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On a Theorem of van Douwen

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A topological Abelian group G is maximally almost periodic (MAP) in the sense of von Neumann if for every $e_G \neq x \in G$ there exists a continuous character χ (i.e., a continuous homomorphism into the one-dimensional torus $\mathbb{T}$) such that $\chi(x) \neq 1$. Every maximally almost periodic Abelian group G embeds as a dense subgroup in a compact group which is called the Bohr compactification of G and is usually denoted by bG. The topology that the group G inherites from its Bohr compactification is then called the Bohr topology of G. Following van Douwen, we will write $G^{\#}$ to denote a discrete Abelian group G endowed with its Bohr topology which turns to be the maximal totally bounded group topology of G.

In his paper [3] van Douwen proved (Theorem 1.1.3 (a)) the following remarkable theorem:

Theorem 1. (van Douwen) *If G is an Abelian group, then every infinite subset A of $G^{\#}$ has a relatively discrete subset D with $|D| = |A|$ that is C-embedded in $G^{\#}$ and is C^*-embedded in bG.*

Van Douwen's proof is very technical and, therefore, it is difficult to place the key ideas used in it. The goal of this paper is to present an alternative and much simpler proof of that result and, over all, to make it clear the main concepts we have used in our approach. This is achieved mainly with Proposition 1 and Proposition 2, below, which intend to make clear which are the foundations that lie in our approximation to the problem and suggest the main line for subsequent extensions of van Douwen's theorem.

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Let I_1 and I_{-1} be two closed and disjoint intervals in the one-dimensional torus $\mathbb{T}$ which contain at least one n-th root of the unity for each $n \geq 2$. We will say that an Abelian group G satisfies property P if every infinite subset A of G contains a subset B, with $|B| = |A|$ such that, for every map $\phi : B \to \{1, -1\}$, there exists a character χ of G satisfying that $\chi(b) \in I_{\phi(b)}$ for every $b \in B$.

PROPOSITION 1. *If G is an Abelian group all whose quotient groups satisfy property P, then van Douwen's theorem holds for G.*

We will split the proof of this Proposition in several lemmas in order to dissociate technical dificulties from the essence of the theorem.

Throughout Lemma 1 we adopt the notation of [3]. Given a topological space X, we say that a set A is X-embedded in another topological space B if $A \subseteq B$ and every continuous function of A into X admits a continuous extension to B.

LEMMA 1. *Let G be a discrete Abelian group and let A be an independent subset of G such that all the elements in A have the same order, then A is discrete and C-embedded in $G^{\#}$.*

Proof. That A is closed and discrete in $G^{\#}$ whenever G is discrete and A is independent, is proved in [4].

Let us denote by H the subgroup generated by A. By [1, Theorem 6.3] the group $H^{\#}$ is $\mathbb{Z}$-embedded and $\mathbb{R}$-embedded in $G^{\#}$. On the other hand, since A is discrete in $G^{\#}$, an application of Lemma 2.1.1 of [3] shows that, in order to prove that A is $\mathbb{R}$-embedded in $G^{\#}$, it is enough to prove that A is $\mathbb{Z}$-embedded in $G^{\#}$. Summing up, to show that A is $\mathbb{R}$-embedded in $G^{\#}$, (i.e. C-embedded) it will be enough to prove that A is $\mathbb{Z}$-embedded in $H^{\#}$.

So, let $f : A \longrightarrow \mathbb{Z}$ be any function of A into $\mathbb{Z}$. For each $n \in f(A) \subseteq \mathbb{Z}$, pick exactly one $a_n \in A$ with $f(a_n) = n$ and denote by L the subgroup generated by the set $\{a_n \colon n \in f(A)\}$. Next, define a mapping $g : A \longrightarrow \{a_n \colon n \in f(A)\}$, by

$$g(a) = a_n \quad \text{when} \quad f(a) = n\,.$$

Since the family $\{f^{-1}(n) \colon n \in f(A)\}$ defines a partition of A, it is clear that g is well defined. Since A is an independent subset of G and all the elements in A (and hence in $g(A)$) have the same order, it is possible to extend g to a homomorphism $\overline{g} : H \longrightarrow L$. As it is well known, $\overline{g}$ will be a continuous as a mapping of $H^{\#}$ into $L^{\#}$.

