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Irreducible Characters of a Sylow p-Subgroup of the Orthogonal Group

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1. Introduction

We let p be an odd prime and $q = p^t$ be a power of p. For any prime power d, we denote by $\mathbf{F}_d$ the field with d elements. For any positive integer m, we denote by $O^+(2m, q)$ the orthogonal group of invertible linear transformations on a vector space of dimension $2m$ over $\mathbf{F}_q$ which preserve the bilinear form represented by the matrix

$$K = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

We denote by $U(n, q)$ the unitriangular group of degree n over $\mathbf{F}_q$. This group consists of all lower-triangular $n \times n$ matrices with entries in $\mathbf{F}_q$ which have every diagonal entry equal to 1. We let

$$P(2m, q) = O^+(2m, q) \cap U(2m, q).$$

$P(2m, q)$ is a Sylow p-subgroup of $O^+(2m, q)$. For ease of notation we will write P instead of $P(2m, q)$. Previtali, making use of a result of Isaacs, has shown that the degrees of the irreducible characters of P are all powers of q (see [3], [2]). In [4], he proves that

$$\{\Gamma(1) \colon \Gamma \in \operatorname{Irr}(P)\} = \{q^b : 0 \leq b \leq f(m)\}$$

and gives upper and lower bounds for the value of the function $f(m)$. Herein we announce that $f(m)$ is equal to Previtali's lower bound and give the number of irreducible characters of P which have degree $q^{f(m)}$.

2. Preliminary Results

We will need the following theorem, which follows partly from standard Clifford theory and includes a result of Gallagher. (See Corollary 6.17 and Problem 6.18 of [1].)

THEOREM 2.1. *We let G be a finite group, N be a normal abelian subgroup of G, and H be a subgroup of G which complements N, i.e. $H \cap N = 1$ and $G = HN$. For all elements λ of* $\operatorname{Irr}(N)$, *we let $T_G(\lambda)$ be the inertia subgroup of λ in G and $S_G(\lambda) = T_G(\lambda) \cap H \cong T_G(\lambda)/N$. Then*

$$\{\Gamma(1) \colon \Gamma \in \operatorname{Irr}(G)\} = \{\gamma(1)[G : T_G(\lambda)] \colon \lambda \in \operatorname{Irr}(N), \gamma \in \operatorname{Irr}(S_G(\lambda))\}.$$

DEFINITION. We let $\Upsilon = \{D \in M_m(F_q) \colon D^T = -D\}$.

Partitioning the elements of P into $m \times m$ blocks, we get

$$P = \left\{ \begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in U(2m,q) : B^{-1} = A^T \text{ and } A^T C \in \Upsilon \right\}.$$

We define a normal abelian subgroup N of P and its complement H in P by

$$N = \left\{ \begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix} : C \in \Upsilon \right\},$$
$$H = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} : A \in U(m,q) \right\}.$$

DEFINITION. For each D in Υ, we define $\Lambda_D \colon N \to \mathbb{C}$ by

$$\Lambda_D \left(\begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix} \right) = \omega^{T(\operatorname{tr} CD)}$$

where ω is a primitive p^{th} root of 1 and $T \colon \mathbf{F}_q \to \mathbf{F}_p$ is the trace mapping from an extension field to the ground field.

LEMMA 2.2. $\operatorname{Irr}(N) = \{\Lambda_D \colon D \in \Upsilon\}$ *and $\Lambda_D = \Lambda_{D'}$ if and only if $D = D'$.*

For each element D of Υ, we define $S(D) = S_P(\Lambda_D)$. Then

$$S(D) = \{A \in U(m,q) \colon ADA^T = D\}.$$

We let $\mathcal{O}(D) = \{ADA^T \colon A \in U(m,q)\}$. Viewing D as the matrix of an alternating bilinear form $(\cdot,\cdot)$ on a vector space V of dimension m over $\mathbf{F}_q$ and A as a change of basis matrix, we get

$$S(D) = \{A \in U(m,q) \colon (uA, vA) = (u,v) \text{ for all } u \text{ and } v \text{ in } V\}.$$

For all D, there exists A in $U(m,q)$ such that ADA^T has at most one non-zero entry in each row and column. ADA^T is unique in $\mathcal{O}(D)$ with this property, which we call property $(*)$. We also say that the corresponding basis of V has property $(*)$.

