EXTRACTA MATHEMATICAE Vol. 12, Núm. 3, 251 – 253 (1997)

Spaces of Operators as Continuous Function Spaces

T.S.S.R.K. Rao

Indian Statistical Institute, R.V. College Post, Bangalore 560 059
India, tss@isibang.ernet.in

(Research paper presented by R. Payá)

AMS *Subject Class.* (1991): 46B20, 47D15 *Received March* 10, 1997

1. Introduction

Let X and Y be Banach spaces. Let $\mathcal{L}(X,Y)$ and $\mathcal{K}(X,Y)$ denote spaces of bounded and compact linear operators respectively. Let X_1 denote the closed unit ball of X and $\partial_e X_1$ the set of extreme points of X_1. For any $x^{**} \in X^{**}$, $y^* \in Y^*$ let $x^{**} \otimes y^*$ denote the functional defined on $\mathcal{L}(X,Y)(\mathcal{K}(X,Y))$ by $(x^{**} \otimes y^*)(T) = x^{**}(T^*(y^*))$. Let $\Omega = X_1^{**} \times Y_1^*$. If one equipps the component spaces with the w^*-topology, Ω is a compact space in the product topology. A well known and useful isometric embedding of $\mathcal{K}(X,Y)$ into $C(\Omega)$ is given by the mapping $\phi : \mathcal{K}(X,Y) \to C(\Omega)$ defined by $\phi(T)(x^{**}, y^*) = (x^{**} \otimes y^*)(T)$ (see [4]). It is easy to see that ϕ is not onto ($\phi(T)(0, y^*) = 0 = \phi(T)(x^{**}, 0)$ for any T). This raises the question, is it possible to choose smaller compact subsets of the respective dual balls so that this correspondence is onto? It is easy to see that if $X^* = C(\Omega_1)$ and $Y = C(\Omega_2)$ (isometrically) for some compact sets Ω_1, Ω_2 then $\Omega_1 \times \Omega_2$ is a natural choice that makes the above embedding onto.

In this short note we show that if $\mathcal{K}(X,Y)$ is isometric to $C(\Omega)$ for a compact space Ω then X^* is isometric to $C(\Omega_1)$ and Y is isometric to $C(\Omega_2)$ for some compact sets Ω_1 and Ω_2. In the case of bounded operators we show that if $\partial_e X_1$ is non-empty, then that $\mathcal{L}(X,Y)$ is isometric to a $C(\Omega)$ will imply that X is isometric to an abstract L-space and Y is isometric to a $C(\Omega_1)$.

Our methods involve the L^1 predual theory, for which we shall refer to [2]. We also follow the notation of Lacey's book. Our results are valid for both real and complex scalar fields.

2. Main Result

THEOREM. *Let X and Y be Banach spaces. Suppose $\mathcal{K}(X,Y)$ is isometric to $C(\Omega)$ for some compact set Ω. Then there are compact sets Ω_1 and Ω_2 such that Ω is homeomorphic to $\Omega_1 \times \Omega_2$ and X^* is isometric to $C(\Omega_1)$ and Y is isometric to $C(\Omega_2)$.*

Proof. Fix $x_0 \in X_1, y_0 \in Y_1$ and $x_0^* \in X_1^*, y_0^* \in Y_1^*$ such that $x_0^*(x_0) = 1 = y_0^*(y_0)$. Note that $x^* \to x^* \otimes y_0 (y \to x_0^* \otimes y)$ is an isometric embedding of X^* (Y) into $\mathcal{K}(X,Y)$ and $T \to T^*(y_0^*) \otimes y_0 (T \to x_0^* \otimes T(x_0))$ is a norm one projection onto the range of this embedding. Since $C(\Omega)$ is a L^1-predual space and since the range of a norm one projection of a L^1- predual is a L^1-predual (see [5]), we conclude that both X^* and Y are L^1-predual spaces. Since any dual L^1-predual space is isometric to a continuous function space, we have that X^* is isometric to $C(\Omega_1)$ for some compact (hyperstonean) space Ω_1. Note that $\mathcal{K}(X,Y)$ can now be idetified with $C(\Omega_1, Y)$ (space of Y-valued continuous functions). To show that Y is isometric to a $C(\Omega_2)$ for a compact set Ω_2, by Proposition 6.2 of [8], it is enough to show that $\partial_e Y_1^*$ is w^*-closed and $\partial_e Y_1$ is non-empty.