Now, the set $\{a_n : n \in f(A)\}$ is a discrete closed subset of $L^{\#}$ (note that it is also an independent subset of L), thus, by [3, Lemma 2.1.2], the restriction of the map f to $\{a_n : a \in f(A)\}$ can be extended continuously over $L^{\#}$ to a function, say, $\overline{f} : L^{\#} \longrightarrow \mathbb{Z}$. The composition $\overline{f} \circ \overline{g}$ provides then the required extension of f over $H^{\#}$. ∎

LEMMA 2. *Let G be a discrete Abelian group and let A be an uncountable subset of G such that $o(x) \leq m < \omega$ for all $x \in A$. Then, there is a subset B of A, with $|B| = |A|$ which is discrete and C-embedded in $G^{\#}$.*

Proof. We proceed by induction on $m < \omega$. If $m = 2$, we can take a maximal independent subset $B \subseteq A$. Since for every x in A there must exist some $n \in \mathbb{Z}$ such that $0 \neq nx \in < B >$ and $o(x) = 2$ for all $x \in A$, it is clear that $|B| = |A|$. It is then enough to apply Lemma 1 to B.

Suppose now that the result is proved for every group G and every subset A of G provided that $o(x) \leq n < \omega$ for all $x \in A$, and consider $A \subseteq G$ such that $o(x) \leq n+1$ for all $x \in A$. By the inductive hypothesis, it can be assumed that $o(x) = n + 1$ for all $x \in A$.

Take a maximal independent B subset of A. If $|B| = |A|$, then we can apply Lemma 1 to B and the proof is complete.

Thus, we can assume that $|B| < |A|$. Define $H =< B >$. Then, (by the maximality of B) for each $x \in A$, there must exist $m < n + 1$ such that $mx \in < B >$. If $p : G \longrightarrow G/H$ denotes the canonical epimorphism, then $o(p(x)) \leq n$ for all $x \in A$. Since $|A| \leq |p(A)| \cdot |B|$ and $|B| < |A|$, it follows that $|A| = |p(A)|$. By the inductive assumption, there is a discrete subset $\widetilde{B}$ of $p(A)$ with $|\widetilde{B}| = |p(A)| = |A|$ which is C-embedded in $(G/H)^{\#}$.

Now, for every $\tilde{x} \in \widetilde{B}$ pick exactly one element $x \in A$ such that $p(x) = \tilde{x}$. It is obtained thus, a subset $B = \{x \in A : \tilde{x} \in \widetilde{B}\}$ with $|B| = |A|$. It is easily verified that B is C-embedded in $G^{\#}$. ∎

The following Lemma will be necessary to reduce the general problem to the cases studied in the previous ones. The idea of considering a clopen partition of the group $G^{\#}$ is inspired in Proposition 2.4 of [4].

LEMMA 3. *Let G be a discrete Abelian group all whose quotient groups verify property P and let A be an infinite subset of G. If H is any subgroup of G such that $|H| < |A|$, then there exists a closed and open neighbourhood V of H in $G^{\#}$ such that $|A \setminus V| = |A|$.*