3. A Description of the Group $S(D)$

In the proof of the following theorem, we first consider the case where the form $(\cdot,\cdot)$ is non-degenerate, using an induction argument. We then move on to the case where the radical of V with respect to the form is non-zero. We find two types of generators of $S(D)$ in the non-degenerate case, and then another type associated with the radical.

THEOREM 3.1. *We let $(\cdot,\cdot)$ be an alternating bilinear form on a vector space V of dimension m over $\mathbf{F}_q$ and we let $\{e_1, e_2, \ldots, e_m\}$ be a basis of V which has property $(*)$ with respect to the form $(\cdot,\cdot)$. We let D be the $m \times m$ matrix over $\mathbf{F}_q$ whose (i,j)-entry is given by (e_i, e_j). Letting $r = 2s$ be the rank of the matrix D, we define rows $l_1, l_2, \ldots, l_{m-r}$ to be the zero rows of D, where $l_1 < l_2 < \ldots < l_{m-r}$, and*

$$\Omega = \{1, 2, \ldots, m\} - \{l_1, l_2, \ldots, l_{m-r}\}.$$

We define a permutation π on the set Ω by $\pi(i) = j$ if and only if the (i,j)-entry of D is non-zero. We define

$$\Xi_\pi = \{(i,j) \in \Omega \times \Omega \colon i < j \text{ and } \pi(i) > \pi(j) > i\}$$

and

$$N(\pi) = |\Xi_\pi|.$$

Then

$$|S(D)| = q^{s + N(\pi) + m - l_1 + m - l_2 + \cdots + m - l_{m-r}}.$$

Moreover, $S(D)$ is generated by the set of all matrices of the following three forms:

- (1) $I + \lambda E_{\pi(i)i}$ *where i is any element of Ω for which $\pi(i) > i$;*
- (2) $I + \lambda E_{ji} - \lambda \frac{(e_{\pi(i)}, e_i)}{(e_{\pi(j)}, e_j)} E_{\pi(i)\pi(j)}$ *where (i,j) is any element of Ξ_π;*
- (3) $I + \lambda E_{jl_t}$ *where l_t is any element of $\{1, 2, \ldots, m\} - \Omega$ and $l_t < j \leq m$.*

An easy induction argument gives us the following information about the possible orders of the group $S(D)$ and we deduce the rest from our characterization of the generators of $S(D)$.

COROLLARY 3.2. $\{|S(D)| \colon D \in \Upsilon\} = \left\{q^a \colon [\frac{m}{2}] \leq a \leq \frac{m(m-1)}{2}\right\}$. *Furthermore, if $S(D)$ is of minimal order then $S(D)$ is abelian.*

4. CONCLUSION

Each D in Υ for which $|S(D)| = q^{[\frac{m}{2}]}$ gives rise to $q^{[\frac{m}{2}]}$ irreducible characters of P of degree $q^{\frac{m(m-1)}{2} - [\frac{m}{2}]}$, one for every element of $\operatorname{Irr}(S(D))$. However, if D' is such that $S(D')$ is of greater than minimal order, then it is possible to find an abelian subgroup $M(D')$ of $S(D')$ such that $|M(D')| = q^{[\frac{m}{2}]+1}$. If Θ is an element of $\operatorname{Irr}(P)$ lying over $\Lambda_{D'}$, then

$$\Theta(1) < q^{\frac{m(m-1)}{2} - [\frac{m}{2}]}.$$

Thus the irreducible characters of P of maximal degree lie over irreducible characters Λ_D of N for which $|S(D)|$ is minimal. We count the number of elements D of Υ which have property $(*)$ and also satisfy $|S(D)| = q^{[\frac{m}{2}]}$. We multiply this number by $q^{[\frac{m}{2}]}$ to find the number of irreducible characters of P of maximal degree. The arguments above and Theorem 2 of [3] yield the following.

THEOREM 4.1. $f(m) = \frac{m(m-1)}{2} - [\frac{m}{2}]$, *i.e.*

$$\left\{\Gamma(1) : \Gamma \in \operatorname{Irr}(P)\right\} = \left\{q^b : 0 \leq b \leq \frac{m(m-1)}{2} - \left[\frac{m}{2}\right]\right\}.$$

If $m = 2s$ is even, then there are $q^{s+1}(q-1)^{s-1}$ elements of $\operatorname{Irr}(P)$ of maximal degree. If $m = 2s+1$ is odd, then there are $q^s(q-1)^s$ elements of $\operatorname{Irr}(P)$ of maximal degree.

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