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A Matrix Analysis of the Stability of the Clenshaw Algorithm

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1. Introduction

Finite series of Chebyshev polynomials are often used in several fields of pure and applied mathematics [8]. In this paper we present an error analysis of the Clenshaw algorithm (Clenshaw [2]), which is an elegant and economic summation technique for finite series of Chebyshev polynomials.

Let be $p_n^I(x) = \sum_{i=0}^n c_i\, T_i(x)$ or $p_n^{II}(x) = \sum_{i=0}^n c_i\, U_i(x)$ where $T_i(x)$ and $U_i(x)$ are, respectively, the Chebyshev polynomials of the first and second kind. The Clenshaw algorithm can be expressed as

$$
\begin{aligned}
& q_{n+1} = q_{n+2} = 0, \\
(1) \quad & q_j = 2\,x\,q_{j+1} - q_{j+2} + c_j, \qquad \text{for} \quad j = n, \dots, 1, \\
& p_n^I(x) = x\,q_1 - q_2 + c_0 = q_0, \qquad \text{or} \quad p_n^{II}(x) = 2\,x\,q_1 - q_2 + c_0 = q_0.
\end{aligned}
$$

Elliot [3], Newbery [5], Oliver [6, 7] and Schonfelder and Ramaz [9] present an error analysis of the evaluation of Chebyshev series by using the Clenshaw algorithm and some variations of it. In this paper, we present new error bounds based on the matrix formulation of the Clenshaw's algorithm, which allows its generalization to other families of polynomials. This approach permits to follow the analysis of triangular systems [4], but adapted to the evaluation of Chebyshev series.

2. Preliminaries

In what follows $\widehat{a}$ represents the computed value of a, u the unit roundoff of the computer and, given a matrix A, $|A|$ stands for the matrix whose elements are $|a_{ij}|$. Besides, we assume that the computations are carried out in a floating–point arithmetic that obeys the models (see [4] and [10])

$$\text{(2)}\quad \mathrm{fl}(x \,\mathrm{op}\, y) = (x \,\mathrm{op}\, y)\,(1+\rho), \qquad \mathrm{fl}(x \,\mathrm{op}\, y) = \frac{(x \,\mathrm{op}\, y)}{1+\alpha}, \qquad |\rho|\,,\, |\alpha| \le u,$$

where op $\in \{+, -, \times, \div\}$.

In the analysis, it will be useful a new family of polynomials.

DEFINITION 1. The absolute Chebyshev polynomials of the second kind, $U_i^A(x)$, are defined by the triple recurrence relation

$$\begin{array}{ll} U_0^A(x) = 1, & U_1^A(x) = 2\,|x|\,, \\ U_i^A(x) = 2\,|x|\,|U_{i-1}(x)| + |U_{i-2}(x)|\,. & \end{array} \tag{3}$$

Trivially we have that $|U_i(x)| \le U_i^A(x)$.

Let T be the banded matrix:

$$T = \begin{pmatrix} 1 & -2x & 1 & & & & \\ & 1 & -2x & 1 & & 0 & \\ & & 1 & -2x & \ddots & & \\ & & & 1 & -2x & \ddots & \\ & & & & \ddots & \ddots & 1 \\ & 0 & & & & \ddots & -2x \\ & & & & & & 1 \end{pmatrix}. \tag{4}$$

The Clenshaw algorithm in the case of Chebyshev polynomials of the second kind is equivalent, using matrix formulation, to solve the triangular banded system (3 diagonals) of linear equations $T\,q = c$, where T is given by (4), $q = (q_0, q_1, \dots, q_n)^T$ of Eq. (1), $c = (c_0, c_1, \dots, c_n)^T$ (the coefficients vector) and $p_n(x) = q_0$. In the case of Chebyshev polynomials of the first kind the linear system is similar, but now using a matrix $\widetilde{T}$, that is the same as T but $\widetilde{t}_{1,2} = -x$ instead of $-2x$.

We only present the analysis of the algorithm with the matrix T because the difference between T and $\widetilde{T}$ does not change the error bounds.

PROPOSITION 2. *The matrix T verifies the following properties*

(i) *The inverse matrix of* T, $T^{-1} = U$, *is given by*

$$U = (u_{ij}) = \begin{cases} 0, & j < i, \\ U_{j-i}(x), & j \geq i, \end{cases} \tag{5}$$

where $U_k(x)$ *is the Chebyshev polynomial of the second kind of degree* k.

(ii) $|T|\,|U| = |U|\,|T|$,

(iii) $|T|\,|U| = U^A + |U| - I \leq 2\,U^A$ *where*

$$U^A = (u_{ij}^A) = \begin{cases} 0, & j < i, \\ 1, & j = i, \\ U_{j-i}^A(x), & j > i. \end{cases} \tag{6}$$

Proof. (i) The matrix T is an upper triangular matrix with $t_{ii} = 1$, and $(t_{1j} = t_{1+k,j+k}$ $(1 < j,\ 0 \leq k \leq n + 2 - j)$; therefore its inverse matrix, U, has the same structure, that is to say,

$$U = (u_{ij}) = \begin{cases} 0, & j < i, \\ 1, & j = i, \\ u_{j-i}^*, & j > i. \end{cases}$$

In order to obtain the inverse matrix we must solve the recurrence equation

$$u_{j-i+2}^* - 2\,x\,u_{j-i+1}^* + u_{j-i}^* = 0,$$

with $u_0^* = 1$ and $u_1^* = 2x$, which is the triple recurrence relation that verifies the Chebyshev polynomials of the second kind [1, 8].

