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Snarked Sums of Banach Spaces*

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1. Introduction

Against tradition for Banach spacers, linear maps in this paper have *not* to be assumed continuous: they are (usually) not. More yet: when a map is bounded we shall say so explicitly. If with such a map F the expression

$$\|(y,z)\|_F = \|y + Fz\| + \|z\|$$

becomes (equivalent to) a norm has to be carefully considered. Actually, the preceding map $\|\cdot\|$ is not a norm and we shall be lucky enough if it is equivalent to a norm. So, the reader should not assume that the topologies involved are locally convex at all. All expressions not explicitly explained have been taken from [2].

2. The Kalton - Peck fit

The direct sum $Y \bigoplus Z$ of two Banach spaces Y and Z is the algebraic product $Y \times Z$ endowed with the norm $\|(y,z)\| = \|y\| + \|z\|$. It is a part of the syllabus that Y and Z, identified with the subspaces $\{(y,0) : y \in Y\}$ and $\{(0,z) \colon z \in Z\}$ are closed subspaces of $Y \bigoplus Z$ such that $Y + Z = Y \times Z$ and $Y \cap Z = \{0\}$, precisely what makes Y complemented in $Y \bigoplus Z$.

When some element is modified above then the whole structure collapses and Y ceases to be complemented in $Y \bigoplus Z$. The usual reason for this to happen is that the alleged complement of Y, i.e. $\{(0,z) : z \in Z\}$, is not closed

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in $Y \bigoplus Z$ (and therefore it is not Z). There are other possibilities: that the norm itself be *twisted* by some (homogeneous) map $F\colon Z \to Y$, in the sense:

$$\|(y,z)\|_F = \|y + Fz\| + \|z\|;$$

or that $\{(y,0) : y \in Y\}$ is no longer a copy of Y.

These were the ways explored by Kalton [9], Kalton and Peck [10] and Ribe [16], originating the theory of twisted sums. A *twisted sum* (by opposition to direct sum) of Y and Z is a topological vector space X admitting Y as a (nonnecessarily complemented) subspace in such a way that the corresponding quotient X/Y is isomorphic to Z.

Just to keep up spirits let us display a twisted sum. Consider a dense hyperplane H on an infinite dimensional Banach space X. Algebraically the space X is nothing but $< u > \times H$, and nevertheless the norm on $< u > \times H$ is not the product norm; it has been "twisted" in the following sense: although the natural imbedding $< u > \times H$ is continuous, the projection $p(\lambda u, h) = \lambda u$ is not. For the same reason, the quotient norm

$$\|h\| = \inf\{\|h + \lambda u\| \colon \lambda \in \mathbb{R}\}$$

that makes H a Banach space is not the norm of H.

That everything here can be straightened is also clear. Indeed, there is a continuous projection onto $< u >$, which has the form $p(\lambda u, h) = \lambda u + Lh$ for some linear map $L\colon H \to \mathbb{R}$. Maybe it is not excessively awkward to mention now, as a warning, that this map L has to be obtained through the Hahn-Banach theorem, either directly or as a closed complement of $< u >$. To do so also amends the norm: $\|\lambda u + h\|$ is (equivalent to) the product norm $\|\lambda u + Lh\| + \|h\|$. Threfore, one can think of the norm of X as the product norm "twisted" by some linear map $L\colon Z \to \mathbb{R}$.

Assume that both the subspace Y and the quotient space Z have been fixed. Which maps $F\colon Z \to Y$ are suitable as "twisters" for the product norm? This question is actually three questions: When gives the map $\|\cdot\|_F\colon Y \times Z \to \mathbb{R}$ defined as

$$\|(y,z)\|_F = \|y + Fz\| + \|z\|$$

a reasonable vector topology on $Y \times Z$ (taking as closed unit ball the set of all points (y,z) with $\|(y,z)\|_F \le 1$)? When is $\|\cdot\|_F$ equivalent to a norm?. When is itself a norm?

