoted by A^{-1}. An element $a \in A$ will be called a *left* (resp. *right*) *topological divisor of zero* if there exists a sequence (z_n) in A such that $\|z_n\| = 1$ for all n and $az_n \to 0$ (resp. $z_n a \to 0$) as $n \to \infty$. A *topological divisor of zero* is both a left and right topological divisor of zero (see §4). It can be shown that the set of topological divisors of zero in A is a closed set. A left or right topological divisor of zero cannot be invertible. The *singular spectrum* of $a \in A$ is the set $\tau(a, A) = \{\lambda \in \mathbb{C} : \lambda - a \text{ is a topological divisor of zero}\}$. If the usual spectrum of $a \in A$ is denoted by $\sigma(a, A)$ then it is familiar [19, Theorem 2.5, p. 397] that

$$\partial\sigma(a, A) \subset \tau(a, A) \subset \sigma(a, A) \tag{1}$$

with ∂K the topological boundary of a subset K of $\mathbb{C}$. Therefore, the singular spectrum of a is a compact nonempty subset of $\mathbb{C}$. The *generalised exponentials* [9] form the component of 1 in the topological group A^{-1}:

$$\operatorname{Exp} A = \{e^{c_1} e^{c_2} \cdots e^{c_k} : k \in \mathbb{N}, c \in A^k\}.$$

For $a \in A$ the *exponential spectrum* of a in A is the set $\epsilon(a, A) = \{\lambda \in \mathbb{C} : \lambda - a \notin \operatorname{Exp} A\}$. It is then familiar [9, Theorem 1] that

$$\partial\epsilon(a, A) \subset \tau(a, A) \subset \sigma(a, A) \subset \epsilon(a, A) \subset \eta\sigma(a, A) \tag{2}$$

where ηK denotes the connected hull of a compact subset K of $\mathbb{C}$. In view of Theorem 1.2 of [11] and (2) we have for every $a \in A$

$$\eta\tau(a, A) = \eta\sigma(a, A) = \eta\epsilon(a, A). \tag{3}$$

We will use the symbol $\operatorname{acc} K$ to denote the set of accumulation points of K and the symbol $\operatorname{iso} K$ for the set of isolated points of K. By an ideal in A we mean a two sided ideal in A. An ideal J in A is called *inessential* [1, p. 106] whenever it follows from $b \in J$ that $\operatorname{acc} \sigma(b, A) \subset \{0\}$.

The radical of A will be denoted by $\operatorname{Rad} A$ and A is said to be semisimple if $\operatorname{Rad} A = \{0\}$. It follows from (1) that every element in the radical of A is a topological divisor of zero. An element $a \in A$ is quasinilpotent if $\sigma(a, A) = \{0\}$. The set of these elements will be denoted by $\operatorname{QN}(A)$. Recall that if J is a closed ideal in A then $b \in A$ is called *Riesz* relative to J if $b + J \in \operatorname{QN}(A/J)$, (see [3, section R.1]). An element $a \neq 0$ in a semisimple Banach algebra A is called *rank one* [17] if there exists a linear functional t_a on A such that $axa = t_a(x)a$ for all $x \in A$. If A and B are Banach algebras, the linear operator $T : A \to B$ is called a *homomorphism* if $T(ab) = TaTb$ for all $a, b \in A$ and $T1 = 1$. It is said to be *bounded below* if $\inf\{\|Ta\| : \|a\| \geq 1\} > 0$.

The paper is organised as follows: We prove first that if x and y are nonzero elements in A and if λ is a nonzero complex number then $\lambda - xy$ is a topological divisor of zero if and only if $\lambda - yx$ is a topological divisor of zero. In Section 2 we investigate how the singular spectrum depends on the algebra and in Section 3 we investigate the behaviour of the singular spectrum under perturbation by certain elements.

