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A non-Semiprime Associative Algebra with Zero Weak Radical

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1. Introduction

The weak radical, $W\text{–}Rad(A)$ of a non-associative algebra A, has been introduced by A. Rodríguez Palacios in [3] in order to generalize the Johnson's uniqueness of norm theorem to general complete normed non-associative algebras. (See also [2] for another application of this notion). In [4], he showed that if A is a semiprime non-associative algebra with DCC on ideals, then $W\text{–}Rad(A) = 0$. In the first part of this paper we give an example of a non-semiprime associative algebra A with DCC on ideals and $W\text{–}Rad(A) = 0$. As a consequence, we shall see that, in the class of all associative algebras, the subclass $\mathcal{S} = \{A : W\text{–}Rad(A) = 0\}$ is not a semisimple class relative to a radical in the sense of Amitsur-Kurosh. In the second part of this paper, we shall establish the coincidence between the weak radical and the maximal nilpotent ideal in a finite dimensional Jordan algebra.

2. Preliminaries and notations

(i) Let A be an associative algebra, we say that an element a of A is quasi-invertible (q.i.) in A if there exists an element b in A such that $a+b = ab = ba$. A subalgebra B of A is said to be full in A if every element of B q.i. in A is q.i. in B.

(ii) Let A be a non-associative algebra over a field K and $\mathcal{L}_K(A)$ the (associative) K-algebra of the endomorphisms of the vector space A. For every $a \in A$, we denote by L_a and R_a the elements of $\mathcal{L}_K(A)$ defined by :

$$L_a: \begin{array}{l} A \longrightarrow A \\ x \longmapsto ax, \end{array} \qquad R_a: \begin{array}{l} A \longrightarrow A \\ x \longmapsto xa, \end{array}$$

called respectively the left and the right multiplication by a. The multiplication algebra, $M(A)$ of A is defined to be the subalgebra of $\mathcal{L}_K(A)$ generated by the set $\{L_a, R_a : a \in A\}$. The full multiplication algebra, $FM(A)$, of A is the smallest full subalgebra of $\mathcal{L}_K(A)$ containing the set $\{L_a, R_a : a \in A\}$.

(*iii*) The weak radical, $W\text{–}Rad(A)$, of a non-associative algebra A, is the largest $FM(A)$-invariant subspace of $\{a \in A : L_a, R_a \in J(FM(A))\}$, where $J(FM(A))$ denotes the Jacobson radical of $FM(A)$. Clearly $W\text{–}Rad(A)$ is an ideal of A and, when A is a non-commutative Jordan algebra, $W\text{–}Rad(A)$ is contained in the Jacobson-McCrimmon radical of A [3], but even in the associative case, this inclusion may be strict (see [4]).

In what follows the term ideal will always mean a two-sided ideal. An algebra is said to be having DCC (Descending Chain Condition) on ideals if every non empty set of ideals has a minimal element. An algebra A is said to be semiprime if for every nonzero ideal I of A we have $I^2 \neq 0$.

3. Counter-example in the case of associative algebras with DCC on ideals

The starting point of our counter-example is the Weyl algebra B over a field K, generated by two elements x and y such that $yx - xy = 1$. Every element f of B can be written $f = \sum_{k=0}^{n} a_k(x)y^k$, where the $a_k(x)$ are polynomials in $K[x]$, moreover $fx - xf = \frac{\partial f}{\partial y}$, where $\frac{\partial f}{\partial y}$ denotes the partial differentiation of f with respect to y. It can be shown that if the characteristic of K is zero, then B is a simple algebra [5, Corollary 1.6.34].