The first question finds an answer in Kalton and Peck's theory of quasi-linear maps (see [10]), which shall be explained next; the second question was solved in [1] (see also [4]) and we shall see how this question leads to the definition of 0-linear map; the third question was solved by D. Yost (see [1] and [4]).

He has bought a large map representing the sea,
Without the least vestige of land:
And the crew were much pleased when they found out it to be
A map they could all understand.

3. The Enflo - Lindenstrauss - Pisier fit.

It was shown in [10] that twisted sums of Y and Z are in correspondence with quasi-linear maps $F : Z \to Y$. These are homogeneous maps verifying that for some constant $K > 0$ and all points $x, y \in Z$

$$\|F(y + x) - Fy - Fx\| \leq K(\|y\| + \|x\|).$$

Given a quasi-linear map F, the formula $\|(y, z)\|_F = \|y + Fz\| + \|z\|$ defines a quasi-norm on $Y \times Z$, that becomes in this way a twisted sum of Y and Z denoted by $Y \bigoplus_F Z$; in particular, Y is isometric to the subspace $Y_0 = \{(y, 0) : y \in Y\}$ of $Y \bigoplus_F Z$ and Z to the corresponding quotient. Moreover, all twisted sums can be obtained in this manner. No, there is no simple way to make that expression a norm (even $K = 1$ would produce a quasi-norm with constant 2). Although the quasi-norm $\|\cdot\|_F$ is, sometimes, equivalent to a norm on $Y \times Z$, it cannot always be "straightened" to be made equivalent to the product norm. Things could go worse since it may even happen that it cannot be even made equivalent to a norm whatsoever.

Thus, one of the unexpected results of the theory is that *twisted sums of Banach spaces need not be locally convex*; this was shown by Kalton, Ribe and Roberts (see [11], [16]) who independently produced an example of a twisted sum of $\mathbb{R}$ and l_1 where $\mathbb{R}$ is not complemented.

Kalton's analysis turned to properties of the Banach spaces Y and Z making *all* their twisted sums locally convex; he obtained [9] that *If Y and Z are B-convex, every twisted sum of Y and Z is locally convex.* Let us ask instead: when is a given twisted sum of Y and Z locally convex? This question is better considered if one undoes all the way, back to the cornerstone of three-space problems, the paper [8] of Enflo, Lindenstrauss and Pisier.

To obtain a solution to Palais problem, i.e., a non-Hilbert space which we shall call ELP admitting a subspace isometric to l_2 in such a way that ELP/l_2

is also isometric to l_2, they proceed as follows. Let $h(n)$ be the Hilbert space of dimension n. They consider homogeneous maps $f\colon h(n) \to h(n^2)$ having the property that whenever $z_1, \dots, z_n$ is a finite set of points such that $\sum z_i = 0$ then

$$\|\sum fz_i\| \leq \sum \|z_i\|.$$

With such a map at hand they endow the product space $h(n^2) \times h(n)$ with the norm having as unit ball the closed convex hull of all points $(f(x), x)$ with $\|x\| \leq 1$ and $(x, 0)$ with $\|x\| \leq 1$. It turns out that $h(n^2)$ is isometric to $\{(x, 0)\colon \|x\| \leq 1\}$ while the corresponding quotient space is isometric to $h(n)$. With a proper choice of the maps f_n (necessary to make the projection constants onto $h(n^2)$ increase with n) they construct the norm $\|\cdot\|_n$ on the product space $h(n^2) \times h(n)$ as indicated. Let us denote $ELP(n)$ this finite dimensional resultant Banach space. The space $ELP = l_2(ELP(n))$ contains an uncomplemented isometric copy of the Hilbert space $l_2(h(n^2))$ such that $ELP/l_2(h(n^2))$ is isometric to the Hilbert space $l_2(h(n))$.