LEMMA 1.1. *Let $0 \neq x, y \in A$ and let $0 \neq \lambda \in \mathbb{C}$. If (z_n) is a sequence in A with $\|z_n\| = 1$ for all n and $(\lambda - xy)z_n \to 0$ if $n \to \infty$, then $yz_n \not\to 0$.*

Proof. Note that

$$\|\lambda z_n\| - \|x\|\|yz_n\| \leq \|\lambda z_n\| - \|xyz_n\| \leq \|(\lambda - xy)z_n\|.$$

If $yz_n \to 0$ as $n \to \infty$, then it follows from our hypothesis that $|\lambda| \leq 0$, which is a contradiction. ∎

It follows from the lemma that there exist an $\epsilon > 0$ and a subsequence $(yz_{k(n)})$ of (yz_n) such that $\|yz_{k(n)}\| \geq \epsilon$ for all n.

THEOREM 1.2. *Let $0 \neq x, y \in A$ and $0 \neq \lambda \in \mathbb{C}$. Then $\lambda - xy$ is a topological divisor of zero if and only if $\lambda - yx$ is a topological divisor of zero, i.e. $\tau(xy, A)\backslash\{0\} = \tau(yx, A)\backslash\{0\}$.*

Proof. If $\lambda - xy$ is a topological divisor of zero, it is a left topological divisor of zero. Hence there is a sequence (z_n) in A with $\|z_n\| = 1$ for all n and $(\lambda - xy)z_n \to 0$ as $n \to \infty$. Then the sequence $(\frac{yz_n}{\|yz_n\|})$ (or passing to a subsequence if necessary) has the property that $\|\frac{yz_n}{\|yz_n\|}\| = 1$ for all n and

$$(\lambda - yx)\frac{yz_n}{\|yz_n\|} = \frac{y}{\|yz_n\|}(\lambda - xy)z_n \to 0$$

as $n \to \infty$, (see the remarks following Lemma 1.1). Consequently $\lambda - yx$ is a left topological divisor of zero. It follows similarly that $\lambda - yx$ is a right topological divisor of zero and so $\lambda - yx$ is a topological divisor of zero. ∎

The above result for the spectrum is well known [1, Lemma 3.1.2]. We do not know if the corresponding result for the exponential spectrum is true in general. However, we refer to the observation of Murphy [15, Proposition 4.3] that $\varepsilon(ab)\backslash\{0\} = \varepsilon(ba)\backslash\{0\}$ provided that either a or b is the limit of invertible elements. Theorem 1.2 was proved by Bruce Barnes [2, Theorem 3] in the Banach algebra $\mathcal{B}(X)$ of bounded linear operators on a Banach space X.

2. Subspaces

In this section we investigate how the singular spectrum depends on the algebra.

PROPOSITION 2.1. ([9, 1.6]) *Let A and B be Banach algebras and $T : A \to B$ a bounded homomorphism. If T is bounded below then $\tau(a, A) \subset \tau(Ta, B)$ for all $a \in A$.*

Proof. If $a \in A$ and if $\lambda \in \tau(a, A)$ then $\lambda - a$ is a topological divisor of zero in A. If $\lambda - a$ is a left topological divisor of zero in A then there is a sequence (z_n) in A with $\|z_n\| = 1$ for all n and $(\lambda - a)z_n \to 0$ as $n \to \infty$. Since T is a continuous and bounded below homomorphism we have that

$$y_n = \frac{Tz_n}{\|Tz_n\|}, \quad \text{for} \quad n = 1, 2, \ldots$$

is a sequence in B with $\|y_n\| = 1$ for all n and $(\lambda - Ta)y_n \to 0$ as $n \to \infty$. Hence $\lambda - Ta$ is a left topological divisor of zero. It follows similarly that $\lambda - Ta$ is a right topological divisor of zero and consequently, $\lambda \in \tau(Ta, B)$. ∎

Let A and B be Banach algebras and $T : A \to B$ a homomorphism (not necessarily bounded). If $\omega \in \{\sigma, \epsilon\}$ then $\omega(Ta, B) \subset \omega(a, A)$ for all $a \in A$ [12, Theorem 3]. If in addition T is bounded below then $\omega(a, A) \subset \eta\omega(Ta, B)$ for all $a \in A$ [13, Corollary 2.2].