Proof. Let $p : G \longrightarrow G/H$ be the canonical mapping. Since $|H| < |A|$ and $|A| \leq |p(A)| \cdot |H|$, it follows that $|A| = |p(A)|$. On the other hand the group G/H verifies property P. Consequently, it may be found a subset $B_1 \subseteq p(A)$ with $|B_1| = |p(A)|$, a characater $\chi \in \widehat{(G/H)}$ and a neighbourhood U of the identity in $\mathbb{T}$ such that $\chi^{-1}(U) \cap B_1 = \emptyset$. Recalling that the topology of $(G/H)^\#$ is zero-dimensional ([3, Theorem 4.8]), it is possible to find a closed and open neighbourhood W of the identity in $(G/H)^\#$ such that $W \subseteq \chi^{-1}(U)$. Then observing that

$$|A \setminus p^{-1}(W)| \geq |p(A \setminus (p^{-1}(W))| \geq |p(A \setminus p^{-1}(G/H \setminus B_1))| \geq |B_1| = |A|$$

the lemma follows by considering $V = p^{-1}(W)$. ∎

Now, we are ready to finish the proof of Proposition 1.

Proof of Proposition 1. Let G be an Abelian group and let A be an infinite subset of G. Since G satisfies property P, it will contain a subset B with $|B| = |A|$ with the properties stated above. Take $b \in B$. Then there exist two closed disjoint intervals in $\mathbb{T}$ and a character $\chi \in \widehat{G}$ such that $\chi(b) \in I_1$ and $\chi(B \setminus \{b\}) \subseteq I_{-1}$. Then $\chi^{-1}(\mathbb{T} \setminus I_{-1})$ is a neighbourhood of b in $G^\#$ which intersects B exactly in the point b. This is to say that B is discrete in $G^\#$.

On the other hand, to see that B is C^*-embedded in bG, it is enough to prove that every pair of disjoint subsets of B, say B_1 and B_{-1}, is completely separated, i.e., there is a continuous function $f \in C(bG)$ such that $0 \leq f \leq 1$, $f(B_1) = \{0\}$ and $f(B_1) = \{1\}$. Applying again that G satisfies property P, we can find two closed disjoint intervals in $\mathbb{T}$, I_1 and I_{-1} and a character $\chi \in \widehat{G}$ such that $\chi(B_1) \subseteq I_1$ and $\chi(B_{-1}) \subseteq I_{-1}$. Denoting by d the usual metric of the complex plane, we can define $g : \mathbb{T} \to \mathbb{R}$ by

$$g(t) = \frac{d(t, I_1)}{d(t, I_1) + d(t, I_{-1})} \quad .$$

Clearly, g is continuous, $0 \leq g \leq 1$, $g(I_1) = \{0\}$ and $g(I_{-1}) = \{1\}$. Every character of G can be extended to a continuous character of the Bohr compactification. So, if $\tilde{\chi}$ is the extension of χ to bG, the function $g \circ \tilde{\chi}$ shows that B_1 and B_{-1} are completely separated. Thus B is C^*-embedded in bG. Note that B cannot be relatively compact in $G^\#$, because $cl_{bG}B$ is homeomorphic to βB, the Stone-Čech compactification of B, $cl_{G^\#}B$ is a subset of $< B >$, since every subgroup is closed in $G^\#$, and $| <B> | = |B|$.

Suppose now that A is countable. Being countable, $G_0 =<B>^\#$ is realcompact and it follows that every functionally bounded subset of G_0 is relatively compact. Hence, there is $f \in C(G_0, \mathbb{R})$ such that $f_{|_B}$ is unbounded.

That is, there is a subset $D = \{y_n\}_{n=1}^{\infty} \subseteq B$ such that $|f(y_{n+1})| > |f(y_n)| + 1$ for every $n < \omega$. Let now $V_n = \{x \in G_0 : |f(x) - f(y_n)| < 1/2\}$. Since G_0 is completely regular we can take, for every $n < \omega$, $f_n \in C(G_0)$ with the properties $0 \leq f_n \leq 1$, $f_n(y_n) = 1$ and $f_n(G_0 \setminus V_n) = \{0\}$. If now g is any real valued continuous function on D, the function $\sum_{n<\omega} g \cdot f_n$ is a continuous extension of g to G_0. Since G_0 is C-embedded in $G^{\#}$ ([1, theorem 6.3]) it follows that D is discrete and C-embedded in $G^{\#}$ and C^*-embedded in bG.