SNARKED SUMS. Let $F\colon Z \to Y$ be a quasi-linear map; the Snarked sum of Y and Z defined by F, which we denote $Y\boldsymbol{S}_F Z$, is the product space $Y \times Z$ endowed with the norm $\|\cdot\|_S$ having as unit ball the set:

$$C = \operatorname{conv}\{(y, 0)\colon \|y\| \leq 1\} \cup \{(F(z), z)\colon \|z\| \leq 1\},$$

or, in other words, the convex hull of the unit ball of $\|\cdot\|_F$. It is therefore clear that $Y \bigoplus_F Z$ is locally convex if and only if coincides with $Y\boldsymbol{S}_F Z$. This coincidence means that for some constant K one has $\|\cdot\|_S \leq \|\cdot\|_F \leq K\|\cdot\|_S$. Therefore, if $(y, 0) \in C$, i.e., if for some finite set (z_j) of elements of Z and some finite set (y_i) of elements of Y (all having norm lesser than or equal to 1) one has

$$\begin{aligned} y &= \sum \lambda_i y_i + \sum \mu_j F z_j \\ 0 &= \phantom{\sum \lambda_i y_i + {}} \sum \mu_j z_j, \end{aligned}$$

whith $\sum \lambda_i + \sum \mu_j = 1$, $\lambda_i \geq 0$ and $\mu_j \geq 0$, then $\|y\| = \|(y, 0)\| \leq K$. This means that $\|\sum \lambda_i y_i + \sum \mu_j F z_j\| \leq K$, and therefore $\|\sum \mu_j F z_j\| \leq K + 1$. By homogeneity, what has been proved is that F verifies the following property that we called

0-LINEARITY: For some constant $K(F)$ and every finite set (z_j) of elements of Z such that $\sum z_j = 0$ one has

$$\|\sum F z_j\| \leq K(F) \sum \|z_j\|.$$

Therefore: *A twisted sum $Y \bigoplus_F Z$ is locally convex if and only if F is 0-linear.*

This was charming, no doubt: but they shortly found out
That the Captain they trusted so well
Had only one notion for crossing the ocean,
And that was to tingle his bell.

4. The Ribe fit

Snarked sums are charming, no doubt: but one shortly finds out that a Snarked sum of Y and Z may not be a twisted sum of Y and Z. To verify this, let $Y_0 = \{(y, 0) \colon y \in Y\}$, who should be a copy of Y. It is easy to verify that Z is isometric to $(Y \boldsymbol{S}_F Z)/Y_0$. But, contrarily to twisted sums (where Y_0 is isometric to Y), it may happen that Y_0 is not isomorphic to Y.

The subspace Y_0 is K-isomorphic to Y if and only if $\|\cdot\|_F$ and $\|\cdot\|_S$ are K-equivalent.

Once the surprise passes by, one realizes that one of the implications is obvious since Y is isometric to Y_0 in $\|\cdot\|_F$. And, conversely, if one assumes that Y_0 is K-isomorphic to Y then we shall prove that F is 0-linear with constant K; being thus $Y \bigoplus_F Z$ locally convex and isomorphic to $Y \boldsymbol{S}_F Z$.

To verify that F is 0-linear, let $z_1, \dots, z_n$ be a finite set of elements of Z such that $\sum z_i = 0$. Let us call $d = \sum \|z_i\|$. The points $(F(d^{-1}z_i), d^{-1}z_i)$ belong to $d^{-1}\|z_i\|\, C$. Thus

$$(\sum_{i=1}^{n} F(d^{-1}z_i), 0) = \sum_{i=1}^{n} (F(d^{-1}z_i), d^{-1}z_i) \in C$$

Since the subspace $\{(y, 0) \colon y \in Y\}$ of $Y \bigoplus_F Z$ is isometric to Y, one has

$$\|\sum_{i=1}^{n} F(d^{-1}z_i)\| = \|(\sum_{i=1}^{n} F(d^{-1}z_i), 0)\|_F \leq K \, \|(\sum_{i=1}^{n} F(d^{-1}z_i), 0)\|_C \leq K,$$

that by homogeneity yields

$$\|\sum_{i=1}^{n} F(z_i)\| \leq Kd.$$

But the principal failing ocurred in the sailing,
And the Bellman, perplexed and distressed,
Said he had hoped, at least, when the wind blew due East
That the ship would not travel due west!