We are now going to define a homomorphism $\rho : B \longrightarrow \mathcal{L}_K(B)$ and an antihomomorphism $\theta : B \longrightarrow \mathcal{L}_K(B)$ such that $\forall a, b \in B$, $\rho_a \circ \theta_b = \theta_b \circ \rho_a$. For this, it suffices to define ρ_x, ρ_y, θ_x and θ_y in $\mathcal{L}_K(B)$ such that the following equalities (1) and (2) hold:

$$[\rho_y, \rho_x] = [\theta_x, \theta_y] = 1_B, \tag{1}$$

(where 1_B denotes the identity operator on B), and:

$$[\rho_x, \theta_x] = [\rho_x, \theta_y] = [\rho_y, \theta_x] = [\rho_y, \theta_y] = 0, \tag{2}$$

where $[u,v]$ denotes the commutator $u \circ v - v \circ u$ in $\mathcal{L}_K(B)$. If we take the values:

$$\begin{cases} \rho_x = L_x & \theta_x = L_x + R_y \\ \rho_y = L_x - R_x + L_y - R_y & \theta_y = -R_x - R_y \end{cases}$$

then using the fact that $[L_y, L_x] = [R_x, R_y] = 1_B$ and $[L_a, R_b] = 0$ for all $a, b \in B$, one can easily show that the equalities (1) and (2) are satisfied.

PROPOSITION 1. *With the same notations as above and K of characteristic zero, let A be the vector space given by $A = B \times B$ and the multiplication defined in A by the rule:*

$$(a,b)(c,d) = (ac, \rho_a(d) + \theta_c(b)).$$

Then:

(i) A is an associative algebra and $J = \{O\} \times B$ is the unique ideal of A other than $\{0\}$ and A. J is also the Jacobson radical of A and is nilpotent of index two.

(ii) There exists $u \in M(A)$ and $a \in J$ such that u is invertible in $\mathcal{L}_K(A)$ and $u^{-1} \circ L_a$ is not quasi-invertible in $\mathcal{L}_K(A)$.

In particular A is a non semiprime associative algebra with DCC on ideals and whose weak radical is zero.

Proof. (i) The fact that A is an associative algebra is easily proved since we have defined a B-B-bimodule structure over the vector space $M = B$ by $a \circ m = \rho_a(m)$ and $m \circ a = \theta_a(m)$ for every $a \in B$ and $m \in M$. Thus the multiplication

$$(a,m)(b,n) = (ab, a \circ n + m \circ b)$$

defines an associative algebra structure in A (see[1]).

The subspace $J = \{0\} \times B$ is an ideal of A and $J^2 = 0$. We are now going to show that J is the unique proper and nonzero ideal of A. First, let I be a nonzero ideal contained in J, then $I = \{0\} \times L$ where L is a nonzero subspace of B. For every $b \in L$ we have:

(a) $(x,0)(0,b) = (0, \rho_x(b)) = (0, xb) \in I$, i.e. $xb \in L$.

(b) $(0,b)(x,0) = (0,\theta_x(b)) = (0,xb+by) \in I$, i.e. $xb+by \in L$ and thus $by \in L$.

(c) $(0,b)(y,0) = (0,\theta_y(b)) = (0,-bx-by) \in I$, i.e. $-bx-by \in L$ and thus $bx \in L$.

(d) $(y,0)(0,b) = (0,\rho_y(b)) = (0, xb-bx+yb-by) \in I$, i.e $xb-bx+yb-by \in L$ and thus $yb \in L$.

Consequently $xb, yb, bx, by \in L$. Thus L is invariant under left and right multiplication by the generators of B and is therfore a nonzero ideal of B. Since B is simple, we obtain $L = B$, i.e. $I = J$. This implies that J is minimal. On the other hand the algebra isomorphism $A/J \cong B$ and the simplicity of B imply that J is a maximal ideal. Now let I be a proper and nonzero ideal of A. By the minimality of J either $I \cap J = 0$ or $I \cap J = J$. The first possibility cannot happen since it implies $I \oplus J = A$ (by the maximality of J), and then, writing $(1,0) = i + j$, with i in I and j in J, j would be a non-zero idempotent in J contradicting that $J^2 = 0$. So $I \cap J = J$ and $J \subset I$. Now, again by the maximality of J, we have finally that $I = J$. Now it is clear that J is the Jacobson radical of A.

(ii) Let $u = R_{(x,1)} - L_{(x,0)} \in M(A)$. Then for all $(a,b) \in A$:

$$\begin{aligned} u(a,b) &= (a,b)(x,1) - (x,0)(a,b) \\ &= (ax, \rho_a(1) + \theta_x(b)) - (xa, \rho_x(b)) \\ &= (ax, \rho_a(1) + xb + by) - (xa, xb) \\ &= (ax - xa, by + \rho_a(1)) \end{aligned}$$

We are now going to show that u is bijective.