Suppose finally that A (and hence B) is not countable. Since $B = (B \cap tG) \cup (B \cap G \setminus tG)$, it may be assumed that either all the elements of B are torsion free or all of them have finite order.

Suppose first that $B \subseteq tG$, the torsion part of G. Denote in this case by t_nG the subset $\{x \in G\colon o(x) \leq n\}$ and let $B_n = B \cap (t_nG)$. If $|B_n| = |B| = |A|$ for some $n < \omega$, then Lemma 2 applies and we are done.

So, we can suppose that $|B_n| < |B|$ and define $H_n = \langle B_n \rangle$, for all $n < \omega$. Then $H = \bigcup_{n<\omega} H_n$ is a subgroup of G and $|B_n| \leq |H_n| < |B| = |H|$ for all $n < \omega$. Since $H^{\#}$ is C-embedded in $G^{\#}$ (see [1]), it will be enough to prove that B is C-embedded in $H^{\#}$.

Since $B = \bigcup_{n<\omega} B_n$, it follows that $|B|$ has countable cofinality. Thus it may be chosen a sequence of regular cardinals $\kappa_1 < \cdots < \kappa_n < \cdots$ such that $\sup_n \kappa_n = |B|$. Recall that a cardinal number m is called "regular" if there is no way of partitioning a set of power m into less than m pieces of smaller power or, equivalently, if the cofinality of m is exactly m (see [2], Chapter 1).

By induction on $n < \omega$ we will construct an integer m_n, a closed and open neighbourhood U_n of H_n in $H^{\#}$ and a discrete C-embedded subset D_n of $B_{m_n} \setminus U_{n-1}$ such that

1. $0 < m_1 < m_2 < \cdots < \cdots m_n < \cdots$
2. $\emptyset = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_n \subseteq \cdots$
3. $|D_n| = \kappa_n$, for all $n < \omega$ and
4. $|B \setminus U_n| = |B|$ for all $n < \omega$.

Given that κ_1 is a regular cardinal, there is $m_1 \in \omega$ such that $|B_{m_1}| = |H_{m_1}| \geq \kappa_1$. Then, Lemma 2 applied to B_{m_1} provides a subset $D_1 \subseteq B_{m_1}$ with $|D_1| = \kappa_1$ which is discrete and C-embedded in $G^{\#}$. And Lemma 3 allows us to define a closed and open neighbourhood U_1 of H_{m_1} such that $|B \setminus U_1| \geq |B|$.

Now suppose that m_n, D_n and U_n have been already defined so that they hold the inductive assumptions 1-4.

There is then $m_{n+1} \in \omega$ such that $m_{n+1} > m_n$ and

$$|B_{m_{n+1}}| = |H_{m_{n+1}}| \geq \kappa_{n+1} \ .$$

Lemmas 2 and 3, now applied to the subset $B \setminus U_n$ yield a subset $D_{n+1} \subseteq B_{m_{n+1}} \setminus U_n$ with $|D_{n+1}| = \kappa_{n+1}$ which is discrete and C-embedded in $G^{\#}$ and a closed and open neighbourhood V of H_{n+1} such that $|B \setminus (U_n \cup V)| \geq |B|$. Defining the set $U_{n+1} = U_n \cup V$ the inductive process is complete.