5. The "À la Helly" fit

The principal failing that occurred in the chapter preceding should not distract ourselves from the interesting fact that duality theory exists for Snarked sums: after all they are Banach spaces. And more yet: the norm in $(Y\boldsymbol{S}_F Z)^*$ need not definition.

$$\|\mu\| = \sup_{\|(y,z)\|\leq 1} |\mu(y,z)|.$$

The only thing open to reflection is the form of the duality. On the one hand, $(Y\boldsymbol{S}_F Z)^*$ has to be a Snarked sum of Z^* and Y^*, which identifies μ with some couple (z^*, y^*). Duality then has to be:

$$< (z^*, y^*), (y,z) >= y^*(y) + z^*(z) + Tw(y,z)$$

where Tw designs something linear that "twists" the product-like action

$$y^*(y) + z^*(z).$$

We determine the nature of Tw. Knowing μ on points $(y,0)$ and (Fz,z) determines its behaviour over C and thus on all $Y\boldsymbol{S}_F Z$.

Since μ admits restriction to Y^*, μ should act on Y_0 as y^* :

$$< (z^*, y^*), (y,0) >= y^*(y) + Tw(y,z) = y^*(y)$$

which implies $Tw(y,0) = 0$. Thus, Tw can be interpreted a linear map on Z :

$$z \to Tw(0,z).$$

This is not everything Tw does: since μ is continuous

$$\|\mu\| = \sup_{(y,z)\in C} |\mu(y,z)| < \infty$$

and it is enough to calculate this supremum on points $(y,0)$ and (Fz,z). What happens on points $(y,0)$ is so boring that we only look at points (Fz,z) :

$$\sup_{\|z\|\leq 1} |y^*(Fz) + z^*(z) + Tw(Fz,z)| < \infty,$$

which implies, since z^* is continuous, that

$$\sup_{\|z\|\leq 1} |y^*(Fz) + Tw(Fz, z)| < \infty,$$

or else

$$\sup_{\|z\|\leq 1} |y^*(Fz) + Tw(0, z)| < \infty.$$

All together now: Tw is an element of Z' at finite distance of y^*F. This is a nice identification of the elements $\mu \in (Y\boldsymbol{S}_F Z)^*$ as: $\mu = (z^*, y^*)$ where

$$\begin{aligned} y^* &= \mu|Y_0 \\ z^* &= y^*F - Tw. \end{aligned}$$

See [1] for a far reaching approach to this topic.

The Bellman looked uffish, and wrinkled his brow.
"If only you'd spoken before!
It's excessively awkward to mention it now,
With the Snark, so to speak, at the door!

6. The Hunting

Duality theory begins with the Hahn-Banach theorem. This asserts that given a Snarked sum $\mathbb{R}\boldsymbol{S}_F Z$ induced by a 0-linear map $F\colon Z \to \mathbb{R}$ the identity of $\mathbb{R}$ can be continuously extended to a functional

$$L\colon \mathbb{R}\boldsymbol{S}_F Z \to \mathbb{R}.$$

Since $L(\lambda, z) = L(\lambda, 0) + L(0, z) = \lambda + L(0, z)$, it is possible to define a linear (not necessarily continuous) map $l : Z \to \mathbb{R}$ by $l(z) = L(0, z)$. This makes L to adopt the form $L(\lambda, z) = \lambda + l(z)$. Moreover, it is possible to define a linear (not yet continuous) map $T : \mathbb{R}\boldsymbol{S}_F Z \to \mathbb{R} \bigoplus Z$ by means of $T(\lambda, z) = (\lambda + l(z), z)$. This map is continuous:

$$\|T(\lambda, z)\| = |\lambda + lz| + \|z\| \leq \|L\| \, \|(\lambda, z)\|_F + \|z\| \leq (\|L\| + 1)\|(\lambda, z)\|_F.$$

Since T is clearly surjective and injective (its inverse map is $(\lambda, z) \to (\lambda - lz, z)$) the open mapping theorem asserts that it is an isomorphism.

