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Norm and Pointwise Topologies Need not to be Binormal

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1. Introduction

Let $m(\Gamma)$ denotes the space of all bounded functions on the ordinal Γ. The *pointwise* topology on $m(\Gamma)$ is the topology of coordinate convergence in the $m(\Gamma)$, we use U pointwisely open, F pointwisely closed ... with respect to this topology and we use $\mathcal{U}$ norm open, $\mathcal{F}$ norm closed ... with respect to the *norm* topology on $m(\Gamma)$.

Let $m_0(\Gamma)$ denotes the subspace of functions $\alpha(n) \in m(\Gamma)$ that converge to 0 for n approaching Γ, i.e. for each $\varepsilon > 0$ there exists $\Lambda < \Gamma$ such that $|\alpha(n)| < \varepsilon$ for $n > \Lambda$. The space $m_0(\Gamma)$ inherit both pointwise and norm topologies from $m(\Gamma)$.

We say that topologies *blue* and *green* on a space X are *binormal* if for each pair of disjoint subsets F and $\mathcal{F}$ of X, F blue closed, $\mathcal{F}$ green closed, there are disjoint subsets G and $\mathcal{G}$ of X, G blue open, $\mathcal{G}$ green open, such that $\mathcal{F} \subset G$, $F \subset \mathcal{G}$ ([2], [1, p. 85]).

2. Result

Theorem 2.1. *Let $X = m(\Gamma)$ for $\Gamma \geq \omega$ or $X = m_0(\Gamma)$ for $\Gamma > \omega$. Then the norm and the pointwise topologies on the space X are not binormal.*

Proof. Set $F = \{\alpha \in X, \alpha = (a_I)_{I\in\Gamma}, a_0 = 0, a_I \in \{0,1\}\}$. We see that F is pointwisely closed (for $\alpha \in X \setminus F$ we find a pointwisely open neighborhood disjoint with F). Set $\mathcal{F} = \{\alpha \in X, \alpha = (a_I)_{I\in\Gamma}, a_0 = 1/n, a_I =$

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1 exactly for n natural $I, a_I = 0$ elsewhere }. Obviously F and $\mathcal{F}$ are disjoint. We claim:

CLAIM: $\mathcal{F}$ is norm closed.

Proof of Claim. Let $\alpha \in X \setminus \mathcal{F}$, then

(i) if $a_I \neq 0, 1$ for $I > 0$ the situation is very simple, we find a norm neighborhood of α disjoint with $\mathcal{F}$;

(ii) if $a_0 \neq 0, 1/n$ similarly;

(iii) for $a_0 = 0$ we have two cases:

case (1): when $a_I = 1$ for infinitely many indexes - the $1/2$ norm neighborhood of α is disjoint with $\mathcal{F}$;

case (2): when $a_I = 1$ for n indexes - the $1/2n$ norm neighborhood of α is disjoint with $\mathcal{F}$;

(iv) for $a_0 = 1/n$ we have three cases:

case (1): when $a_I = 1$ for infinitely many indexes - $1/2$ norm neighborhoodof α is disjoint with $\mathcal{F}$;

case (2): when $a_I = 1$ for n indexes - impossible for $\alpha \in X \setminus \mathcal{F}$;

case (3): when $a_I = 1$ for $m \neq n$ indexes - $|1/n - 1/m|/2$ norm neighborhood of α is disjoint with $\mathcal{F}$.

The claim is proved. ∎

It remains to prove that each pointwisely open set U containing norm closed set $\mathcal{F}$ meets each norm open set $\mathcal{U}$ containing pointwisely closed set F.

Let such U is given, we construct $\alpha_\omega \in F$ such that each norm neighborhood $\mathcal{U}$ of α_ω meets U. We construct by induction:

(1) We take $\alpha_1 \in \mathcal{F}$ with the value 1 on the index set $A_1 = \{1\}$. In U the point α_1 has a pointwise neighborhood U_1 controlling 'ones' on the index set A_1 and controlling 'zeroes' on an index set B_1.

(n) For $n > 1$ we take $\alpha_n \in \mathcal{F}$ with exactly n values 1 on the index set $A_n \subset \mathbb{N}$, $A_{n-1} \subset A_n$, $B_{n-1} \cap A_n = \emptyset$. In U the point α_n has a pointwise neighborhood U_n controlling 'ones' on the index set A_n and controlling 'zeroes' on an index set B_n, $B_{n-1} \subset B_n$.

(ω) We take $\alpha_\omega \in F$ with the value 1 exactly on the index set $A_\omega = \bigcup_{n \in \mathbb{N}} A_n$.

We see that any ε-norm neighborhood $\mathcal{U}$ of α_ω meets the pointwise neighborhood U_n (for $1/n < \varepsilon$) in the point $\alpha = (1/n, \dots$ the rest like $\alpha_\omega \dots)$. It means that U meets $\mathcal{U}$.

The theorem is proved. ∎

References

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