To see the former, let $\{y_\alpha^*\}$ be a net in $\partial_e Y_1^*$ such that $y_\alpha^* \to y^*$ in the w^*-topology. For any $w \in \Omega_1$, $\delta(w) \otimes y_\alpha^* \in \partial_e C(\Omega_1, Y)_1^*$ and $\delta(w) \otimes y_\alpha^* \to \delta(w) \otimes y^*$ in the w^*-topology of $C(\Omega_1, Y)^*$. As $\partial_e C(\Omega_1, Y)_1^*$ is a w^*-closed set by hypothesis, we get that $\delta(w) \otimes y^* \in \partial_e C(\Omega_1, Y)_1^*$. Therefore $y^* \in \partial_e Y_1^*$. Thus $\partial_e Y_1^*$ is a w^*-closed set. Let $g \in \partial_e C(\Omega_1, Y)_1$ correspond to the constant function 1 in $C(\Omega)$. Since $C(\Omega)$ has the extreme point intersection property (see [3] for this concept), we have that $C(\Omega_1, Y)$ has the extreme point intersection property. Thus by Theorem 3 of [6] (which should read as "$C(K,X)$ has the E.P.I.P iff X has the E.P.I.P and the extreme points of $C(K,X)_1$ take extremal values"), we get that g takes values in $\partial_e Y_1$ and in particular $\partial_e Y_1$ is non-empty. Thus Y is isometric to $C(\Omega_2)$. Therefore we have that $C(\Omega_1, C(\Omega_2))$ is isometric to $C(\Omega)$. Hence by the classical Banach-Stone theorem we get that Ω is homeomorphic to $\Omega_1 \times \Omega_2$. ∎

Concerning the analogous question for the space of bounded operators we have only some partial answers. It is known (see [7], page 252) that if X is an abstract L-space and $Y = C(K)$ for some extremally disconnected compact set K then $\mathcal{L}(X,Y)$ is a $C(\Omega)$ space. On the other hand if $\mathcal{L}(X,Y)$ is a $C(\Omega)$ space, since X^* and Y embed into $\mathcal{L}(X,Y)$ (as in the case of $\mathcal{K}(X,Y)$) as ranges of norm one projections, one has that X is an abstract L-space (see

[7]) and Y is a L^1-predual space.

PROPOSITION. *Suppose X is a Banach space such that $\partial_e X_1$ is non-empty. If $\mathcal{L}(X, Y)$ is isometric to a $C(\Omega)$ space then X is isometric to a L-space and Y is isometric to $C(\Omega_1)$ for some compact set Ω_1.*

Proof. Let x_0 be an extreme point of X_1. Since X is isometric to a L-space (from the preceding discussion), the one dimensional space line$\{x_0\}$ is the range of a L-projection in X. Thus by Proposition 6.3 of [3] we get that Y is isometric to the range of a M-projection in $C(\Omega)$. Therefore by a well-known result in M-structure theory (see [1], page 3) it follows that Y is isometric to $C(\Omega_1)$ for a clopen subset Ω_1 of Ω. ∎

Remark. This argument shows that for any discrete set Γ and for any Banach space Y, if $\mathcal{L}(\ell^1(\Gamma), Y)$ (which can also be isometrically identified as the ℓ^∞ direct sum of $|\Gamma|$-many copies of Y) is a $C(\Omega)$ space, then Y is also of the same type. I do not know an example of a L-space X with $\partial_e X_1$ empty and a compact set K which is not extremally disconnected but $\mathcal{L}(X, C(K))$ is isometric to a $C(\Omega)$.

REFERENCES

[1] HARMAND, P., WERNER, D., WERNER, W., "M-ideals in Banach spaces and Banach algebras", L.N.M. Vol. 1547, Springer, Berlin, 1993.

[2] LACEY, H.E., "The isometric theory of classical Banach spaces", Springer, Berlin, 1974.

[3] LIMA, Å., Intersection properties of balls in spaces of compact operators, *Ann. Inst. Fourier,* **28** (1978), 35–65.

[4] LIMA, Å., OLSEN, G., Extreme points in duals of complex operator spaces, *Proc. Amer. Math. Soc.,* **94** (1985), 437–440.

[5] LINDENSTRAUSS, J., WULBERT, D.E., On the classification of the Banach spaces whose duals are L_1 spaces, *J. Funct. Anal.,* **4** (1969), 332–349.

[6] RAO, T.S.S.R.K., On the extreme point intersection property, in "Function spaces, the second conference", Ed. K. Jarosz, Lecture Notes in Pure and Appl. Math. 172, Marcel Dekker, 1995, 339–346.

[7] SCHAEFER, H.H., "Banach lattices and positive operators", Springer, Berlin, 1974.

[8] WULBERT, D.E., Real structure in complex L_1-preduals, *Trans. Amer. Math. Soc.,* **235** (1978), 165–181.