COROLLARY 2.2. *If A is a closed subalgebra of a Banach algebra B then $\tau(a, A) \subset \tau(a, B)$ for all $a \in A$.*

In the analogous results for the spectrum and exponential spectrum we only require that A is a subalgebra of B: Let A and B be Banach algebras such that $1 \in A \subset B$. If $\omega \in \{\sigma, \epsilon\}$ then $\omega(a, B) \subset \omega(a, A)$ for all $a \in A$ [12, Proposition 5].

If A is a closed subalgebra of a Banach algebra B and A and B do not have the same unit element then we have

PROPOSITION 2.3. *Let B be a Banach algebra with idempotent $0 \neq p \neq 1$. If $A := pBp$ then $\tau(a, A) \subset \tau(a, B)$ for all $a \in A$.*

Proof. If $\lambda \in \tau(a, A)$ then $\lambda p - a$ is a topological divisor of zero in A. If $\lambda p - a$ is a left topological divisor of zero in A then there is a sequence (z_n) in A with $\|z_n\| = 1$ for all n and $(\lambda p - a)z_n \to 0$ as $n \to \infty$. But then $(\lambda - a)z_n = (\lambda p - a)z_n \to 0$ as $n \to \infty$ in B, i.e. $\lambda - a$ is a left topological divisor of zero in B. It follows similarly that $\lambda - a$ is a right topological divisor of zero in B and so $\lambda \in \tau(a, B)$. ∎

Let B be a Banach algebra with idempotent $0 \neq p \neq 1$. If $A = pBp$ then one can show that $\sigma(a, B) = \sigma(a, A) \cup \{0\}$ for all $a \in A$. For an analogous result for the exponential spectrum we refer to [12, Proposition 10]. We need the next lemma to prove Theorem 2.5.

LEMMA 2.4. *Let A and B be Banach algebras. If the bounded homomorphism $T : A \to B$ is bounded below then $\epsilon(a, A) \subset \eta\epsilon(Ta, B)$ for every $a \in A$.*

Proof. Let $\lambda \in \partial\,\epsilon(a, A)$. In view of $\operatorname{int}\sigma(a, A) \subset \operatorname{int}\epsilon(a, A)$, (see (2)), it follows again from (2) that $\lambda \in \partial\sigma(a, A)$. Since T is bounded below it follows from [10, (3.1)] that $\lambda \in \sigma(Ta, B)$ and so again by (2) $\lambda \in \epsilon(Ta, B)$. Our conclusion follows from [11, Theorem 1.2]. ∎

THEOREM 2.5. *Let A and B be Banach algebras. If the bounded homomorphism $T : A \to B$ has closed range and if $\omega \in \{\tau, \sigma, \epsilon\}$ then*

$$\bigcap_{Tb=0} \omega(a+b, A) \subset \eta\, \omega(Ta, B).$$

Proof. For the spectrum this result is familiar [9, Theorem 3]. If we factorise $T : A \to A/J \xrightarrow{T^\wedge} B$, where $J := T^{-1}(0)$, then the natural homomorphism $a \longmapsto a + J$ is onto, while $T^\wedge$ is one-one with closed range and hence bounded below. Let $\omega = \epsilon$. By [9, Theorem 2]

$$\bigcap_{b\in J} \epsilon(a+b, A) = \epsilon(a+J, A/J).$$

Since $T^\wedge$ is bounded below we have by Lemma 2.4 that $\epsilon(a + J, A/J) \subset \eta\epsilon(T^\wedge(a+J), B) = \eta\epsilon(Ta, B)$. If we combine these observations

$$\bigcap_{b\in J} \epsilon(a+b, A) \subset \eta\epsilon(Ta, B).$$

Let $\omega = \tau$. By (2) and the remarks above

$$\bigcap_{b\in J} \tau(a+b, A) \subset \bigcap_{b\in J} \epsilon(a+b, A) = \epsilon(a+J, A/J).$$