Thus the Hahn-Banach theorem can be read, in terms of Snarked sums, as: *every Snarked sum $\mathbb{R}\boldsymbol{S}_F Z$ is isomorphic to the direct sum $\mathbb{R} \bigoplus Z$.* But it is more interesting to go ahead and observe that although nor F – the 0-linear map that induces the Snarked sum – neither l are continuous, their distance (actually the distance to -l) in the classical norm is finite:

$$\|Fz + lz\| \leq \|Fz + lz\| + \|z\| = \|T^{-1}(0, z)\|_F \leq \|T^{-1}\| \, \|z\|.$$

Therefore, the following interpretation of the Hahn-Banach theorem is also possible: *Every* 0*-linear map $F : Z \to \mathbb{R}$ is at finite distance from some linear map $l : Z \to \mathbb{R}$.*

The hunting of that linear map, which necessarily involves a proof of the Hahn-Banach theorem, appears in [1]. Actually, this is one of the central topics we are discussing: every time that the copy of Y in a snarked twisted sum $Y \boldsymbol{S}_F Z$ is complemented there is some linear map $L : Z \to Y$ at finite distance from F (the proof is a rewriting of what we did for $\mathbb{R}$). When this happens one says that the twisted sum $Y \bigoplus_F Z$ splits.

Thus, the existence of a linear map at finite distance of a given 0-linear map $F \colon Z \to Y$ is equivalent to the fact that the copy of Y inside $Y \bigoplus_F Z$ is complemented. When such linear map exist, the hunting of a specimen is a rewarding task.

They sought it with thimbles, they sought it with care;
They pursued it with forks and hope;
They threatened his life with a railway-share;
They charmed it with smiles and soap.

7. The Sobczyk - Lindenstrauss fit

Two such situations are presented now.

The first one we would like to mention is Sobczyk's theorem: *c_0 is complemented in every separable space X containing it.* Therefore, from the snarked point of view, this can be read as: every 0-linear map $F : Z \to c_0$ from a separable space into c_0 is at finite distance from some linear map $l : Z \to c_0$ (observe that X is separable if and only if $X/c_0 = Z$ is separable).

The second is due to Lindenstrauss, who proved [14] that every Snarked twisted sum of a space complemented in its bidual and a $\mathfrak{L}_1$-space splits. Again, this means that every 0-linear map $F : \mathfrak{L}_1 \to Y$ from a $\mathfrak{L}_1$-space into a space Y complemented in its bidual is at finite distance from some linear map $l : \mathfrak{L}_1 \to Y$.

The hunting of the suitable linear map for these two cases can be followed at [1].

But if ever I meet with a Boojum, that day,
In a moment (of this I am sure),
I shall softly and suddenly vanish away-
And the notion I cannot endure!

8. The Kalton - Roberts fit

If one looks for linear maps at finite distance one may find 0-linear maps at finite distance. This is the question of when all twisted sums of two given Banach spaces are Snarked sums, and is also interesting. Unfortunately, it is not easy to construct quasi-linear or 0-linear maps (it is not easy even to construct linear or bounded maps!). A quasi-linear not 0-linear map $F : l_1 \to \mathbb{R}$ was constructed by Ribe [16]. A method to construct quasi-linear maps between certain sequence spaces was given in [10]. A general method appears in [3]. Since Kalton [9] showed that *Twisted sums of B-convex Banach spaces are locally convex*, then an appeal to the statement al the end of section 3 shows: *A quasi-linear map acting between B-convex Banach spaces is* 0*-linear.*

On the other hand, B-convex Banach spaces are not the border: Kalton and Roberts [13] proved: *a twisted sum of $\mathbb{R}$ and a $\mathfrak{L}_\infty$-space is locally convex*, which yields: *A quasi-linear map from a $\mathfrak{L}_\infty$-space into $\mathbb{R}$ is* 0*-linear.*

However, direct proofs for those results would be welcome.