Consider now $D = \bigcup_{n<\omega} D_n$. Clearly $|D| = |B| = |A|$; let us see that D is C-embedded in $G^{\#}$ or, equivalently, that it is $\mathbb{Z}$-embedded in $G^{\#}$. To this end let $f : D \longrightarrow \mathbb{Z}$ be arbitrarily chosen and define $f_n = f_{|D_n}$. Since every D_n is C-embedded, there is $g_n \in C(H^{\#}, \mathbb{Z})$ that extends continuously f_n. Now, since $\{U_n \setminus U_{n-1}\}_{n<\omega}$ is a closed and open partition of $H^{\#}$ we conclude that, denoting by $\chi_{U_n \setminus U_{n-1}}$ the corresponding characteristic function, the map $g : H^{\#} \longrightarrow \mathbb{Z}$ defined by

$$g = \sum_{n<\omega} g_n \chi_{U_n \setminus U_{n-1}}$$

is the desired continuous extension of f ot $H^{\#}$. Thus D (which is a subset of B) is discrete and C-embedded in $G^{\#}$ and C^*-embedded in bG.

Suppose now that the order of all the elements in B is infinite. Consider again a maximal independent subset C contained in B. If $|C| = |B| = |A|$ it suffices to apply Lemma 1 to finish the proof. Otherwise, we define $H = <C>$ and $\pi : G \longrightarrow G/H$. Then, $|B| = |\pi(B)|$ and $o(\pi(x)) < \infty$ for all $x \in B$. Applying the foregoing paragraphs to $\pi(B)$ it follows the existence of a subset $\widetilde{C} \subseteq \pi(B)$ with $|\widetilde{C}| = |\pi(B)| = |B|$ which is discrete and C-embedded in $(G/H)^{\#}$. Now it is easy to find, as in Lemma 1, a subset $C \subseteq \pi^{-1}(\widetilde{C}) \cap B$ with $|C| = |B|$ which verifies the required properties. The proof is now done. ∎

We will split the proof of the Theorem in two parts, first we will prove the finitely generated case.

PROPOSITION 2. *Every finitely generated Abelian group satisfies van Douwen's theorem.*

Proof. Like every finitely generated Abelian group, G is isomorphic to $\mathbb{Z}^n \times F$ for some non-negative integer n and some finite Abelian group F.

Let A be an infinite subset of G. Let $\pi_i : G \longrightarrow H_i$ where H_i is isomorphic with $\mathbb{Z}$ if $1 \leq i \leq n$ and H_{n+1} is isomorphic to F. Clearly there will exist i, with $1 \leq i \leq n$ such that the set $A_i = \pi_i(A) \subseteq \mathbb{Z}$ is infinite.

Consider a sequence $\{z_n\}_{n=1}^{\infty} \subseteq A_i$ such that $|z_n| \geq |4z_{n-1}|$ and let $\{I_n\}_{n=1}^{\infty}$ be a sequence of intervals in $\mathbb{T}$ such that $I_n = I_1$ or $I_n = I_{-1}$ for every $n < \omega$. Here I_1 and I_{-1} are two closed and disjoint intervals in $\mathbb{T}$ of length $\rho(I_i) > 2\pi/3$ centered in 1 and -1 respectively. Clearly each of them contains n-th rooths of the unity for every $n \geq 2$. Denote by $\{t_n\}_{n=1}^{\infty}$ the sequence of middle points of the intervals I_n.

Let J_1 be an interval in $\mathbb{T}$ such that $z_1(J_1) = I_1$ and such that the length of J_1, $\rho(J_1)$ is $2\pi/3|z_1|$.

