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Matched Pairs and Extensions of Lie Bialgebras

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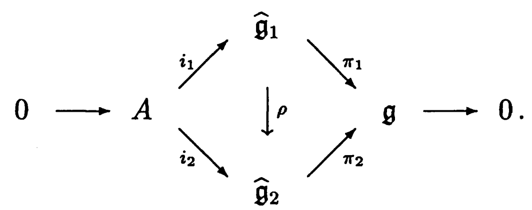
1. Introduction

A Lie bialgebra is a Lie algebra also equipped with a Lie coalgebra compatible structure ([2], [3]). A Lie bialgebra morphism is a Lie algebra morphism which is also a Lie coalgebra morphism. Let $\mathfrak{g}$ and A be finite dimensional Lie bialgebras over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A Lie bialgebra $\widehat{\mathfrak{g}}$ is called an extension of $\mathfrak{g}$ by A if there exists an exact sequence $0 \to A \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$ where i and π are Lie bialgebra morphisms. Two extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by A are called equivalent if there exists a Lie bialgebra morphism $\rho : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the following diagram commutes:



We denote by $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ the set of all inequivalent Lie bialgebra extensions of $\mathfrak{g}$ by A.

For an arbitrary commutative Lie bialgebra A and a general Lie bialgebra $\mathfrak{g}$ an explicit description of $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ can be found in ([1]). If $A \neq \mathbb{K}$ then the set $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ is not in general a group: it is described by a non-abelian cohomology of Lie bialgebras.

In this work we suppose that $\mathfrak{g}$ is a co-commutative Lie bialgebra to avoid non-abelian cohomology. The set $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ naturally admits an abelian group structure and it is isomorphic to the second cohomology group of a differential complex constructed out of $\mathfrak{g}$ and A. More precisely, the data of an element of $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ induces a matched structure on the pair $(\mathfrak{g}, A)$. For such a fixed structure we give an explicit description of $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ and define a cohomology of the matched pair $(\mathfrak{g}, A)$ such that its second group is isomorphic to $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$.

For a particular matched structure on $(\mathfrak{g}, \mathfrak{g}^*)$ we realize the double $\mathfrak{D} = \mathfrak{g} \bowtie \mathfrak{g}^*$ of $\mathfrak{g}$ as the co-central trivial extension of $\mathfrak{g}$ by $\mathfrak{g}^*$.

In the entire sequel $(\mathfrak{g}, A)$ denote an abelian pair of Lie bialgebras i.e. $\mathfrak{g}$ is a co-commutative Lie bialgebra and A is a commutative Lie bialgebra. All vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ considered here are finite dimensional.

2. Description of $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ for a Fixed Matched Pair

Let (E) : $0 \to A \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$ be an extension of Lie bialgebras. By definition we obtain the following two Lie algebra extensions:

$$(\mathrm{E}_1) : 0 \to A \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0\,,$$

$$(\mathrm{E}_2) : 0 \to \mathfrak{g}^* \xrightarrow{\pi^*} \widehat{\mathfrak{g}}^* \xrightarrow{i^*} A^* \to 0\,,$$

where $\mathfrak{g}^*$ (resp. A^*) is the dual Lie algebra of $\mathfrak{g}$ (resp. A) and i^* (resp. π^*) denotes the transpose of i (resp. π).

Extension (E_1) induces a (left) $\mathfrak{g}$-module structure $\varphi : \mathfrak{g} \to \mathrm{End}(A)$ and similarly extension (E_2) determines an A^*-module structure $F : A^* \to \mathrm{End}(\mathfrak{g}^*)$ on $\mathfrak{g}^*$. Using the fact that $\widehat{\mathfrak{g}}$ is a Lie bialgebra we obtain the following compatibilities between φ and F (see ([1]): $\forall \alpha, \beta \in A^*$, $\forall x, y \in \mathfrak{g}$,

$$\begin{aligned} F(\alpha)^*([x,y]) &= [F(\alpha)^*(x), y] + [x, F(\alpha)^*(y)] \\ &\quad + F(\alpha \circ \varphi(x))^*(y) - F(\alpha \circ \varphi(y))^*(x)\,, \qquad (2.1) \\ [\alpha, \beta] \circ \varphi(x) &= [\alpha \circ \varphi(x), \beta] + [\alpha, \beta \circ \varphi(x)] \\ &\quad + \beta \circ \varphi(F(\alpha)^* x) - \alpha \circ \varphi(F(\alpha)^* y)\,. \qquad (2.2) \end{aligned}$$

Definition. ([3]) A matched pair structure on $(\mathfrak{g}, A)$ is the data of a $\mathfrak{g}$-module structure φ on A and an A^*-module structure F on $\mathfrak{g}^*$ satisfying the identities (2.1) and (2.2).