Injectivity: Let $(a,b) \in A$ such that $u(a,b) = (0,0)$. Then $ax - xa = 0$ and $by + \rho_a(1) = 0$. The first equation implies that $\partial a/\partial y = 0$, so $a = f(x) = \sum_{k=0}^{n} a_k x^k \in K[x]$. Therefore $\rho_a = \sum_{k=0}^{n} a_k L_{x^k}$, hence $\rho_a(1) = a$. The equality $by + \rho_a(1) = 0$ implies that $by + a = 0$ which is impossible unless $a = b = 0$.

Surjectivity: Let $(c,d) \in A$. Since the mapping $\partial/\partial y : B \longrightarrow B$ is surjective, there exists $a \in B$ such that $\partial a/\partial y = ax - xa = c$. On the other hand we have $B = K[x] \oplus By$ as vector spaces. So $a = a_1 + a_2 y$ and $d = d_1 + d_2 y$, where $a_1, d_1 \in K[x]$. Let $h = d_1 + a_2 y$ and compute $u(h, d_2) = (hx - xh, \rho_h(1) + d_2 y)$. We have $h - a = d_1 - a_1 \in K[x]$, so $hx - xh = \frac{\partial h}{\partial y} = \frac{\partial a}{\partial y} = c$. Since ρ is a homomorphism, we have $\rho_h = \rho_{d_1} + \rho_{a_2 y}$ and $\rho_{a_2 y} = \rho_{a_2} \circ \rho_y$. From the fact that $\rho_y(1) = 0$ it follows that $\rho_h(1) = \rho_{d_1}(1) = d_1$. Finally $\rho_h(1) + d_2 y = d_1 + d_2 y = d$ and $u(h, d_2) = (c,d)$. Consequently u is surjective.

$u \in M(A)$ and is bijective, so u is invertible in $\mathcal{L}_K(A)$ and $u^{-1} \in FM(A)$. On the other hand $u(1,0) = (0,1)$ thus $u^{-1}((0,1)(1,0)) = u^{-1}(0,1) = (1,0)$. This implies that $u^{-1} \circ L_{(0,1)}$ is not quasi-invertible in $FM(A)$. Hence $(0,1) \notin$ $W\!-\!Rad(A)$. But $(0,1) \in J$, thus $W\!-\!Rad(A)$ is strictly contained in J. Then the minimality of J implies that $W\!-\!Rad(A) = 0$.

To conclude, A is a non semiprime associative algebra with DCC on ideals and whose weak radical is zero. ∎

Remark. Let $\mathcal{A}$ be the class of all associative algebras and $\mathcal{S} = \{A \in \mathcal{A} : W - Rad(A) = 0\}$ then $\mathcal{S}$ is not a semisimple class relative to a radical property. For, if $\mathcal{S}$ were a semisimple class, then every ideal of an element of $\mathcal{S}$ would be an element of $\mathcal{S}$ (see [7]). But if we consider the algebra A of our example then $A \in \mathcal{S}$ whereas the ideal $J \notin \mathcal{S}$.

4. Weak radical of finite dimensional Jordan algebras

(i) Let A be a finite dimensional nonassociative algebra over a field K. Then $M(A)$ is finite dimensional and therfore a full subalgebra of $\mathcal{L}_K(A)$, so $FM(A) = M(A)$ and $W\!-\!Rad(A)$ is equal to the largest ideal contained in $\{a \in A : L_a, R_a \in J(M(A))\}$. The subalgebra $W\!-\!Rad(A)^*$ of $M(A)$ generated by the set $\{L_a, R_a : a \in W\!-\!Rad(A)\}$ is contained in $J(M(A))$ and is therfore a nilpotent algebra, so $W\!-\!Rad(A)$ is a nilpotent ideal. (See [6, Theorem 2.4]).