Suppose now that a sequence of intervals in $\mathbb{T}$, $J_1 \ldots J_k$ has been chosen such that $J_i \subseteq J_{i-1}$, $\rho(J_i) \geq 2\pi/3|z_i|$. Consider now the set $z_{k+1}(J_k) = \{e^{2\pi i z_{k+1} t} : e^{2\pi i t} \in J_k\}$. Since $|z_{k+1} \cdot \rho(J_k)| \geq 2\pi + 2\pi/3$, it follows that there exists x_{k+1} with $e^{2\pi i x_{k+1}} \in J_k$ such that $e^{2\pi i z_{k+1} x_{k+1}} = t_{k+1}$. If $\{e^{2\pi i t} : |t - x_{k+1}| < 1/(6|z_{k+1}|)\}$ is contained in J_k we choose this set to be J_{k+1}. Otherwise, $e^{2\pi i (x_{k+1}-1)} \in J_k$ and we can choose $J_{k+1} = \{e^{2\pi i t} : |t - (x_{k+1} - 1)| < 1/(6|z_{k+1}|)\}$ which will have to be contained in J_k. Moreover since $e^{2\pi i z_{k+1} x_{k+1}} = e^{2\pi i z_{k+1}(x_{k+1}-1)} = t_{k+1}$, (the center of I_{k+1}) and the length of J_{k+1} is exactly $2 \cdot \pi/(3 \cdot |z_{k+1}|)$ it follows that in any case, $z_{k+1}(J_{k+1}) \subseteq I_{k+1}$.

It has been thus constructed a sequence $\{J_n\}_{n=1}^{\infty}$ of closed intervals in $\mathbb{T}$ such that $z_n(J_n) \subseteq I_n$. Since the intervals satisfy the finite intersection property, one point $x_0 \in \bigcap_{n=1}^{\infty} J_n$ may be chosen. This point can be identified with a character of $\mathbb{Z}$ such that $x_0(z_n) \in I_n$for all $n < \omega$.

Now choose a sequence $\{y_n\} \subseteq A$ such that $\pi_i(y_n) = z_n$. The character group $\widehat{G}$ of G is exatly $\mathbb{T} \times F$, so if $\chi \in \widehat{G}$ is the character of G with i-th. projection equal to x_0 and j-th. projection equal to 0 for all $j \neq i$, then it also holds that $\chi(y_n) \in I_n$ for all $n < \omega$. Since the subset A and the sequence $\{I_n\}_{n=1}^{\infty}$ were arbitrarily chosen, this means that G satisfies property P. Every quotient of a finitely generated group is also finitely generated, hence by Proposition 1 G also satisfies van Douwen's theorem. ∎

THEOREM 2. *Let G be a discrete Abelian group. Every subset A of $G^{\#}$ has a relatively discrete subset B, with $|B| = |A|$, such that B is C-embedded in $G^{\#}$ and C^*-embedded in bG.*

Proof. In view of Proposition 1 it suffices to prove that every discrete Abelian group G satisfies property P. In order to do this, consider I_1 and I_{-1} two closed disjoint intervals in $\mathbb{T}$, both of them containing at least one n-rooth of the unity for each $n \geq 2$.

Given any subset C of A and any mapping $\phi : C \to \{1, -1\}$ we define

$$N(\phi, C) = \{\chi \in \widehat{G} : \chi(c) \in I_{\phi(c)} \text{ for all } c \in C \}.$$

Now, we introduce a family $\mathcal{X}$ of subsets of A, as follows:

$$\mathcal{X} = \{C \subseteq A : \quad N(\phi, C) \neq \emptyset \text{ for all } \phi \in \{1, -1\}^C\}.$$

To show that the family $\mathcal{X}$ is not empty, take $x \in A$ and consider $\phi : B \to \{1, -1\}$. If x has infinite order, then we can define $\chi(x) \in I_{\phi(x)}$ arbitrarily and extend it to the subgroup generated by x. If $o(x) = n$, then we choose $t \in I_{\phi(x)}$ such that $t^n = 1$, and then it can be defined a homomorhism χ of $<x>$ into $\mathbb{T}$ with $\chi(x) = t$. Since $\mathbb{T}$ is divisible, the resulting homomorphism can be extended in both cases to a character of G ([5, A.14]). This character belongs to $N(\phi, \{x\})$. Therefore $\mathcal{X}$ is not an empty family.

The set $\mathcal{X}$ can be ordered in the following way: given B_1 and B_2 in $\mathcal{X}$, $B_1 \preceq B_2$ if and only if:

1. $B_1 \subseteq B_2$ and
2. $(N(\phi, B_2))_{|B_1} = (N(\phi_{|B_1}, B_1))_{|B_1}$ for all $\phi \in \{1, -1\}^{B_2}$.