If (E)$'$ is a Lie bialgebra extension equivalent to the given extension (E), then the induced Lie algebra extensions $(\mathrm{E}_1)'$ and $(E_2)'$ are equivalent to (E_1) and (E_2) respectively. As a consequence, extension (E)$'$ induces the same matched structure (φ, F) on $(\mathfrak{g}, A)$. Therefore, we can conclude the following:

PROPOSITION. *The data of an equivalence class of Lie bialgebra extensions induces a matched pair structure on the pair* $(\mathfrak{g}, A)$.

Let (φ, F) be a fixed (but arbitrary) matched pair structure on $(\mathfrak{g}, A)$. Our aim is to describe the set $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ of Lie bialgebra inequivalent extensions of $\mathfrak{g}$ by A inducing the given matched structure on the pair $(\mathfrak{g}, A)$.

Let $0 \to A \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$ be an extension of $\mathfrak{g}$ by A. By choosing a linear section s of π we identify the vector spaces $\widehat{\mathfrak{g}}$ and $\mathfrak{g} \times A$; $(x, a) \equiv s(x) + i(a)$, where $x \in \mathfrak{g}$ and $a \in A$. The Lie bracket on $\widehat{\mathfrak{g}} = \mathfrak{g} \times A$ is given by: $\forall x, y \in \mathfrak{g}$, $\forall a, b \in A$,

$$[(x,a),(y,b)] = ([x,y], \varphi(x)(b) - \varphi(y)(a) + \gamma(x,y))\,, \tag{2.3}$$

where $\gamma : \mathfrak{g} \times \mathfrak{g} \to A$, $\gamma(x,y) = [s(x), s(y)] - s([x,y])$, is a 2-cocycle, $\gamma \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A)$, of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$-module A. The fact that i and π are Lie bialgebra morphisms implies that the Lie bracket on $\widehat{\mathfrak{g}}^* \cong \mathfrak{g}^* \times A^*$ is necessarily of the following form: $\forall \xi, \eta \in \mathfrak{g}^*$, $\forall \alpha, \beta \in A^*$,

$$[(\xi,\alpha),(\eta,\beta)] = (F(\alpha)(\eta) - F(\beta)(\xi) + \Omega(\alpha,\beta), [\alpha,\beta])\,, \tag{2.4}$$

where $\Omega : A^* \times A^* \to \mathfrak{g}^*$ is a 2-cocycle, $\Omega \in \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$, of the Lie algebra A^* with values in the A^*-module $\mathfrak{g}^*$. The compatibility between these two brackets on the Lie bialgebra $\widehat{\mathfrak{g}}$ implies the following condition ([1]): $\forall \alpha, \beta \in A^*$, $\forall x, y \in \mathfrak{g}$,

$$\begin{aligned}
&\langle \Omega(\alpha,\beta), [x,y]\rangle + \langle \gamma(x,y), [\alpha,\beta]\rangle\rangle = \\
&\qquad \langle F(\alpha)(\beta \circ \widetilde{\gamma}(x)) - F(\beta)(\alpha \circ \widetilde{\gamma}(x)) + \Omega(\alpha, \beta \circ \varphi(x)) - \Omega(\beta, \alpha \circ \varphi(x), y\rangle \\
&\qquad - \langle F(\alpha)(\beta \circ \widetilde{\gamma}(y)) - F(\beta)(\alpha \circ \widetilde{\gamma}(y)) + \Omega(\alpha, \beta \circ \varphi(y)) - \Omega(\beta, \alpha \circ \varphi(y), x\rangle
\end{aligned} \tag{2.5}$$

$\widetilde{\gamma} : \mathfrak{g} \to \mathrm{L}(\mathfrak{g}, A)$ is given by $\langle \widetilde{\gamma}(x), y\rangle = \gamma(x,y)$, where $\mathrm{L}(\mathfrak{g}, A)$ is the vector space of linear maps of $\mathfrak{g}$ in A.