(ii) As before, both matrices, T and U, are upper triangular matrices with 1 in the principal diagonal and where each diagonal has equal each element. Consequently, they commute among them. Obviously, we have the same situation for $|T|$ and $|U|$.

(iii) After a little manipulation we obtain $|T|\,|U| = A$, where

$$A = (a_{ij}) = \begin{cases} 0, & j < i, \\ 1, & j = i, \\ |U_1(x)| + 2\,|x|\,, & j = i + 1, \\ |U_{j-i}(x)| + 2\,|x|\,|U_{j-i-1}(x)| + |U_{j-i-2}(x)|\,, & j > i + 1. \end{cases}$$

The result follows from the definition of the polynomials $U_i^A(x)$, the property (ii) and the inequality $|U_i(x)| \leq U_i^A(x)$. ∎

3. Backward and forward error bounds

In this section we introduce the *backward* and *forward* error bounds for the Clenshaw algorithm. We begin with the backward error bound.

THEOREM 3. *Let $T\,q = c$ be the system of linear equations equivalent to the Clenshaw algorithm, then the solution $\widehat{q}$ calculated by substitution satisfies*

$$(T + \Delta T)\,\widehat{q} = c, \qquad |\Delta T| \leq 2u\,|T| + \mathcal{O}(u^2). \tag{7}$$

Proof. To solve the system is equivalent to follow the Clenshaw algorithm, thus

$$q_i = 2\,x\,q_{i+1} - q_{i+2} + c_i.$$

Now, taking into account the rounding errors in the computation and using (2) we obtain

$$\widehat{q}_i = \frac{(2\,x\,\widehat{q}_{i+1}\,(1+\alpha_i) - \widehat{q}_{i+2})(1+\beta_i) + c_i}{(1+\gamma_i)},$$

where $|\alpha_i|\,, |\beta_i|\,, |\gamma_i| < u$. Therefore, we have

$$(\widehat{q}_i - 2\,x\,\widehat{q}_{i+1} + \widehat{q}_{i+2}) + \widehat{q}_i\,\gamma_i - 2\,x\,\widehat{q}_{i+1}\,(\alpha_i + \beta_i + \alpha_i\,\beta_i) + \widehat{q}_{i+2}\,\beta_i = c_i.$$

Finally, we obtain the result by only retaining the terms up to first order in u. ∎

The forward error bound is given by the following result.

THEOREM 4. *Let $T\,q = c$ be the system of linear equations equivalent to the Clenshaw algorithm, then the value $\widehat{q}_0$ given by the algorithm satisfies*

$$|\widehat{q}_0(x) - q_0(x)| \leq 4\,u \sum_{j=0}^{n} \rho_j(x)\,|c_j| + \mathcal{O}(u^2), \tag{8}$$

where $\rho_j(x) = \sum_{i=0}^{j} |U_i(x)|\,U^A_{j-i}(x)$.

Proof. From Theorem 3 we have $(T + \Delta T)\,\widehat{q} = c$, with $|\Delta T| \le 2\,u\,|T|$. Now, by using the relations $\widehat{q}(x) = (T + \Delta T)^{-1}\,c$, $q(x) = T^{-1}\,c$ and

$$(T + \Delta T)^{-1} = U - U \cdot \Delta T \cdot U + \mathcal{O}((\Delta T)^2) = U - U \cdot \Delta T \cdot U + \mathcal{O}(u^2),$$

we obtain

$$\begin{aligned} |q(x) - \widehat{q}(x)| &\le 2\,u\,|U|\,|T|\,|U|\,|c| + \mathcal{O}(u^2) \\ &= 2\,u\,|U|\,|U|\,|T|\,|c| + \mathcal{O}(u^2) \\ &\le 4\,u\,|U|\,|U^A|\,|c| + \mathcal{O}(u^2), \end{aligned}$$

where we have used Proposition 2.

Taking the first component of the vector $|q(x) - \widehat{q}(x)|$ and after some manipulation we obtain

$$\begin{aligned} |q_0(x) - \widehat{q}_0(x)| &\le 4\,u \sum_{j=0}^{n} \left(\sum_{i=0}^{j} |U_i(x)|\, U_{j-i}^A(x) \right) |c_j| + \mathcal{O}(u^2) \\ &= 4\,u \sum_{j=0}^{n} \rho_j(x)\,|c_j| + \mathcal{O}(u^2). \end{aligned}$$

∎

The above results can be extended to any set of polynomials $\{p_k(x)\}$ that satisfies a triple recurrence relation of the type

$$p_k(x) - a(x)\,p_{k-1}(x) - b(x)\,p_{k-2}(x) = 0 \tag{9}$$

where $p_0(x) = 1$ and $p_1(x) = a(x)$. In this case we obtain a matrix P whose inverse verifies

$$P^{-1} = \left(p_{ij}^{-1}\right) = \begin{cases} 0, & j < i, \\ p_{j-i}(x), & j \ge i. \end{cases}$$

The proof is similar to Proposition 2. The Theorems 3 and 4 can also be easily extended.

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