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Solvability of a Riemann Linear Conjugation Problem on a Fractal Surface†

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One of the classical boundary value problems for analytic functions (see [3], [5]) and for the solutions of more general elliptic systems in the plane (see [1]) is the Riemann conjugation problem (the simplest particular case of this problem is the so-called jump problem); it consists of finding a function analytic in the interior and exterior of a given smooth closed Jordan curve, such that the solution has a prescribed condition across the curve.

In order to solve the Riemann linear conjugation problem one of the most important method that can be found in the literature is the Kats' method (see [6], [7], [8]) that does not use contour integration and can thus be used on fractal curves.

The purpose of this note is to establish a three dimensional extension of Kats' technique.

Let S be a simple closed surface bounding a domain Ω^+ in $\mathbb{R}^3$ and let $\Omega^- = \overline{\mathbb{R}^3} \setminus \Omega^+$. Let D denote the homogeneous operator $\sum_{j=1}^3 e_j \partial/\partial x_j$, where e_j are the generators of the real quaternionic algebra.

The jump problem for quaternionic functions is the problem of finding a function $u(x)$ such that $Du = 0$ in $\mathbb{R}^3 \setminus S$ satisfying the boundary condition

$$\begin{aligned} &u^+(t) - u^-(t) = g(t), \quad t \in S, \\ &u^-(\infty) = 0, \end{aligned} \tag{1}$$

where g is a given function on S and $u^{\pm}(t)$ are the limit values of the desired function u at a point t as this point is approached from Ω^+ and from Ω^- respectively.

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DEFINITION 1. A function $u \in C^1(\mathbb{R}^3 \setminus S)$ is called a quasi–solution of the problem (1) if there exist the limit values of the function u and they satisfy the condition (1).

For example, if $g \in C(S)$ then a quasi–solution of the jump problem can be obtained from the formula:

$$u_0(x) = g^\omega(x) = X_{\Omega^+}(x)\Psi_0 g(x), \tag{2}$$

where X_{Ω^+} is the characteristic function of the set Ω^+ and Ψ_0 is the Whitney extension operator.

Concerning solvability of the problem (1), we have the next theorem.

THEOREM 1. *The jump problem (1) is solvable if and only if there exists a quasi–solution u such that the support of Du is compact and $Du \in L^p(\mathbb{R}^3)$ for $p > 3$.*

The proof of this theorem is based on the fact that the function

$$u_0(x) = u(x) - \int_{\mathbb{R}^3} e(x-y)Du(y)\,dy, \tag{3}$$

where $e(z) = -\dfrac{z}{4\pi|z|^3}$, is a solution of (1).

Let $H_\nu(S)$ denote the set of functions that satisfy a Hölder condition on S with exponent ν and let $\alpha(S)$ be the cell dimension of S (see [2]). The next result easily follows from the construction of the Whitney partition.

LEMMA 1. *Suppose that $0 < \nu < 1$ and $\alpha = \alpha(S) < 3$. If $g \in H_\nu(S)$ then $Dg^\omega \in L^p$ for $p < (3-\alpha)/(1-\nu)$.*

An immediate consequence of Lemma 1 and the Borel–Pompei's formula (see [4, p. 28]) is the following theorem.

THEOREM 2. *If $g \in H_\nu(S)$ with $1 \geq \nu > \alpha/3$ then the problem (1) is solvable.*

The following theorem is the main result of the paper.

THEOREM 3. *Suppose that $\alpha = \alpha(S) < 3$, $g \in H_\nu(S)$ and that the Hausdorff dimension $\alpha_H(S)$ of the set S is such that*

$$\alpha_H(S) - 2 < \mu < \frac{3\nu - \alpha}{3 - \alpha}.$$

Then, the function (3), with $u \equiv g^{\omega}$, is the unique solution of the jump problem (1) of class

$$H_{\mu} = \left\{ u(x) : \ Du = 0 \ on \ \overline{\mathbb{R}^3} \setminus S, \ and \ u^{\pm} \in H_{\mu}(\overline{\Omega}^{\pm}) \right\}.$$

Proof. Having into account a three dimensional extension of E.P. Dolzhenko's theorem and in virtue of Theorem 2, the proof is an easy traslation of the one given in [7, Theorem 2]. ∎

References

[1] BEGEHR, H.G., GILBERT, R.P., On Riemann boundary value problem for certain linear elliptic systems in the plane, *J. Differential Equations,* **32** (1979), 1–14.

[2] FALCONER, K.J., "The Geometry of Fractal Sets", Cambridge Univ. Press, Cambridge, 1985.

[3] GAKHOV, F.D., "Boundary Value Problems", Nauka, Moscow, 1980.

[4] GÜRLEBECK, K., SPRÖSSIG, W., "Quaternionic Analysis and Elliptic Boundary Value Problem", Birkhaüser–Verlag, Basel, 1990.

[5] JIAN, K.L., "Boundary Value Problems for Analytic Functions", World Scientific Publish., Singapure–New Jersey–London–Hong Kong, 1993.

[6] KATS, B.A., On the Riemann boundary value problem on a fractal curve, *Russian Acad. Sci. Dokl. Math.,* **48** (3) (1994), 559–561.

[7] KATS, B.A., The Riemann problem on a closed Jordan curve, *Soviet Math. (Iz. VUZ),* **27** (1983), 83–98.

[8] KATS, B.A., A version of Riemann boundary value problem on a fractal curve, *Izv. Vyssh. Uchebn. Zaved. Mat.,* **4** (383) (1994), 10–20.