Here, given a subset D of $\widehat{G}$ and a subset Y of G, we denote by $D_{|Y}$ the subset $\{\chi_{|Y} : \chi \in D\}$.

The set $\mathcal{X}$ with the order introduced above is inductive. Indeed, assume that $\{C_\delta : \delta \in \Delta\}$ is a chain in $\mathcal{X}$ and define $C = \bigcup_{\delta \in \Delta} C_\delta$. Given any mapping $\phi : C \to \{1, -1\}$ it is easily verified that $N(\phi, C) = \bigcap_{\delta \in \Delta} N(\phi_{|C_\delta}, C_\delta)$. Hence to show that C belongs to $\mathcal{X}$ it is enough to show that the family $\{N(\phi_{|C_\delta}, C_\delta) : \delta \in \Delta\}$ has the finite intersection property. But this is clear since for any finite sequence $\delta_1 \ldots \delta_n$, the sets can be ordered such that $C_{\delta_1} \preceq \ldots \preceq C_{\delta_n}$ and then $\bigcap_{i=1}^n N(\phi_{|C_{\delta_i}}, C_{\delta_i}) = N(\phi_{|C_{\delta_n}}, C_{\delta_n}) \neq \emptyset$. In order to check that the set C is an upper bound of the chain, take δ an arbitrary element of Δ and any element $\chi \in N(\phi_{|B_\delta}, B_\delta)$. For every $\delta \in \Delta$ define

$$P_\delta = \{\psi \in N(\phi_{|C_\delta}, C_\delta) : \psi_{|C_\delta} = \chi_{|C_\delta}\}.$$

Since $\{C_\delta : \delta \in \Delta\}$ is a chain, it is easy to verify that the family of compact sets $\{P_\delta\}_{\delta \in \Delta}$ has the finite intersection property and an element $\psi \in \bigcap_{\delta \in \Delta} P_\delta$ can be chosen. Then $\psi \in N(\phi, C)$ and $\psi_{|B_\delta} = \chi_{|B_\delta}$, showing that $B_\delta \preceq C$ and that C is an upper bound of the chain. Hence we may apply Zorn's Lemma to find a maximal set B in $\mathcal{X}$ for the order $\preceq$.

Suppose now that $A \not\subseteq < B >$, then there must exist $a \in A\backslash < B >$. Consider $B^* = B \bigcup \{a\}$.

Take any mapping $\phi : B^* \to \{1, -1\}$ and let $\chi \in N(\phi_{|_B}, B)$. If $< a > \cap < B > = \{0\}$, then any function coinciding with χ in $< B >$ and mapping a into $I_{\phi(a)}$ can be extended to a homomorphism ψ of G into $\mathbb{T}$. On the contrary, if $< a > \cap < B > \neq \{0\}$, choose n such that na generates $< a > \cap < B >$. Since $I_{\phi(a)}$ contains one n-rooth of the unity, say t_n, we can define a function coinciding with χ in $< B >$ and mapping a onto t_n and extend it to a homomorphism ψ of G into $\mathbb{T}$. So, in either case, we obtain that $\psi \in N(\phi, B^*)$ and that $\psi_{|_B} = \chi_{|_B}$. Thus, $B^* \in \mathcal{X}$ and $B \prec B^*$. This goes against the maximality of B and implies that $A \subseteq < B >$.

If B is infinite, it follows that $|A| \leq | < B > | = |B|$. Hence $|A| = |B|$ and the proof is done.

If B is finite, we may apply Proposition 2 to the group $< B >$. Since the Bohr compactification of $< B >$ is contained in bG and $< B >^{\#}$ is C-embedded in $G^{\#}$ ([1, 6.3]) the result also follows. ∎

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