(ii) Recall that a Jordan algebra A, over a field of characteristic not two, is a commutative algebra satisfying the Jordan identity:

$$(x^2y)x = x^2(yx) \quad \forall x, y \in A \tag{J}$$

Linearization of (J) yields the identity:

$$[a, b, cd] + [c, b, ad] + [d, b, ac] = 0 \quad \forall a, b, c, d \in A \tag{J'}$$

where $[x, y, z]$ denotes the associator $(xy)z - x(yz)$. If $a = c$ then the preceding identity becomes $2[a, b, ad] + [d, b, a^2] = 0$. Since $char(K) \neq 2$, we have:

$$a(b(ad)) = (ab)(ad) + \frac{1}{2}[d, b, a^2] \tag{J''}$$

(iii) If I and J are two ideals of a Jordan algebra A, then the sets $I(IJ) + I^2J$ and $A(IJ) + IJ$ are also ideals of A, where MN denotes the subspace of

A generated by the set $\{xy \in A : x \in M,\ y \in N\}$, for every subspaces M and N of A. (Use (J') or see [8, Lemma 4.3.3]).

(*iv*) Let I be an ideal of a Jordan algebra A. Consider the sequence $(I_k)_{k\geq 1}$ defined by:

$$I_1 = I, \qquad I_{k+1} = A(II_k) + II_k$$

By (*iii*), $(I_k)_{k\geq 1}$ is a decreasing sequence of ideals, and we have the the following:

PROPOSITION 2. *Let A be a finite dimensional Jordan algebra and I a nilpotent ideal of A. Then there exists k such that $I_k = 0$.*

To prove this result, we need the following lemmas:

LEMMA 1. *Let A be a Jordan algebra, I and J two ideals of A, $a_1, a_2, \ldots, a_n$ $(n > 1)$ a sequence of elements of A and $b_1, b_2, \ldots, b_n$ a sequence of elements of I. Suppose that $a_1 = a_n$. Then:*

$$(\prod_{i=1}^{n} L_{a_i} L_{b_i})J \subset IJ.$$

Proof. We proceed by induction on n. If $n = 2$, then $a_1 = a_2$. For every $c \in J$, by the identity (J"), $(\prod_{i=1}^{n} L_{a_i} L_{b_i})(c) = a_1(b_1(a_1(b_2c))) = (a_1b_1)(a_1(b_2c)) + \frac{1}{2}[b_2c, b_1, a_1^2]$. We have $(a_1b_1)(a_1(b_2c)) \in IJ$. On the other hand, $[b_2c, b_1, a_1^2] = ((b_2c)b_1)a_1^2 - (b_2c)(b_1a_1^2)$ with $(b_2c)(b_1a_1^2) \in IJ$ and $((b_2c)b_1)a_1^2 \in A(I(IJ))$. Since

$$A(I(IJ)) \subset A(I(IJ) + I^2J) \subset I(IJ) + I^2J \subset IJ, \tag{3}$$

we have $((b_2c)b_1)a_1^2 \in IJ$, so $a_1(b_1(a_1(b_2c))) \in IJ$.

Suppose that the lemma is true for $n-1$, and put $(\prod_{i=1}^{n} L_{a_i} L_{b_i})(c) = a_1(b_1(a_2(b_2m)))$ where $m = (\prod_{i=3}^{n} L_{a_i} L_{b_i})(c) \in J$. The identity (J') yields:

$$a_1(b_1(a_2(b_2m))) = (a_1b_1)(a_2(b_2m)) + [a_2, b_1, a_1(b_2m)] + [b_2m, b_1, a_2a_1]$$

with $(a_1b_1)(a_2(b_2m)) \in IJ$ and $[b_2m, b_1, a_2a_1] \in A(I(IJ) + I^2J) \subset IJ$ by (3).