DEFINITION. We say that $\gamma \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A)$ and $\Omega \in \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$ are compatible if the identity (2.5) is satisfied. We denote by $\mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times_c \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$ the subspace of $\mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$ consisting of all compatible cocycles.

So we have established that to any extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by A we can associate an element (γ, Ω) of $\mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times_c \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$. Conversely, the data of such a pair gives an extension $\mathfrak{g} \times A$ of $\mathfrak{g}$ by A defined by the formulae (2.3) and (2.4).

A change of the chosen section s to another section $s' = s + i \circ \theta$ of π with $\theta \in \mathrm{L}(\mathfrak{g}, A)$ transforms the pair (γ, Ω) to $(\gamma + \delta\theta, \Omega - \partial\theta^*)$. Here $\delta\theta$ denotes the coboundary operator of the 1-cochain $\theta : \mathfrak{g} \to A$ of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$-module A and $\partial\theta^*$ is the coboundary of the 1-cochain $\theta^* : A^* \to \mathfrak{g}^*$ of the Lie algebra A^* with values in the A^*-module $\mathfrak{g}^*$.

The map $(\mathfrak{g} \times A, s) \to (\mathfrak{g} \times A, s'), (x, a) \mapsto (x, a + \theta(x))$ between trivializations of $\widehat{\mathfrak{g}}$ defined by s and s' respectively is an equivalence of extensions. An isomorphism ρ defining an extension equivalence between $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by A trivialized by $(\mathfrak{g} \times A, \gamma, \Omega)$ and $(\mathfrak{g} \times A, \gamma', \Omega')$ respectively is always of the above form. We deduce that $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ are equivalent if and only if there exists $\theta \in \mathrm{L}(\mathfrak{g}, A)$ such that $\gamma' = \gamma + \delta\theta$ and $\Omega' = \Omega - \partial\theta^*$.

So we have established the following result:

THEOREM. *There is a bijective correspondence between* $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, A)$ *and the quotient* $\mathcal{B}(\mathfrak{g}, A)$ *of* $\mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times_c \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*)$ *by* $\{(\delta\varphi, -\partial\varphi^*) \mid \varphi \in \mathrm{L}(\mathfrak{g}, A)\}$.

The set $\mathcal{B}(\mathfrak{g}, A)$ is an abelian group for the natural addition:

$$((\gamma, \Omega)) + ((\gamma', \Omega')) = ((\gamma + \gamma', \Omega + \Omega')),$$

where double parentheses denote equivalent classes in $\mathcal{B}(\mathfrak{g}, A)$.

3. $\mathcal{B}(\mathfrak{g}, A)$ IS A SECOND COHOMOLOGY GROUP

Let (φ, F) be a fixed matched pair structure on $(\mathfrak{g}, A)$. The $\mathfrak{g}$-module structure φ on A naturally induces a $\mathfrak{g}$-module structure, which we also denote by φ, on $\wedge^q A$ for every integer $q \geq 2$:

$$\varphi(x)(a_1 \wedge a_2 \wedge \dots a_q) = \sum_{k=1}^{q} a_1 \wedge a_2 \wedge \dots a_{k-1} \wedge \varphi(x)(a_k) \wedge a_{k+1} \wedge \dots \wedge a_q .$$

Similarly, $\wedge^p \mathfrak{g}^*$ is endowed via F with A^*-module structure for every integer $p \geq 2$. We denote by ∂ (resp. δ) the coboundary operator of the Lie algebra A^* (resp. $\mathfrak{g}$) with values in the A^* (resp.$\mathfrak{g}$) module $\wedge^q A$ (resp. $\wedge^p \mathfrak{g}^*$). An element ω of $\wedge^p \mathfrak{g}^* \otimes \wedge^q A$ is a p-cochain of the Lie algebra $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\wedge^q A$. This element ω is also regarded (via its transpose) as a q-cochain of the Lie algebra A^* with values in the A^*-module $\wedge^p \mathfrak{g}^*$. In this identification, the following result is an immediate fact by definition of δ and ∂.