Since $\partial\epsilon(a + J, A/J) \subset \tau(a + J, A/J)$, (2), it follows from [11, Theorem 1.2] that $\epsilon(a+J, A/J) \subset \eta\tau(a+J, A/J)$. This together with Proposition 2.1 gives

$$\bigcap_{b\in J} \tau(a+b, A) \subset \eta\tau(a+J, A/J) \subset \eta\tau(T^\wedge(a+J), B) = \eta\, \tau(Ta, B).$$

∎

There are examples to show that the connected hull η in the above theorem cannot be omitted.

3. PERTURBATION RESULTS

In this section we study the behaviour of the singular spectrum under perturbation by rank one elements, inessential elements and Riesz elements.

LEMMA 3.1. *Let $A \neq \mathbb{C}$ be a semisimple Banach algebra and $a \in A^{-1}$. If b is of rank one then $a + b$ is a topological divisor of zero if and only if $t_b(a^{-1}) = -1$.*

Proof. If $a+b$ is a topological divisor of zero then $a+b \notin A^{-1}$ and so in view of [17, Lemma 2.7, Lemma 2.8] and [9, 1.5]

$$a+b = a(1+a^{-1}b) \quad \Rightarrow \quad -1 \in \sigma(a^{-1}b, A) = \{0, t_b(a^{-1})\} = \tau(a^{-1}b, A),$$

and so $t_b(a^{-1}) = -1$. Conversely, if $t_b(a^{-1}) = -1$ then

$$-1 \in \sigma(a^{-1}b, A) = \{0, t_b(a^{-1})\} = \tau(a^{-1}b, A).$$

Hence $1 + a^{-1}b$ is a topological divisor of zero. If $1+a^{-1}b$ is a left topological divisor of zero there is a sequence (z_n) in A with $\|z_n\| = 1$ for all n and $(1+a^{-1}b)z_n \to 0$ as $n \to \infty$. Since $a+b = a(1+a^{-1}b)$ it follows that $(a+b)z_n \to 0$ and so $a+b$ is a left topological divisor of zero. Also, if $t_b(a^{-1}) = -1$ then in view of Theorem 1.2, $-1 \in \tau(ba^{-1}, A)$ and so $1+ba^{-1}$ is a topological divisor of zero in A. Since $1+ba^{-1}$ is a right topological divisor of zero in A it follows in the same way as above that $a+b$ is a right topological divisor of zero in A. We have shown that $a+b$ is a topological divisor of zero in A. ∎

THEOREM 3.2. *Let A be a semisimple Banach algebra and $a \in A$. If $b \in A$ is rank one then acc $\tau(a+b, A) \subset \eta\tau(a, A)$.*

Proof. By the previous lemma

$$\tau(a+b, A)\backslash\sigma(a, A) = \{\lambda \in \mathbb{C}\backslash\sigma(a, A) : t_{-b}((\lambda - a)^{-1}) = -1\}.$$

Since $t_{-b}((\lambda-a)^{-1}) + 1$ is an analytic function of λ and the set $\tau(a+b, A)$ compact, it follows from [5, Theorem IV.3.7] that $\tau(a+b, A)\backslash\eta\sigma(a, A)$ consists of isolated points of $\tau(a+b, A)$. In view of (3), $\eta\sigma(a, A) = \eta\tau(a, A)$ so that $\operatorname{acc}\tau(a+b, A) \subset \eta\tau(a, A)$. ∎