We have $[a_2, b_1, a_1(b_2m)] = (a_2b_1)(a_1(b_2m)) - a_2(b_1(a_1(b_2m)))$ and $(a_2b_1)(a_1(b_2m)) \in IJ$. It remains to show that $a_2(b_1(a_1(b_2m))) \in IJ$. If we apply the induction hypothesis to the sequences: $a_1, a_3, \ldots, a_n = a_1 \in A$ and $b_2, \ldots, b_n \in I$, we obtain $a_1(b_2m) \in IJ$ thus $a_2(b_1(a_1(b_2m))) \in A(I(IJ)) \subset IJ$ again by (3). Hence $a_2(b_1(a_1(b_2m))) \in IJ$. ∎

COROLLARY 1. *Let A be a Jordan algebra, I and J are two ideals of A, $a_1, \ldots, a_n$ a sequence of elements of A and $b_1, \ldots, b_n$ a sequence of elements of I. Suppose further that $a_j = a_k$ for some j, k with $j \neq k$. Then $(\prod_{i=1}^{n} L_{a_i} L_{b_i}) J \subset IJ$.*

Proof. This is a consequence of Lemma 1 and the fact that

$$(\prod_{i=1}^{m} L_{x_i} L_{y_i}) IJ \subset IJ$$

for any finite sequences $x_1, \ldots, x_m$ in A and $y_1, \ldots, y_m$ in I (proved by induction on m using (3)). ∎

LEMMA 2. *Let A be a finite dimensional Jordan algebra and I,J be ideals of A. Consider the sequence of subspaces $I_{[k]}$ defined by $I_{[1]} = J$ and $I_{[k+1]} = A(II_{[k]})$. Then there exists $k \in N$ such that $I_{[k]} \subset IJ$.*

Proof. The subspace $I_{[k]}$ is generated by the set $\{(\prod_{i=1}^{k} L_{a_i} L_{b_i})(c) : a_i \in A,\ b_i \in J,\ c \in J\}$. Let $(e_1, \ldots, e_n)$ be a basis of A. Then $I_{[k]}$ is generated by the set $\{(\prod_{q=1}^{k} L_{e_{i_q}} L_{b_q})(c)\}$. If we take $k > n$ then there exist q and q' such that $q \neq q'$ and $i_q = i_{q'}$. By Corollary 1 we have $(\prod_{q=1}^{k} L_{e_{i_q}} L_{b_q})(c) \in IJ$. Thus, for $k > dim(A)$, a generating set of $I_{[k]}$ is contained in IJ. Hence $I_{[k]} \subset IJ$. ∎

Proof of Proposition 2. Let I be a nilpotent ideal. Since the sequence $(I_k)_{k \geq 1}$ is decreasing and the algebra A is finite dimensional, there exists k such that $I_k = I_{k+1}$. Write $J = I_k$. Then $A(IJ) + IJ = J$. Our aim is to prove that $J = 0$. First we prove that $J \subset I_{[k]} + IJ$ for every k. This is true for $k = 1$ and by induction : $J \subset A(IJ) + IJ \subset A(I(I_{[k]} + IJ)) + IJ \subset I_{[k+1]} + A(I(IJ)) + IJ$ by (3). Then $J \subset I_{[k+1]} + IJ$. Now by Lemma 2, there exists k such that $I_{[k]} \subset IJ$. This implies that $J \subset IJ$ and thus $J = IJ$. By induction we obtain $J \subset I^k$ for every k. But I is nilpotent, hence $J = 0$. ∎

PROPOSITION 3. *Let A be a finite dimensional Jordan algebra, then W-Rad(A) is the largest nilpotent ideal of A.*

Proof. We have already seen, in general finite dimensional case, that $W-Rad(A)$ is a nilpotent ideal. For the reverse inclusion, let I be a nilpotent ideal of A. Consider the set $\mathcal{A} = \{u \in \mathcal{L}_K(A) : u \circ L_a(I_k) \subset I_{k+1}\ \forall a \in I\}$. This is clearly a subalgebra of $\mathcal{L}_K(A)$ and contains the set $\{L_x : x \in A\}$. Thus $M(A) \subset \mathcal{A}$. Hence for every $u \in M(A)$ and $a \in I$ $u \circ L_a(I_k) \subset I_{k+1}$. By induction we have $(u \circ L_a)^k(A) \subset I_k$. By Proposition 2, there exists k such that $I_k = 0$ and hence $(u \circ L_a)^k = 0$. It follows that for every $u \in M(A)$ and $a \in I$ $u \circ L_a$ is nilpotent. Thus for every $a \in I$ $L_a \in J(M(A))$. Consequently $I \subset W\text{–}Rad(A)$. ∎

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