LEMMA.
$$\mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times_c \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*) = \{(\gamma, \Omega) \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, A) \times \mathcal{Z}^2_{\mathrm{alg}}(A^*, \mathfrak{g}^*) \mid \delta\Omega + \partial\gamma = 0\}.$$

As $(\mathfrak{g}, A)$ is a matched pair of Lie bialgebras we obtain the following differential double complex:

$$\begin{array}{ccccccc}
 & & \delta\big\downarrow & & \delta\big\downarrow & & \\
\xrightarrow{\partial} & & \wedge^p\mathfrak{g}^* \otimes \wedge^q A & \xrightarrow{\partial} & \wedge^p\mathfrak{g}^* \otimes \wedge^{q+1} A & \xrightarrow{\partial} & \\
 & & \delta\big\downarrow & & \delta\big\downarrow & & \\
\xrightarrow{\partial} & & \wedge^{p+1}\mathfrak{g}^* \otimes \wedge^q A & \xrightarrow{\partial} & \wedge^{p+1}\mathfrak{g}^* \otimes \wedge^{q+1} A & \xrightarrow{\partial} & \\
 & & \delta\big\downarrow & & \delta\big\downarrow & &
\end{array}$$

A. Masuoka ([4]) obtained this double complex from a double complex constructed on universal enveloping algebras of $\mathfrak{g}$ and A which also form a matched pair of Hopf algebras.

Let us denote by (T, D) the total differential complex associated to our double complex: $T^n = \oplus_{p,q\geq 1}^{p+q=n+1} \wedge^p \mathfrak{g}^* \otimes \wedge^q A$ and $D\mid_{\wedge^p\mathfrak{g}^*\otimes\wedge^q A} = \delta + (-1)^p\partial$.

DEFINITION. The cohomology $\mathrm{H}^{(*)}_{\mathrm{big}}(\mathfrak{g}, A)$ of a matched pair $(\mathfrak{g}, A)$ is the cohomology of the total complex (T, D) restricted to the intersection of the kernels of all vertical and horizontal operators.

As a consequence of the preceding lemma and theorem, we conclude the following:

THEOREM. *Set $\mathcal{B}(\mathfrak{g}, A)$ is the second cohomology group $\mathrm{H}^2_{\mathrm{big}}(\mathfrak{g}, A)$ of the matched pair $(\mathfrak{g}, A)$.*

This result is analogous to the work of W. M. Singer ([5]) on Hopf algebras. Let H_1 be a commutative Hopf algebra and H_2 a co-commutative Hopf algebra and let $\mathrm{Ext}(H_2, H_1)$ denote the set of all inequivalent Hopf algebra extensions of H_2 by H_1. An element of $\mathrm{Ext}(H_2, H_1)$ determines a matched structure on the pair (H_2, H_1). For such a fixed structure on the pair (H_2, H_1), the set $\mathrm{Ext}(H_2, H_1)$ is an abelian group isomorphic to the second cohomology group of a differential complex constructed out of the given matched pair (H_2, H_1), similarly as for Lie bialgebras.

4. The Double of a Co-commutative Lie bialgebra.

The double $\mathfrak{D} = \mathfrak{h} \bowtie \mathfrak{h}^*$ of a Lie bialgebra $\mathfrak{h}$ is the vector space $\mathfrak{D} = \mathfrak{h} \oplus \mathfrak{h}^*$ endowed with the following Lie bracket and Lie co-bracket:

$$[(x,\xi),(y,\eta)]_{\mathfrak{h}\oplus\mathfrak{h}^*} = ([x,y]_{\mathfrak{h}} + \mathrm{coad}_\xi\, y - \mathrm{coad}_\eta\, x, [\xi,\eta]_{\mathfrak{h}^*} + \mathrm{coad}_x\, \eta - \mathrm{coad}_y\, \xi)\,,$$

$$[(\xi,x),(\eta,y)]_{\mathfrak{h}^*\oplus\mathfrak{h}} = ([\xi,\eta]_{\mathfrak{h}^*}, -[x,y]_{\mathfrak{h}})\,.$$