Taking three as the subject to reason about-
A convenient number to state-
We add Seven, and Ten, and then multiply out
By One Thousand diminished by Eight

9. The Dierolf - Díaz - Domanski - Fernández fit

Taking three as the subject to reason about, recall that a property P is said to be a three-space property if whenever $0 \to Y \to X \to Z \to 0$ is a short exact sequence where Y and Z have P then also X has P. The following problems were apparently open:

Problem 1. Is "being isomorphic to a dual space" a three-space property?

PROBLEM 2. Is "being complemented in its bidual" a three-space property?

An example in [7] solves in the negative both questions in the setting of Fréchet spaces: it is constructed an exact sequence $0 \to Y \to F \to l_1 \to 0$ where Y is a Fréchet Montel space (hence reflexive) and where F is not complemented in its bidual (hence, not a dual space). Moreover, they show that if $0 \to Y^* \to X \to R \to 0$ is an exact sequence of Banach spaces with R reflexive and Y^* a dual space then X is also a dual space and the sequence is a dual sequence. This last result refers to a question attributed to Vogt [17]: *if* $0 \to Y^* \to X \to Z^* \to 0$ *is an exact sequence, has it to be a dual sequence?*

Two partial results, in a sense dual one of the other, are: *An exact sequence* $0 \to Y^* \to X \to R \to 0$ *where* R *is reflexive is a dual sequence*; and: *An exact sequence* $0 \to R \to X^* \to Z \to 0$, *where* R *is reflexive, is a dual sequence.*

Problems 1, 2 and Vogt's question have been answered in the negative also in the Banach space setting (see [1] for full details and proofs). Indeed, *There exists an exact sequence* $0 \to l_2 \to D \to W^* \to 0$ *where* D *is not isomorphic to a dual space.* Also, *There exists an exact sequence* $0 \to Y \to X \to Z^{**} \to 0$ *where* Y *is complemented in its bidual and* X *is not.*

A partial positive answer to problem 2 can be obtained when the reflexive space is placed at the end of the sequence (see also [7]). *If* $0 \to Y \to X \to R \to 0$ *is an exact sequence with* R *reflexive and* Y *complemented in its bidual then* X *is complemented in its bidual.*

The paper [1] displays a rather interesting nonlinear duality theory in which, however, there remain several intriguing conundrums to guess.

In the midst of the words he was trying to say,
in the midst of his laughter and glee,
He had softly and suddenly vanished away-
For the Snark was a Boojum, you see.

10. THE ORLICZ FIT

One is perhaps tempted to believe that locally convex twisted sums and Snarked sums are different names for the same thing. In a sense, that is true. Nevertheless, between Kalton and Peck solution [10] to Palais problem and that of Enflo, Lindenstrauss and Pisier [8] the differences are not small change. Kalton and Peck show that the map $F : l_2 \to l_2$ given by a suitable extension of the following map defined over the finite sequences

$$F(x) = \sum_i x_i \log|x_i| - \big(\sum_i x_i\big) \log\big|\sum_i x_i\big|$$

is quasi-linear and is not at finite distance from linear maps. Since Hilbert spaces are B-convex, the twisted sum $l_2 \bigoplus_F l_2$ is locally convex (hence a Snarked sum). The solution of Enflo, Lindenstrauss and Pisier has been already shown: it is a true Snarked sum.

Now, the vectors $(0, e_n)$ span in $l_2 \bigoplus_F l_2$ a copy of the Orlicz space l_M, where $M(x) = x \log x$. In particular, that sequence does not satisfy an upper 2-estimate (it is not weakly 2-summable, if one prefers). However, the space ELP constructed as an l_2-sum of finite dimensional spaces verifies that every weakly null sequence contains a weakly 2-summable subsequence (see [5]; and also [6]). In the terms of [6], ELP is in the class W_2 while $l_2 \bigoplus_F l_2$ is not. It was Raquel Gonzalo who observed these facts.

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