The inclusion in Theorem 3.2 may be strict: It follows from Example 23 in [12] that there exists a semisimple Banach algebra A and elements $a, b \in A$ such that b is rank one and

$$\epsilon(a+b, A) = \{z \in \mathbb{C} : |z| = 1\} \subset \{z \in \mathbb{C} : |z| \leq 1\} = \epsilon(a, A).$$

This together with (2) and the fact that $\eta\epsilon(a, A) = \eta\tau(a, A)$ give

$$\operatorname{acc}\tau(a+b, A) = \{z \in \mathbb{C} : |z| = 1\} \subset \{z \in \mathbb{C} : |z| \leq 1\} = \eta\tau(a, A).$$

THEOREM 3.3. *Let J be a closed inessential ideal in a Banach algebra A and $a \in A$. If $b \in J$ then* $\operatorname{acc}\tau(a+b, A) \subset \eta\tau(a, A)$.

Proof. It follows from [14, Theorem 5.3] and [1, Theorem 5.7.4 (iii)] that

$$\operatorname{acc}\sigma(a+b,A) \subset \eta\sigma(a+b+J,A/J) = \eta\sigma(a+J,A/J).$$

This together with (1) gives

$$\operatorname{acc}\tau(a+b,A) \subset \eta\sigma(a+J,A/J) \subset \eta\sigma(a,A) = \eta\tau(a,A).$$

∎

COROLLARY 3.4. *If $a \in A$ and if $b \in \operatorname{Rad} A$, then $\operatorname{acc}\tau(a+b,A) \subset \eta\tau(a,A)$.*

In view of Corollary 3.4 note that the spectrum as well as the exponential spectrum of an element is invariant under perturbation by radical elements [1, Theorem 5.3.1] and [12, Section 5]. One can use Example 1 in [6] to show that the inclusion in Corollary 3.4 may be strict.

THEOREM 3.5. *Let J be a closed inessential ideal in a Banach algebra A and $a \in A$. If $b \in A$ is Riesz relative to J and $ab = ba$ then $\operatorname{acc}\tau(a+b,A) \subset \eta\tau(a,A)$.*

Proof. It follows from [14,Theorem 5.3] and [1, Theorem 5.7.4 (iii)] that $\operatorname{acc}\sigma(a+b,A) \subset \eta\sigma(a+b+J,A/J)$. Since $b+J \in \mathrm{QN}(A/J)$ and $b+J$ and $a+J$ commute in A/J, $\sigma(a+b+J,A/J) = \sigma(a+J,A/J)$. If we combine these remarks with (3) then

$$\begin{aligned}\operatorname{acc}\tau(a+b,A) &\subset \eta\sigma(a+b+J,A/J) = \eta\sigma(a+J,A/J)\\ &\subset \eta\sigma(a,A) = \eta\tau(a,A).\end{aligned}$$

∎

The commutativity condition in the above theorem cannot be omitted: It follows from Example 1 in [7] that there exists a Banach algebra A and elements $a,b \in A$ such that $ab \neq ba$ and b is Riesz relative to some closed inessential ideal in A. Furthermore, $\sigma(a+b,A) = \sqrt{2}\mathbb{D}$ while $\sigma(a,A) = \mathbb{D}$ with $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. By (1) $\{\lambda \in \mathbb{C} : |\lambda| = \sqrt{2}\} \subset \operatorname{acc}\tau(a+b,A)$ and since $\eta\tau(a,A) = \eta\sigma(a,A) = \mathbb{D}$, (3), it follows that $\operatorname{acc}\tau(a+b,A) \not\subset \eta\tau(a,A)$.

4. Remarks

Many authors (e.g. [4], [16], [18]) define a *topological divisor of zero* as either a *left topological divisor of zero* or a *right topological divisor of zero*. The results in this paper also apply to this definition of a topological divisor of zero as well as to a modified version of the spectrum, defined as the intersection of the left and right spectrum.

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