In the following $\mathrm{coad}_\eta\, x$ denotes the coadjoint action of $\eta \in \mathfrak{h}^*$ on $x \in \mathfrak{h} \cong (\mathfrak{h}^*)^*$ ($\mathfrak{h}$ is finite dimensional) and $\mathrm{coad}_x\, \eta$ is the coadjoint action of $x \in \mathfrak{h}$ on $\eta \in \mathfrak{h}^*$. Endowed with the preceeding structures the double $\mathfrak{D} = \mathfrak{h} \bowtie \mathfrak{h}^*$ is in fact a Lie bialgebra.

Let $\mathfrak{g}$ be a co-commutative Lie bialgebra. The commutative (not co-commutative) Lie bialgebra $A = \mathfrak{g}^*$ is a $\mathfrak{g}$-module for the coadjoint action ($\theta = \mathrm{coad}$) and $\mathfrak{g}^*$ is considered as a trivial $A^* = \mathfrak{g}$-module i.e. $F = 0$. With these data the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a matched pair of Lie bialgebras; condition (2.1) is trivially verified and condition (2.2) is reduced to a Jacobi identity in $\mathfrak{g}$. The brackets (2.3) and (2.4) of an extension $\mathfrak{g} \oplus \mathfrak{g}^*$ of $\mathfrak{g}$ by $\mathfrak{g}^*$ described by the zero class $((0,0)) \in \mathrm{H}^2_{\mathrm{big}}(\mathfrak{g}, \mathfrak{g}^*)$ are given by:

$$[(x,\xi),(y,\eta)]_{\mathfrak{g}\oplus\mathfrak{g}^*} = ([x,y], \mathrm{coad}_x(\eta - \varphi(y)) - \mathrm{coad}_y(\xi - \varphi(x)) + \varphi([x,y]))\,,$$

$$[(\xi,x),(\eta,y)]_{\mathfrak{g}^*\oplus\mathfrak{g}} = (-\varphi^*([x,y]), [x,y]),$$

where φ is an element of $\mathrm{L}(\mathfrak{g}, \mathfrak{g}^*)$. If $\varphi = 0$ then we obtain the double $\mathfrak{g} \bowtie \mathfrak{g}^*$ with the opposite bracket on its dual $\mathfrak{g}^* \oplus \mathfrak{g}$.

A co-central extension $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$ by A is an extension of $\mathfrak{g}$ by A such that the dual extension $\widehat{\mathfrak{g}}^*$ is a central extension of A^* by $\mathfrak{g}^*$. This holds since here $F = 0$.

Proposition. *The double $\mathfrak{g} \bowtie \mathfrak{g}^*$ of a co-commutative Lie bialgebra is the trivial co-central extension of $\mathfrak{g}$ by the $\mathfrak{g}$-module $\mathfrak{g}^*$ for the coadjoint action, except for one sign.*

Remark. The double $\mathfrak{h} \bowtie \mathfrak{h}^*$ of an arbitrary Lie bialgebra $\mathfrak{h}$ is not an extension of $\mathfrak{h}$ by $\mathfrak{h}^*$; the natural projection $\mathfrak{h} \bowtie \mathfrak{h}^* \to \mathfrak{h}$ is not a Lie algebra morphism.

References

[1] Benayed, M., Extensions abéliennes des bigèbres de Lie, *Journal of Lie Theory* **8** (1998), 173 – 182.

[2] Drinfeld, V.G., Quantum groups, in "Proceeding of the International Congress of Mathematicians", Berkeley, USA, 1986, 798 – 820.

[3] Majid, S., "Foundations of Quantum Group Theory", Cambridge University Press, Cambridge, USA, 1995.

[4] Masuoka, A., Extensions of Hopf algebras and Lie bialgebras, Preprint de l'Institut Mathématique de l'Université de Munich, 1996 – 1997.

[5] Singer, W.M., Extension theory for connected Hopf algebras, *Journal of Algebra* **21** (1972), 1 – 16.